J. Appl. Prob. **48,** 56–67 (2011)
Printed in England

HAZARD RATE PROPERTIES OF A GENERAL COUNTING PROCESS STOPPED AT AN INDEPENDENT RANDOM TIME

F. G. BADIA,* *University of Zaragoza*

Abstract

In this work we provide sufficient conditions under which a general counting process stopped at a random time independent from the process belongs to the reliability decreasing reversed hazard rate (DRHR) or increasing failure rate (IFR) class. We also give some applications of these results in generalized renewal and trend renewal processes stopped at a random time.

Keywords: IFR property; DRHR property; generalized renewal process; trend renewal process

2010 Mathematics Subject Classification: Primary 62E10
Secondary 60E15

1. Introduction

An interesting problem in reliability theory is to give conditions under which the reliability properties of a random time T are inherited by $N(T)$, where $\{N(t): t \geq 0\}$ is a stochastic process independent of T. Letting $\{N(t): t \geq 0\}$ be a homogeneous Poisson process, Grandell [7] and Block and Savits [4] analyzed this problem, and showed that in this case the most common reliability classes of T are preserved by $N(T)$. Esary *et al.* [6] analyzed the same problem for a renewal process and showed that if T is increasing failure rate (IFR) then $N(T)$ is discrete increasing failure rate average (IFRA) (see [6, Theorem 5.2(a)]), and that if T is new better than used (NBU) then $N(T)$ is discrete NBU (see [6, Theorem 5.3]). Moreover, in [6] it is postulated as an open problem that $N(T)$ is discrete IFRA when T is IFRA. Recently, it has been shown by Badia and Sangüesa [1] that, for a renewal process, the decreasing failure rate (DFR) and log-convex life functions are preserved. Furthermore, in that paper the authors established conditions under which a renewal process stopped at an independent random time inherits the IFR, decreasing reversed hazard rate (DRHR), IFRA, and decreasing failure rate average (DFRA) reliability classes. The assumptions are the growth conditions of the arrival times under a suitable stochastic order. Ross *et al.* [13] considered a general counting process that includes a nonhomogeneous Poisson process and a renewal process, and established sufficient conditions under which $N(T)$ is IFR (see [13, Theorem 3.1]). In [13] there are a large number of applications of previous results in Markov chains, parallel systems, nonhomogeneous Poisson processes, random sums, etc.

In this paper we obtain, by a different approach, Ross *et al.*'s [13] IFR property for a general counting process stopped at an independent random time. Furthermore, we give similar conditions under which the general counting process stopped at a random independent time

Received 27 May 2010; revision received 24 December 2010.
* Postal address: Department of Statistical Methods, University of Zaragoza, Maria de Luna 3, Zaragoza, 50018, Spain.
Email address: gbadia@unizar.es

is DRHR. The results are stated in Theorems 2 and 3. The key to the proofs are the bivariate characterizations of the hazard rate and reversed hazard rate stochastic orders (see Theorem 1) applied to consecutive arrival times of the process with the bivariate functions considered in Lemmas 2 and 3. These bivariate characterizations were introduced in [16].

The paper is organized as follows. In Section 2 we give definitions of reliability concepts, reliability classes, and stochastic orderings, and formulate useful properties, such as the bivariate characterizations of the hazard rate, reversed hazard rate, and likelihood ratio stochastic orders. The main results are presented in Section 3 along with the definition of the general counting process. As a consequence of Theorems 2 and 3, in Section 3 we obtain similar results for a generalized renewal process stopped at a random independent time. Finally, applications of the results of Section 2 to the trend renewal process appear in Section 4.

2. Preliminaries

For a nonnegative random variable X, the cumulative distribution and the reliability functions are denoted by F_X and $\bar{F}_X = 1 - F_X$, respectively. If X is absolutely continuous, the probability density function is denoted by f_X. We write $X \stackrel{\mathrm{D}}{=} Y$ to indicate that the random variables X and Y have the same distribution. Throughout this paper, the terms 'increasing' and 'decreasing' mean, as usual, 'nondecreasing' and 'nonincreasing', respectively.

We now present the definitions of the hazard rate (see the classic books by Barlow and Proschan [2], [3]), the reversed hazard rate (see [11]), the IFR class (see [2], [3], and [9]), the discrete IFR class (see [9]), the DRHR class (see [5]), and the discrete DRHR class (see [14]).

Definition 1. For any absolutely continuous random variable X, we respectively define its hazard rate (denoted r_X) and reversed hazard rate (denoted q_X) by

$$r_X(x) = \frac{f_X(x)}{\bar{F}_X(x)} \quad \text{and} \quad q_X(x) = \frac{f_X(x)}{F_X(x)}$$

for x such that $\bar{F}_X(x) > 0$ and $F_X(x) > 0$.

Definition 2. A random variable X is said to be IFR if $\bar{F}_X(x+h)/\bar{F}_X(x)$ is decreasing with x for each $h \geq 0$ or, equivalently, if and only if $r_X(x)$ is increasing with x.

Definition 3. Let X be a nonnegative integer-valued random variable. Then X is said to be discrete IFR if and only if

$$\mathrm{P}(X \geq n+1)^2 \geq \mathrm{P}(X \geq n)\,\mathrm{P}(X \geq n+2), \qquad n = 0, 1, \ldots.$$

Definition 4. A random variable X is said to be DRHR if $F_X(x+h)/F_X(x)$ is decreasing with x for each $h \geq 0$ or, equivalently, if and only if $q_X(x)$ is decreasing with x.

Definition 5. Let X be a nonnegative integer-valued random variable. Then X is said to be discrete DRHR if and only if

$$\mathrm{P}(X \leq n+1)^2 \geq \mathrm{P}(X \leq n)\,\mathrm{P}(X \leq n+2), \qquad n = 0, 1, \ldots.$$

The stochastic orders defined below are widely used throughout the paper. The books by Müller and Stoyan [12] as well as Shaked and Shanthikumar [15] provide a comprehensive treatment of stochastic orders.

Definition 6. Let X and Y be nonnegative random variables. Then X is said to be smaller than Y in the usual stochastic order (denoted $X \leq_{\text{st}} Y$) if and only if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all x.

Definition 7. Let X and Y be nonnegative random variables. Then X is said to be smaller than Y in the hazard rate stochastic order (denoted $X \leq_{\text{hr}} Y$) if and only if the function $x \mapsto \bar{F}_Y(x)/\bar{F}_X(x)$ is increasing or, equivalently, if and only if $r_X(x) \geq r_Y(x)$ for all x.

Definition 8. Let X and Y be nonnegative random variables. Then X is said to be smaller than Y in the reversed hazard rate stochastic order (denoted $X \leq_{\text{rhr}} Y$) if and only if the function $x \mapsto F_Y(x)/F_X(x)$ is increasing or, equivalently, if and only if $q_X(x) \leq q_Y(x)$ for all x.

Definition 9. Let X and Y be nonnegative random variables. Then X is said to be smaller than Y in the likelihood ratio stochastic order (denoted $X \leq_{\text{lr}} Y$) if and only if the function $x \mapsto f_Y(x)/f_X(x)$ is increasing with x in the union of the support sets of X and Y, with $a/0 = \infty$ for $a > 0$.

The following lemma is a well-known result about the usual stochastic order (see [12] and [15]).

Lemma 1. *Let X and Y be nonnegative random variables. Then $X \leq_{\text{st}} Y$ if and only if $\mathrm{E}[h(X)] \leq \mathrm{E}[h(Y)]$ for all increasing functions h or $\mathrm{E}[h(X)] \geq \mathrm{E}[h(Y)]$ for all decreasing functions h.*

The bivariate characterizations given below (see [12] and [16]) for the hazard rate, reversed hazard rate, and likelihood ratio stochastic orders, play a significant role in the proof of the main results of this paper.

Theorem 1. *Let X, $X^\star$, Y and $Y^\star$ be nonnegative random variables with $X^\star$ and $Y^\star$ independent, and $X \stackrel{\mathrm{D}}{=} X^\star$ and $Y \stackrel{\mathrm{D}}{=} Y^\star$. Then*

(a) *$X \leq_{\text{hr}} Y$ if and only if*

$$\mathrm{E}\, g(X^\star, Y^\star) \leq \mathrm{E}\, g(Y^\star, X^\star) \quad \textit{for all } g \in \mathcal{G}_{\text{hr}},$$

where $\mathcal{G}_{\text{hr}}$ is the class of bivariate functions satisfying

$$\mathcal{G}_{\text{hr}} = \{g\colon [0, \infty) \times [0, \infty) \to \mathbb{R} \mid \Delta g(x, y) \textit{ is increasing in } x \textit{ for all } x \geq y\}$$

with $\Delta g(x, y) = g(x, y) - g(y, x)$;

(b) *$X \leq_{\text{rhr}} Y$ if and only if*

$$\mathrm{E}\, g(X^\star, Y^\star) \leq \mathrm{E}\, g(Y^\star, X^\star) \quad \textit{for all } g \in \mathcal{G}_{\text{rhr}},$$

where $\mathcal{G}_{\text{rhr}}$ is defined as

$$\mathcal{G}_{\text{rhr}} = \{g\colon [0, \infty) \times [0, \infty) \to \mathbb{R} \mid \Delta g(x, y) \textit{ is increasing in } x \textit{ for all } x \leq y\};$$

(c) *$X \leq_{\text{lr}} Y$ if and only if*

$$\mathrm{E}\, g(X^\star, Y^\star) \leq \mathrm{E}\, g(Y^\star, X^\star) \quad \textit{for all } g \in \mathcal{G}_{\text{lr}},$$

where $\mathcal{G}_{\text{lr}}$ is the class of bivariate functions satisfying

$$\mathcal{G}_{\text{lr}} = \{g\colon [0, \infty) \times [0, \infty) \to \mathbb{R} \mid \Delta g(x, y) \geq 0 \textit{ for all } x \geq y\}.$$

3. DRHR and IFR properties of counting at random times

The results of this section establish sufficient conditions under which a general counting process $\{N(t): t \geq 0\}$ and a random time independent from the process verify that $N(T)$ belongs to the DRHR reliability class (see Theorem 2) or IFR reliability class (see Theorem 3).

The general counting process is defined in the following way. The renewal epochs or arrival times of the counting process are denoted by $S_n,\ n = 1, 2, \dots$, with $S_0 = 0$, and the interarrival times are denoted by $X_n,\ n = 1, 2, \dots$. It is verified that

$$S_n = \sum_{i=1}^{n} X_i.$$

The only assumption in the model concerning the interarrival times is that the X_ns are nonnegative random variables. The process is defined, by means of the arrival times, as

$$N(t) = \max\{n \text{ nonnegative integers } \mid S_n \leq t\}, \qquad t \geq 0,$$

that is, the nth event of the process occurs at a random time $S_n,\ n = 1, 2, \dots$. If the X_ns are independent and identically distributed, the general counting process is a renewal process.

In what follows, $Z_{n,x}$ will denote, for $n = 0, 1, \dots$ and $x \geq 0$, the conditional distribution of X_{n+1} given that $S_n = x$, that is, $Z_{n,x} = X_{n+1} \mid S_n = x$. In addition, it is necessary to assume that S_n and $Z_{n,x}$ are both absolutely continuous random variables. The results remain true if $Z_{n,x}$ and S_n are both discrete random variables.

For $n = 1, 2, \dots$ and $x \geq 0$, let us denote by δ_n and $f_{n,x}$ the probability density functions of S_n and $Z_{n,x}$, respectively. In the case in which $n = 0$, δ_0 is Dirac's delta function defined on the nonnegative real numbers, and $f_{0,x}$ is the probability density function of $S_1 = X_1$.

The auxiliary results presented below are key in the proofs of the main results. They provide conditions under which an adequate bivariate function belongs to the $\mathcal{G}_{\mathrm{hr}}$ or $\mathcal{G}_{\mathrm{rhr}}$ class associated with the bivariate characterizations of the hazard rate and reversed hazard rate stochastic orders (see Theorem 1).

Lemma 2. *Let T be a nonnegative random variable. If*

(a) *T is DRHR; and*

(b) *$Z_{n+1,x} \geq_{\mathrm{st}} Z_{n+1,y}$ for $x \leq y$ $(x, y \geq 0)$ and $n = 0, 1, \dots$,*

then, for $n = 0, 1, \dots$, the bivariate function g_n, defined by

$$g_n(x, y) = F_T(x) \int_0^{\infty} F_T(y + h) f_{n+1,y}(h)\,\mathrm{d}h, \qquad x, y \geq 0,$$

belongs to $\mathcal{G}_{\mathrm{hr}}$.

Proof. Let n be a fixed nonnegative integer, and let x_1, x_2, and y be nonnegative real numbers satisfying $x_2 \geq x_1 \geq y \geq 0$. The property in the lemma is equivalent to verifying that

$$\Delta g_n(x_1, y) \leq \Delta g_n(x_2, y), \qquad x_2 \geq x_1 \geq y \geq 0. \tag{1}$$

We first consider the case in which $F_T(y) = 0$. Then,

$$\Delta g_n(x, y) = F_T(x) \int_0^{\infty} F_T(y + h) f_{n+1,y}(h)\,\mathrm{d}h, \qquad x \geq 0,$$

and condition (1) is verified because F_T is an increasing function.

https://doi.org/10.1239/jap/1300198136 Published online by Cambridge University Press

Now assume that $F_T(y) > 0$. As F_T is an increasing function, $F_T(x_2) \geq F_T(x_1) \geq F_T(y) > 0$ and $\Delta g_n(x_1, y)$ can be written as

$$\Delta g_n(x_1, y) = F_T(x_1)F_T(y) \times \left(\int_0^\infty \frac{F_T(y+h)}{F_T(y)} f_{n+1,y}(h)\, \mathrm{d}h - \int_0^\infty \frac{F_T(x_1+h)}{F_T(x_1)} f_{n+1,x_1}(h)\, \mathrm{d}h \right). \quad (2)$$

By assumption (a) (see Definition 4) we obtain

$$\frac{F_T(y+h)}{F_T(y)} \geq \frac{F_T(x_1+h)}{F_T(x_1)} \geq \frac{F_T(x_2+h)}{F_T(x_2)}, \qquad h \geq 0.$$

Hence,

$$\int_0^\infty \frac{F_T(x_1+h)}{F_T(x_1)} f_{n+1,x_1}(h)\, \mathrm{d}h \geq \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_1}(h)\, \mathrm{d}h.$$

As an immediate consequence of Lemma 1, assumption (b), and the fact that F_T is increasing, we obtain

$$\int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_1}(h)\, \mathrm{d}h \geq \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h.$$

Therefore, putting together these inequalities we obtain

$$\int_0^\infty \frac{F_T(x_1+h)}{F_T(x_1)} f_{n+1,x_1}(h)\, \mathrm{d}h \geq \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h. \quad (3)$$

Repeating the same arguments again leads to

$$\begin{aligned} \int_0^\infty \frac{F_T(y+h)}{F_T(y)} f_{n+1,y}(h)\, \mathrm{d}h &\geq \int_0^\infty \frac{F_T(x_1+h)}{F_T(x_1)} f_{n+1,x_1}(h)\, \mathrm{d}h \\ &\geq \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h. \end{aligned} \quad (4)$$

Thus, by (2) and (3) we obtain

$$\Delta g_n(x_1, y) \leq F_T(x_1)F_T(y) \times \left(\int_0^\infty \frac{F_T(y+h)}{F_T(y)} f_{n+1,y}(h)\, \mathrm{d}h - \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h \right)$$

It follows from (4) that the expression inside the parenthesis is nonnegative. Therefore, as F_T is increasing, we conclude that

$$\begin{aligned} \Delta g_n(x_1, y) &\leq F_T(x_1)F_T(y) \\ &\quad \times \left(\int_0^\infty \frac{F_T(y+h)}{F_T(y)} f_{n+1,y}(h)\, \mathrm{d}h - \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h \right) \\ &\leq F_T(x_2)F_T(y) \\ &\quad \times \left(\int_0^\infty \frac{F_T(y+h)}{F_T(y)} f_{n+1,y}(h)\, \mathrm{d}h - \int_0^\infty \frac{F_T(x_2+h)}{F_T(x_2)} f_{n+1,x_2}(h)\, \mathrm{d}h \right) \\ &= \Delta g_n(x_2, y), \end{aligned}$$

proving (1) for the $F_T(y) > 0$ case.

https://doi.org/10.1239/jap/1300198136 Published online by Cambridge University Press

Lemma 3. *Let T be a nonnegative random variable. If*

(a) *T is IFR; and*

(b) *$Z_{n,x} \leq_{\rm st} Z_{n,y}$ for $x \leq y$ $(x, y \geq 0)$ and $n = 0, 1, \ldots,$*

then, for $n = 0, 1, \ldots,$ the bivariate function $\bar{g}_n$, defined by

$$\bar{g}_n(x, y) = \bar{F}_T(x) \int_0^\infty \bar{F}_T(y + h) f_{n,y}(h)\, \mathrm{d}h, \qquad x, y \geq 0,$$

belongs to $\mathcal{G}_{\rm rhr}$.

Proof. Let x_1, x_2, and y be nonnegative real numbers, and let n be a nonnegative integer. The property stated in the lemma is equivalent to

$$\Delta\bar{g}_n(x_1, y) \leq \Delta\bar{g}_n(x_2, y), \qquad x_1 \leq x_2 \leq y. \tag{5}$$

If $\bar{F}_T(y) = 0$ then, as $\bar{F}_T$ is decreasing and nonnegative, $\bar{F}_T(y + h) = 0$ for all $h \geq 0$ and $\Delta\bar{g}_n(x, y) = 0$ for all $x \geq 0$. Therefore, (5) is verified in this case.

For $\bar{F}_T(y) > 0$, the proof of property (5) is analogous to the proof of property (1) in the $F_T(y) > 0$ case, using the facts that assumptions (a) (see Definition 2) and (b) hold, $\bar{F}_T$ is decreasing and nonnegative, and Lemma 1 is satisfied. The details are omitted.

Now, we are going to prove the two main results in this paper.

Theorem 2. *Let $\{N(t) : t \geq 0\}$ be a general counting process with arrival times $(S_n)_{n=1,2,\ldots}$, and let T be a nonnegative random variable independent of the counting process. Consider the case in which the cumulative distribution function of T has no common discontinuity points with the cumulative distribution functions corresponding to the $(S_n)_{n=1,2,\ldots}$. Then, $N(T)$ is DRHR if the following conditions are fulfilled:*

(a) *T is DRHR;*

(b) *$S_{n+1} \leq_{\rm hr} S_{n+2}$ for $n = 0, 1, \ldots;$*

(c) *$Z_{n+1,x} \geq_{\rm st} Z_{n+2,x}$ for $n = 0, 1, \ldots$ and $x \geq 0$;*

(d) *$Z_{n+1,x} \geq_{\rm st} Z_{n+1,y}$ for $x \leq y$ $(x, y \geq 0)$ and $n = 0, 1, \ldots.$*

Proof. Let n be a nonnegative integer. Conditions (a) and (d) imply by, Lemma 2, that the bivariate function g_n considered in it belongs to the function class associated with the bivariate characterization of the hazard rate stochastic order (i.e. $g_n \in \mathcal{G}_{\rm hr}$). By condition (b) and Theorem 1(a), we obtain

$$\mathrm{E}[g_n(S^\star_{n+1}, S^\star_{n+2})] \leq \mathrm{E}[g_n(S^\star_{n+2}, S^\star_{n+1})], \tag{6}$$

where $S^\star_{n+1}$ and $S^\star_{n+2}$ are independent random variables with $S^\star_{n+1} \overset{\mathrm{D}}{=} S_{n+1}$ and $S^\star_{n+2} \overset{\mathrm{D}}{=} S_{n+2}$. Then (6) can be written as

$$\begin{aligned} &\int_0^\infty F_T(x)\delta_{n+1}(x)\,\mathrm{d}x \int_0^\infty \biggl(\int_0^\infty F_T(y + h) f_{n+1,y}(h)\,\mathrm{d}h\biggr)\delta_{n+2}(y)\,\mathrm{d}y \\ &\qquad \leq \int_0^\infty F_T(x)\delta_{n+2}(x)\,\mathrm{d}x \int_0^\infty \biggl(\int_0^\infty F_T(y + h) f_{n+1,y}(h)\,\mathrm{d}h\biggr)\delta_{n+1}(y)\,\mathrm{d}y. \end{aligned} \tag{7}$$

On the other hand, using condition (c), the fact that F_T is increasing, and Lemma 1, leads to

$$\int_0^\infty F_T(y+h) f_{n+2,y}(h)\,\mathrm{d}h \le \int_0^\infty F_T(y+h) f_{n+1,y}(h)\,\mathrm{d}h, \qquad y \ge 0.$$

Thus,

$$\int_0^\infty \bigg(\int_0^\infty F_T(y+h) f_{n+2,y}(h)\,\mathrm{d}h\bigg)\delta_{n+2}(y)\,\mathrm{d}y \\ \le \int_0^\infty \bigg(\int_0^\infty F_T(y+h) f_{n+1,y}(h)\,\mathrm{d}h\bigg)\delta_{n+2}(y)\,\mathrm{d}y.$$

Furthermore, for $j = 1, 2, \dots,$

$$\mathrm{E}[F_T(S_j)] = \int_0^\infty F_T(x)\delta_j(x)\,\mathrm{d}x = \int_0^\infty \bigg(\int_0^\infty F_T(y+h) f_{j-1,y}(h)\,\mathrm{d}h\bigg)\delta_{j-1}(y)\,\mathrm{d}y \tag{8}$$

and

$$F_T(0) = \mathrm{E}[F_T(S_0)] = \int_0^\infty F_T(x)\delta_0(x)\,\mathrm{d}x, \tag{9}$$

which, together with (7), lead to

$$\mathrm{E}[F_T(S_{n+1})]\,\mathrm{E}[F_T(S_{n+3})] \le \mathrm{E}^2[F_T(S_{n+2})]. \tag{10}$$

By the hypothesis of the theorem we also have

$$\mathrm{P}(N(T) \le n) = \mathrm{E}[F_T(S_{n+1})].$$

Therefore, $N(T)$ is a discrete DRHR random variable (see Definition 5 and (10)).

Theorem 3. *Let $\{N(t) : t \ge 0\}$ be a general counting process with arrival times $(S_n)_{n=1,2,\dots}$, and let T be a nonnegative random variable independent of the counting process. Consider the case in which the cumulative distribution function of T has no common discontinuity points with the cumulative distribution functions corresponding to the $(S_n)_{n=1,2,\dots}$. Then $N(T)$ is IFR if the following conditions are fulfilled:*

(a) *T is IFR;*

(b) *$S_n \le_{\mathrm{rhr}} S_{n+1}$ for $n = 0, 1, \dots;$*

(c) *$Z_{n,x} \le_{\mathrm{st}} Z_{n+1,x}$ for $n = 0, 1, \dots$ and $x \ge 0$;*

(d) *$Z_{n,x} \le_{\mathrm{st}} Z_{n,y}$ for $x \le y$ $(x, y \ge 0)$ and $n = 0, 1, \dots.$*

Proof. Let n be a nonnegative integer. The bivariate function $\bar{g}_n$ defined in Lemma 3 verifies that $\bar{g}_n \in \mathcal{G}_{\mathrm{rhr}}$ by conditions (a) and (d) (see Lemma 3). Therefore, by condition (b) and Theorem 1(b), we have

$$\mathrm{E}[\bar{g}_n(S_n^\star, S_{n+1}^\star)] \le \mathrm{E}[\bar{g}_n(S_{n+1}^\star, S_n^\star)],$$

where $S_n^\star$ and $S_{n+1}^\star$ are independent random variables with $S_n^\star \overset{\mathrm{D}}{=} S_n$ and $S_{n+1}^\star \overset{\mathrm{D}}{=} S_{n+1}$. The expression above can be written as

$$\int_0^\infty \bar{F}_T(x)\delta_n(x)\,\mathrm{d}x \int_0^\infty \bigg(\int_0^\infty \bar{F}_T(y+h) f_{n,y}(h)\,\mathrm{d}h\bigg)\delta_{n+1}(y)\,\mathrm{d}y \\ \le \int_0^\infty \bar{F}_T(x)\delta_{n+1}(x)\,\mathrm{d}x \int_0^\infty \bigg(\int_0^\infty \bar{F}_T(y+h) f_{n,y}(h)\,\mathrm{d}h\bigg)\delta_n(y)\,\mathrm{d}y. \tag{11}$$

Moreover, by Lemma 1, the fact that $\bar{F}_T$ is decreasing, and condition (c), we obtain

$$\int_0^\infty \bar{F}_T(y+h) f_{n+1,y}(h)\,\mathrm{d}h \leq \int_0^\infty \bar{F}_T(y+h) f_{n,y}(h)\,\mathrm{d}h, \qquad y \geq 0.$$

Thus,

$$\begin{aligned}\int_0^\infty &\biggl(\int_0^\infty \bar{F}_T(y+h) f_{n+1,y}(h)\,\mathrm{d}h\biggr)\delta_{n+1}(y)\,\mathrm{d}y \\ &\leq \int_0^\infty \biggl(\int_0^\infty \bar{F}_T(y+h) f_{n,y}(h)\,\mathrm{d}h\biggr)\delta_{n+1}(y)\,\mathrm{d}y.\end{aligned}$$

Combining previous results, (8) and (9) applied to $\bar{F}_T$ instead of F_T, with inequality (11) we deduce that

$$\mathrm{E}[\bar{F}_T(S_n)]\,\mathrm{E}[\bar{F}_T(S_{n+2})] \leq \mathrm{E}^2[\bar{F}_T(S_{n+2})].$$

Also, by the hypothesis of this theorem we have

$$\mathrm{P}(N(T) \geq n) = \mathrm{E}[\bar{F}_T(S_n)].$$

So, $N(T)$ is a discrete IFR random variable (see Definition 3).

As a simple consequence of Theorems 2 and 3, we obtain the corollary below. This corollary has been proved in [1, Theorem 4.7 and Remark 4.8].

Corollary 1. (i) *Let $\{N(t)\colon t \geq 0\}$ be a general counting process with independent and IFR interarrival times $(X_n)_{n=1,2,\dots}$, verifying that $X_n \geq_{\mathrm{st}} X_{n+1}$, $n = 1, 2, \dots$, and let T be a nonnegative random variable independent from the DRHR process. Then $N(T)$ is discrete DRHR.*

(ii) *Let $\{N(t)\colon t \geq 0\}$ be a general counting process with independent and DRHR interarrival times $(X_n)_{n=1,2,\dots}$, verifying that $X_n \leq_{\mathrm{st}} X_{n+1}$, $n = 1, 2, \dots$, and let T be a nonnegative random variable independent from the IFR process. Then $N(T)$ is discrete IFR.*

Proof. Let us first consider the case when the X_ns are IFR, $X_n \geq_{\mathrm{st}} X_{n+1}$, $n = 1, 2, \dots$, and T is DRHR. Since the interarrival times are independent, we have $Z_{n,x} \overset{\mathrm{D}}{=} X_{n+1}$, $n = 0, 1, \dots$, and $x \geq 0$, and conditions (c) and (d) of Theorem 2 are verified. Furthermore, if the interarrival times $(X_n)_{n=1,2,\dots}$ are IFR, by Theorem 1.B.4 of [15], it can be deduced that

$$S_n = \sum_{i=1}^{n} X_i \leq_{\mathrm{hr}} S_{n+1} = \sum_{i=1}^{n+1} X_i, \qquad n = 1, 2 \dots,$$

and condition (b) of Theorem 2 holds. Then, by Theorem 2 we find that $N(T)$ is DRHR. The proof in the other case is similar using Theorem 3 and Theorem 1.B.45 of [15].

4. Application to generalized renewal processes

The generalized renewal process was introduced by Kijima and Sumita [8] to model the failures of a repairable system and has been widely analyzed in the literature. The arrival times $(S_n)_{n=0,1,\dots}$ and interarrival times $(X_n)_{n=1,2,\dots}$ in a generalized renewal process satisfy

$$\mathrm{P}(Z_{n,x} > z) = \mathrm{P}(X_{n+1} > z \mid S_n = x) = \frac{\bar{F}(z+qx)}{\bar{F}(qx)} \tag{12}$$

for a nonnegative integer n, where $q, z, x \geq 0$, $Z_{n,x}$ is the random variable defined in Section 3,

https://doi.org/10.1239/jap/1300198136 Published online by Cambridge University Press

and $\bar{F}$ is the reliability function of $S_1 = X_1$, that is, the time to the first renewal of the process. The corresponding probability density function is denoted by f. The model was defined in terms of the virtual age of the system, represented by V_n, $n = 0, 1, \dots$, which is related to the renewal epochs or failure times of the system by means of the equation

$$V_n = qS_n.$$

The model has the following interpretations, depending on the values of the parameter q. If $q = 0$, the model is the classical renewal process, whereas it corresponds to the nonhomogeneous Poisson process if $q = 1$. For those cases where $0 < q < 1$, the repair restores the system somewhere between as-good-as-new and as-bad-as-old, and is better than new after repair for $q > 1$.

The next result provides necessary conditions under which the arrival times of a generalized renewal process are increasing in the likelihood ratio stochastic order.

Lemma 4. *Let $\{N(t) : t \geq 0\}$ be a generalized renewal process with arrival times $(S_n)_{n=1,2,\dots}$, probability density function of the time to the first renewal f, and a nonnegative real number q. If $0 \leq q \leq 1$ and f is log-concave or $q > 1$ and f is log-convex, then $S_n \leq_{\mathrm{lr}} S_{n+1}$, $n = 0, 1, \dots$.*

Proof. As a consequence of (12), it is simple to obtain the following recursive relationship between the probability density functions of the arrival times of the generalized renewal process:

$$\delta_{n+1}(x) = \int_0^x \frac{f(x + (q-1)y)}{\bar{F}(qy)} \delta_n(y) \, \mathrm{d}y, \qquad x \geq 0,\ n = 0, 1, \dots. \tag{13}$$

According to Definition 9, the claim in the lemma is equivalent to showing that, for $n = 0, 1, \dots$,

$$\frac{\delta_{n+1}(x)}{\delta_n(x)} \tag{14}$$

is increasing in x. The proof of this is accomplished by an induction procedure. Let us assume, without loss of generality, that all the functions involved in this proof are derivable. The derivative of a function h is denoted by h'.

The case in which $n = 0$ is obvious by Definition 9 and the fact that δ_0 is Dirac's delta function. Let n be a nonnegative number greater than 0, and suppose that the function defined in (14) is increasing for $n - 1$. We need to show that the function defined in (14) is also increasing for n. The sign of the derivative of such a function for n is the same as the sign of $\delta'_{n+1}(x)\delta_n(x) - \delta'_n(x)\delta_{n+1}(x)$, which can be expressed, using (13), as

$$\begin{aligned}
&\frac{f(qx)}{\bar{F}(qx)}(\delta_n^2(x) - \delta_{n-1}(x)\delta_{n+1}(x)) \\
&+ \int_0^x \int_0^x \frac{f'(x + (q-1)y)\delta_n(y) f(x + (q-1)u)\delta_{n-1}(u)}{\bar{F}(qy)\bar{F}(qu)} \, \mathrm{d}y \, \mathrm{d}u \\
&- \int_0^x \int_0^x \frac{f'(x + (q-1)u)\delta_{n-1}(u) f(x + (q-1)y)\delta_n(y)}{\bar{F}(qy)\bar{F}(qu)} \, \mathrm{d}y \, \mathrm{d}u, \qquad x \geq 0.
\end{aligned} \tag{15}$$

Using (13) and the induction hypothesis for $n - 1$, we obtain

$$\delta_{n+1}(x) = \int_0^x \frac{f(x + (q-1)y)}{\bar{F}(qy)} \frac{\delta_n(y)}{\delta_{n-1}(y)} \delta_{n-1}(y) \, \mathrm{d}y \leq \frac{\delta_n^2(x)}{\delta_{n-1}(x)},$$

and so the first term in (15) is nonnegative. Moreover, the second term in (15) can be written as

$$\mathrm{E}[B_{q,x}(S^\star_{n-1}, S^\star_n)] - \mathrm{E}[B_{q,x}(S^\star_n, S^\star_{n-1})], \qquad x \geq 0,$$

where $B_{q,x}$ is the bivariate function defined as

$$B_{q,x}(u, y) = \frac{f'(x + (q-1)y) f(x + (q-1)u)}{\bar{F}(qy)\bar{F}(qu)} \mathbf{1}_{[0,x]}(u)\, \mathbf{1}_{[0,x]}(y), \qquad y \geq 0,\ u \geq 0,$$

where $\mathbf{1}_A$ is the indicator function of the set A, and $S^\star_{n-1}$ and $S^\star_n$ are independent variables with the same distribution as S_{n-1} and S_n, respectively. It is simple to verify that, for $u \geq y$, $0 \leq q \leq 1$, and f log-concave,

$$\frac{f'(x + (q-1)y)}{f(x + (q-1)y)} \leq \frac{f'(x + (q-1)u)}{f(x + (q-1)u)}.$$

Thus, $B_{q,x}(u, y) \leq B_{q,x}(y, u)$, $u \geq y$, and, consequently, $-B_{q,x} \in \mathcal{G}_{\mathrm{lr}}$ and, by Theorem 1(c) and the induction hypothesis, we conclude that

$$\mathrm{E}[B_{q,x}(S^\star_{n-1}, S^\star_n)] - \mathrm{E}[B_{q,x}(S^\star_n, S^\star_{n-1})] \geq 0, \qquad x \geq 0.$$

Hence, the second term in (15) is nonnegative and the induction procedure is completed because the function in (14) is increasing in x.

The proof in the case when f is log-convex and $q > 1$ is analogous to the previous case because, for all $x \geq 0$,

$$\frac{f'(x + (q-1)y)}{f(x + (q-1)y)} \leq \frac{f'(x + (q-1)u)}{f(x + (q-1)u)}, \qquad u \geq y.$$

The corollary below gives sufficient conditions for a generalized renewal process stopped at a random time independent from the process to be either DRHR or IFR, using Theorem 2 or 3, respectively. The proof also uses Lemma 4 along with the fact that the likelihood ratio stochastic order implies both the hazard rate and the reversed hazard rate stochastic orders.

Corollary 2. *Let $\{N(t) : t \geq 0\}$ be a generalized renewal process, and let T be a nonnegative random variable independent from the process. Then*

(i) *$N(T)$ is discrete DRHR if*

  (a) *f is log-concave and $0 \leq q \leq 1$; and*

  (b) *T is DRHR;*

(ii) *$N(T)$ is discrete IFR if*

  (a) *f is log-convex and $q > 1$; and*

  (b) *T is IFR.*

Proof. Let us consider the case when f is log-concave, $q \in [0, 1]$, and T is DRHR.

The reliability function of $Z_{n,x}$ for n a nonnegative integer and x a nonnegative real number is given by (12). The right-hand side of (12) does not depend of n and, thus, condition (c) of Theorem 2 is verified (see Definition 6). Moreover, it is well known that, if f is log-concave,

https://doi.org/10.1239/jap/1300198136 Published online by Cambridge University Press

the random variable associated with the time to the first renewal is IFR and then the right-hand side is decreasing in x (see Definition 2). Hence, the random variables $Z_{n,x}$ verify the condition (d) of Theorem 2 (see Definition 6). Furthermore, by Lemma 4, the arrival times satisfy condition (b) of Theorem 1 because the likelihood ratio stochastic order implies the hazard rate stochastic order. So, under the assumptions of this corollary, all the hypotheses of Theorem 2 are verified and so $N(T)$ is a discrete DRHR random variable.

The proof in the other case is similar using Theorem 3 and Lemma 4.

5. Applications to trend-renewal processes

The trend-renewal process (TRP) was introduced in [10] as a generalization of the nonhomogeneous Poisson process. Let us consider the following definition of a TRP.

Definition 10. Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, satisfying $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u < \infty$ for each $t \geq 0$ and $\Lambda(\infty) = \infty$. Furthermore, let F be a positive distribution function (i.e. $F(0) = 0$) with expected value 1. The process $\{N(t) : t \geq 0\}$ with arrival times $S_1, S_2, \dots$ is called TRP$(F, \lambda(\cdot))$ if the time transformed process with arrival times $\Lambda(S_1), \Lambda(S_2), \dots$ is a classical renewal process with common interrenewal distribution function F, that is, $\Lambda(S_i) - \Lambda(S_{i-1})$, $i = 1, 2, \dots$, are independent and identically distributed random variables with distribution function F.

Note that a TRP$(F, \lambda(\cdot))$ with F being an exponential distribution with mean equal to 1 is a nonhomogeneous Poisson process with intensity function $\lambda(\cdot)$.

The next corollary establishes conditions under which a TRP stopped at a random time independent from the process is DRHR or IFR. The results obtained are similar to Theorem 5.2 of Ross *et al.* [13], who analyzed the same question for the nonhomogeneous Poisson process and the IFR reliability class.

Corollary 3. *Let $\{N(t) : t \geq 0\}$ be a TRP$(F, \lambda(\cdot))$, and let T be a nonnegative random variable independent from the process. Then*

(i) *$N(T)$ is discrete DRHR if*

  (a) *F is IFR; and*

  (b) *$q_T(x)/\lambda(x)$ is decreasing with x;*

(ii) *$N(T)$ is discrete IFR if*

  (a) *F is DRHR; and*

  (b) *$r_T(x)/\lambda(x)$ is increasing with x.*

Proof. It is easy to check from Definition 10 that

$$N(t) \stackrel{\mathrm{D}}{=} N^*(\Lambda(t)), \qquad t \geq 0,$$

where $\{N^*(t) : t \geq 0\}$ is a renewal process with common interarrival distribution F. Then, by Corollary 1, $N(T)$ is discrete DRHR or IFR if $\Lambda(T)$ is DRHR or IFR, respectively. Simple calculations allow us to show that the reversed hazard rate and the hazard rate of $\Lambda(T)$ are given by

$$q_{\Lambda(T)}(x) = \frac{q_T(\Lambda^{-1}(x))}{\lambda(\Lambda^{-1}(x))} \quad \text{and} \quad r_{\Lambda(T)}(x) = \frac{r_T(\Lambda^{-1}(x))}{\lambda(\Lambda^{-1}(x))},$$

respectively. The results follow from previous expressions of $q_{\Lambda(T)}$ and $r_{\Lambda(T)}$, conditions (b) of the corollary, Definitions 2 and 4, and the fact that $\Lambda^{-1}(x)$ is increasing in x.

Acknowledgements

This work has been supported by MEC grant MTM2007-63683 from the Spanish Government. I am specially grateful to Carmen Sangüesa and Isolina Alberto, who carefully read a first draft of the manuscript.

References

[1] Badía, F. G. and Sangüesa, C. (2008). Preservation of reliability classes under mixtures of renewal processes. *Prob. Eng. Inf. Sci.* **22,** 1–17.
[2] Barlow, R. E. and Proschan, F. (1965). *Mathematical Theory of Reliability*. John Wiley, New York.
[3] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Reinhart and Winston, New York.
[4] Block, H. W. and Savits, T. H. (1980). Laplace transforms for classes of life distributions. *Ann. Prob.* **8,** 465–474.
[5] Block, H. W., Savits, T. H. and Sing, H. (1998). The reversed hazard rate function. *Prob. Eng. Inf. Sci.* **12,** 69–90.
[6] Esary, J. D., Marshall, A. W. and Proschan, F. (1973). Shock models and wear processes. *Ann. Prob.* **1,** 627–649.
[7] Grandell, J. (1997). *Mixed Poisson Processes*. Chapman and Hall, London.
[8] Kijima, M. and Sumita, U. (1986). A useful generalization of renewal theory: counting processes governed by nonnegative Markovian increments. *J. Appl. Prob.* **23,** 71–88.
[9] Lai, C.-D. and Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*. Springer, New York.
[10] Lindqvist, B. H., Elvebakk, G. and Heggland, K. (2003). The trend-renewal process for statistical analysis of repairable systems. *Technometrics* **45,** 31–44.
[11] Marshall, A. W. and Olkin, I. (2007). *Life Distributions*. Springer, New York.
[12] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks*. John Wiley, Chichester.
[13] Ross, S. M., Shanthikumar, J. G. and Zhu, Z. (2005). On increasing-failure-rate random variables. *J. Appl. Prob.* **42,** 797–809.
[14] Sengupta, D. and Nanda, A. K. (1999). Log-concave and concave distributions in reliability. *Naval Res. Logistics* **46,** 419–433.
[15] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.
[16] Shanthikumar, J. G. and Yao, D. D. (1991). Bivariate characterization of some stochastic order relations. *Adv. Appl. Prob.* **23,** 642–659.