COMPOSITIO MATHEMATICA

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Abstract

Let X and Y be compact hyper-Kähler manifolds deformation equivalent to the Hilbert scheme of length n subschemes of a $K3$ surface. A class in $H^{p,p}(X \times Y, \mathbb{Q})$ is an *analytic correspondence*, if it belongs to the subring generated by Chern classes of coherent analytic sheaves. Let $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ be a rational Hodge isometry with respect to the Beauville–Bogomolov–Fujiki pairings. We prove that f is induced by an analytic correspondence. We furthermore lift f to an analytic correspondence $\tilde{f} : H^*(X, \mathbb{Q})[2n] \to H^*(Y, \mathbb{Q})[2n]$, which is a Hodge isometry with respect to the Mukai pairings and which preserves the gradings up to sign. When X and Y are projective, the correspondences f and $\tilde{f}$ are algebraic.

Contents

1. Introduction

1.1 Algebraicity of rational Hodge isometries

An *irreducible holomorphic symplectic manifold* (IHSM) is a simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a nowhere degenerate holomorphic 2-form. Such manifolds admit hyper-Kähler structures and are examples of compact hyper-Kähler manifolds. When two-dimensional, such a manifold is a $K3$ surface. If X is a Kähler manifold which is deformation equivalent to the Hilbert scheme $S^{[n]}$ of length n subschemes of a $K3$ surface S, then X is an IHSM [Bea83]. The latter are said to be of $K3^{[n]}$ *type*. The second integral cohomology $H^2(X, \mathbb{Z})$ of an IHSM X is endowed with a symmetric integral primitive non-degenerate bilinear pairing of signature $(3, b_2(X) - 3)$ known as the Beauville–Bogomolov–Fujiky (BBF) pairing, where $b_2(X)$ is the second Betti number [Bea83]. A homomorphism $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ between the second cohomologies of two IHSMs is said to be an *integral Hodge isometry*, if it is an isometry with respect to the BBF pairings as well as an isomorphism of Hodge structures. If $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ has these properties it is said to be a *rational Hodge isometry*.

Let X be an IHSM. Given a class α in the even cohomology $H^{ev}(X, \mathbb{Q})$ of X, denote by α_i its graded summand in $H^{2i}(X, \mathbb{Q})$. Let $\alpha^\vee \in H^{ev}(X, \mathbb{Q})$ be the class satisfying $(\alpha^\vee)_i = (-1)^i \alpha_i$. The *Mukai pairing* on the even cohomology $H^{ev}(X, \mathbb{Q})$ is defined by

$$(\alpha, \beta) := \int_X \alpha^\vee \cup \beta.$$

The *Mukai vector* of an object F in the bounded derived category $D^b(X)$ of coherent sheaves on X is $v(F) := ch(F)\sqrt{td_X} \in H^{ev}(X, \mathbb{Q})$. Grothendieck–Riemann–Roch yields

$$(v(E), v(F)) := \chi(E^\vee \otimes F)$$

for objects E and F in $D^b(X)$, where $E^\vee := R\mathcal{H}om(E, \mathcal{O}_X)$ is the derived dual object. When X is of $K3^{[n]}$ type the odd cohomology vanishes and $H^{ev}(X, \mathbb{Z}) = H^*(X, \mathbb{Z})$ (see [Mar07]).

Let X be a $2n$-dimensional IHSM. Denote by $H^*(X, \mathbb{Q})[2n]$ the shifted cohomology of X, where $H^k(X, \mathbb{Q})[2n] = H^{k+2n}(X, \mathbb{Q})$. We say that a homomorphism $f : H^*(X, \mathbb{Q})[2n] \to H^*(Y, \mathbb{Q})[2n]$ is *degree reversing*, if it maps $H^k(X, \mathbb{Q})[2n]$ to $H^{-k}(Y, \mathbb{Q})[2n]$,

for all k. The homomorphism is *degree preserving up to sign*, if it is either degree preserving or degree reversing.

Let X and Y be IHSMs of $K3^{[n]}$ type and $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ a Hodge isometry.

THEOREM 1.1. *There exists an analytic correspondence $\tilde{f} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$, which is an isometry with respect to the Mukai pairings and satisfies one of the following:*

(1) *$\tilde{f}$ is degree preserving and it restricts to $H^2(X, \mathbb{Q})$ as f;*
(2) *$\tilde{f}$ is degree reversing and the composition*

$$H^*(X, \mathbb{Q}) \xrightarrow{c_2(X)^{n-1} \cup} H^*(X, \mathbb{Q}) \xrightarrow{\tilde{f}} H^*(X, \mathbb{Q})$$

   of $\tilde{f}$ with cup product with $c_2(X)^{n-1}$ restricts to $H^2(X, \mathbb{Q})$ as a non-zero rational multiple of f.

In particular, f is algebraic whenever X and Y are projective.

The theorem is proved in §9. When $n = 1$ and X and Y are $K3$ surfaces Theorem 1.1 was proved by Mukai for projective surfaces with Picard number ≥ 11 (see [Muk87]), by Nikulin for projective surfaces with Picard number ≥ 5 (see [Nik87]), announced by Mukai in [Muk02], and proved by Buskin in complete generality [Bus19]. Another proof for projective $K3$ surfaces X and Y, without further restrictions on their Picard numbers, is due to Huybrechts [Huy19].

A more direct relationship between the Hodge isometries f and $\tilde{f}$ in Theorem 1.1 is described in terms of Taelman's Looijenga–Lunts–Verbitsky (LLV) lattice in Theorem 1.4 below.

DEFINITION 1.2. Let X_1 and X_2 be deformation equivalent compact Kähler manifolds. An isomorphism $f : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ is said to be a *parallel-transport operator*, if it arises as follows. There exist a smooth and proper family of compact Kähler manifolds $\pi : \mathcal{X} \to B$ over an analytic space B, points b_1, b_2 in B, isomorphisms $g_i : X_i \to \mathcal{X}_{b_i}$ with the fibers $\mathcal{X}_{b_i}$ of π over b_i, $i = 1, 2$, and a continuous path γ from b_1 to b_2, such that parallel transport along the path γ in the local system $R\pi_*\mathbb{Z}$ induces $g_{2,*} \circ f \circ g_1^* : H^*(\mathcal{X}_{b_1}, \mathbb{Z}) \to H^*(\mathcal{X}_{b_2}, \mathbb{Z})$. When $X_1 = X_2$ the parallel-transport operator f is also called a *monodromy operator*. The *monodromy group* $\mathrm{Mon}(X)$ of a compact Kähler manifold X is the subgroup of $GL(H^*(X, \mathbb{Z}))$ consisting of all monodromy operators.

Recall that a groupoid is a category all of whose morphisms are isomorphisms. Let

$$\mathcal{G} \tag{1.1}$$

be the groupoid, whose objects are pairs (X, ϵ) consisting of an IHSM X and an orientation[1] ϵ of $H^2(X, \mathbb{Q})$. If $\dim(X)$ is divisible by 4, assume[2] that $b_2(X)$ is odd. Morphisms in $\mathrm{Hom}_{\mathcal{G}}((X, \epsilon), (Y, \epsilon'))$ are compositions of three types of rational isometries from $H^*(X, \mathbb{Q})$ to $H^*(Y, \mathbb{Q})$ with respect to the Mukai pairing.

(1) A parallel-transport operator $f : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$.
(2) The isometry $H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ induced by the correspondence $ch(\mathcal{P})\sqrt{td_{X \times Y}}$, where $\mathcal{P} \in D^b(X \times Y)$ is the Fourier–Mukai kernel of an equivalence $D^b(X) \to D^b(Y)$ of derived categories between two deformation equivalent IHSMs X and Y.
(3) An isometry of $H^*(X, \mathbb{Q})$ induced by cup product with $\exp(\lambda) := \sum_{k=0}^{\dim(X)} \lambda^k/k!$ for some $\lambda \in H^2(X, \mathbb{Q})$.

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[1] The morphisms of $\mathcal{G}$ do not depend on the orientation. The orientation is not needed if $n := \dim(X)/2$ is odd. If n is even, the orientation is used in [Tae23] in the definition of the functor $\widetilde{H}$ introduced in (1.4).
[2] The assumption holds for all known IHSMs.

The set of morphisms $\mathrm{Hom}_{\mathcal{G}}((X,\epsilon),(Y,\epsilon'))$ is independent of the orientations and is also denoted by $\mathrm{Hom}_{\mathcal{G}}(X,Y)$. Note that the latter is empty, by definition, if X and Y are not deformation equivalent.

DEFINITION 1.3. Given an object F of positive rank r in the derived category of coherent sheaves, set $\kappa(F) := ch(F)\exp\left(-c_1(F)/r\right)$, where $\exp(x) = \sum_{j=0}^{\infty} x^j/j!$. Then $\kappa(F)^r = ch(F^{\otimes r} \otimes \det(F)^{-1})$, where the tensor power is taken in the derived category. If F is an object of positive rank r in the derived category of twisted coherent sheaves with respect to some Brauer class, then $F^{\otimes r} \otimes \det(F)^{-1}$ is an untwisted object and we define $\kappa(F)$ as the rth root of the Chern character of the latter with degree-0 summand equal to r. If F has negative rank, set $\kappa(F) := -\kappa(F[1])$.

Let

$$\mathcal{G}_{an} \tag{1.2}$$

be the subgroupoid of $\mathcal{G}$ with the same objects. Morphisms of $\mathcal{G}_{an}$ are compositions of two types of rational isometries: (1) parallel-transport operators $f : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$, which are Hodge isometries; (2) Hodge isometries

$$[\kappa(\mathcal{P})\sqrt{td_{X\times Y}}]_* : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q}) \tag{1.3}$$

induced by classes $\kappa(\mathcal{P})\sqrt{td_{X\times Y}}$, where $\mathcal{P}$ is a locally free twisted sheaf over $X \times Y$, such that the Azumaya $\mathcal{O}_{X\times Y}$-algebra $\mathcal{E}nd(\mathcal{P})$ is a deformation (in the sense of Definition 5.13) of the endomorphism algebra of a locally free Fourier–Mukai kernel of an equivalence of derived categories.[3] Morphisms in $\mathcal{G}_{an}$ are induced by analytic correspondences. Indeed, this follows from the Torelli theorem for parallel-transport Hodge isometries (Theorem 5.2 and Lemma 5.3) and by definition for morphisms of type (2) above.

The *rational LLV lattice* of an IHSM X is the vector space

$$\widetilde{H}(X,\mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta$$

endowed with the non-degenerate symmetric bilinear pairing, which restricts to $H^2(X,\mathbb{Q})$ as the BBF pairing, such that α and β are isotropic and orthogonal to $H^2(X,\mathbb{Q})$ and satisfy $(\alpha,\beta) = -1$. An orientation for $H^2(X,\mathbb{Q})$ determines an orientation of $\widetilde{H}(X,\mathbb{Q})$, and so the LLV lattice of objects of $\mathcal{G}$ has a chosen orientation. We assign $\widetilde{H}(X,\mathbb{Q})$ the grading where α has degree -2, β has degree 2, and $H^2(X,\mathbb{Q})$ has degree 0.

Let $\widetilde{\mathcal{G}}$ be the groupoid, whose objects are IHSMs and such that morphisms are isometries of their rational LLV lattices. Results of Taelman's yield a functor

$$\widetilde{H} : \mathcal{G} \to \widetilde{\mathcal{G}} \tag{1.4}$$

(see Definition 3.3).

We show that the morphism (1.3) is degree reversing (Lemma 4.1). It follows that every morphism ϕ in $\mathrm{Hom}_{\mathcal{G}_{an}}((X,\epsilon),(Y,\epsilon'))$ preserves the grading up to sign. Consequently, $\widetilde{H}(\phi)$ preserves the grading of $\widetilde{H}(X,\mathbb{Q})$ up to sign and maps $H^2(X,\mathbb{Q})$ to $H^2(Y,\mathbb{Q})$ and restricts to $H^2(X,\mathbb{Q})$ as a Hodge isometry $\widetilde{H}(\phi)_0$ (Lemma 3.4).

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[3] Note that $[\kappa(\mathcal{P})\sqrt{td_{X\times Y}}]_*$ is indeed a morphism of $\mathcal{G}$, so that $\mathcal{G}_{an}$ is indeed a subgroupoid of $\mathcal{G}$. This is seen as follows. The class $\kappa(\mathcal{P})$ is a characteristic class of the Azumaya algebra $\mathcal{E}nd(\mathcal{P})$ (see [Mar20, § 2.3]). If $(X \times Y, \mathcal{E}nd(\mathcal{P}))$ is a deformation of $(X_0 \times Y_0, \mathcal{E}nd(\mathcal{P}_0))$ and $\mathcal{P}_0$ is a locally free untwisted sheaf, which is the Fourier–Mukai kernel of an equivalence of derived categories, then $[\kappa(\mathcal{P})\sqrt{td_{X\times Y}}]_* = f_2 \circ [\kappa(\mathcal{P}_0)\sqrt{td_{X_0\times Y_0}}]_* \circ f_1$, where f_1 and f_2 are parallel-transport operators, and so morphisms in $\mathcal{G}$. Furthermore, writing $\mathrm{rank}(\mathcal{P}_0) = r$ and $c_1(\mathcal{P}_0) = \pi_{X_0}^*(\lambda_1) + \pi_{Y_0}^*(\lambda_2)$ we see that $[\kappa(\mathcal{P}_0)\sqrt{td_{X_0\times Y_0}}]_*(\bullet) = \exp(-\lambda_2/r) \cup [ch(\mathcal{P}_0)\sqrt{td_{X_0\times Y_0}}]_*(\exp(-\lambda_1/r) \cup (\bullet))$ is the composition of three morphisms in $\mathcal{G}$.

Let $\mathcal{H}dg$ be the groupoid, whose objects are IHSMs, and whose morphisms in $\mathrm{Hom}_{\mathcal{H}dg}(X,Y)$ are rational Hodge isometries $f : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ preserving the orientations of the positive cones (Definition 5.17). We get the functor

$$\widetilde{H}_0 : \mathcal{G}_{an} \to \mathcal{H}dg \tag{1.5}$$

forgetting the orientation of objects and mapping a morphism ϕ to $\widetilde{H}(\phi)_0$ or $-\widetilde{H}(\phi)_0$ (whichever preserves the orientations of the positive cones). Let $\mathcal{G}_{an}^{[n]}$ be the full subgroupoid of $\mathcal{G}_{an}$, whose objects are of $K3^{[n]}$ type. Define the subgroupoid $\mathcal{H}dg^{[n]}$ of $\mathcal{H}dg$ analogously. The following theorem is proved in §9.

THEOREM 1.4. *The functor $\widetilde{H}_0 : \mathcal{G}_{an}^{[n]} \to \mathcal{H}dg^{[n]}$ is full.[4] In particular, the analytic correspondence $\tilde{f}$ of Theorem 1.1 can be chosen to be a morphism in $\mathrm{Hom}_{\mathcal{G}_{an}^{[n]}}((X,\epsilon),(Y,\epsilon'))$ satisfying $\widetilde{H}_0(\tilde{f}) = f$, possibly after replacing f by $-f$, so that it would preserve the orientations of the positive cones.*

Let $f : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ be a rational isometry between two IHSMs of $K3^{[n]}$ type X and Y. We say that f is of *r-cyclic type*, for some positive integer r, if there exists a primitive class $u \in H^2(X,\mathbb{Z})$, satisfying $(u,u) = 2r$ and $(u,\bullet) \in H^2(X,\mathbb{Z})^*$ is primitive as well, and there exists a parallel-transport operator $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, such that $f = -g\rho_u$, where $\rho_u \in O(H^2(X,\mathbb{Q}))$ is the reflection in u given by $\rho_u(x) = x - (2(u,x)/(u,u))u$. Such f is necessarily compatible with the orientations of the positive cones. The proof of Theorem 1.4 relies on the following.

THEOREM 1.5 (Corollary 8.5). *Let $f : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ be a rational Hodge isometry of r-cyclic type between two IHSMs X and Y of $K3^{[n]}$ type. There exists over $X \times Y$ a (possibly twisted) locally free coherent sheaf E of rank $n!r^n$, such that $\phi := [\kappa(E)\sqrt{td_{X \times Y}}]_*$ is a degree-reversing morphism in $\mathrm{Hom}_{\mathcal{G}_{an}^{[n]}}(X,Y)$ and $\widetilde{H}_0(\phi) = f$.*

1.2 Fourier–Mukai kernels of positive rank and the Lefschetz standard conjecture

Theorem 1.5 leads to an alternative short proof of the Lefschetz standard conjecture for projective IHSMs of $K3^{[n]}$ type, originally proved in [CM13]. Let X be a projective IHSM of $K3^{[n]}$ type. Let $\lambda \in H^{1,1}(X,\mathbb{Q})$ be an ample class. Let $h \in \mathrm{End}(H^*(X,\mathbb{Q}))$ multiply $H^k(X,\mathbb{Q})[2n]$ by k. Denote by $e_\lambda \in \mathrm{End}(H^*(X,\mathbb{Q}))$ the Lefschetz operator of cup product with λ and let $e_\lambda^\vee \in \mathrm{End}(H^*(X,\mathbb{Q}))$ be dual Lefschetz operator, i.e. the unique element satisfying

$$[e_\lambda, e_\lambda^\vee] = h, \quad [h, e_\lambda^\vee] = -2e_\lambda^\vee.$$

Choose an IHSM Y, such that there exists a rational Hodge isometry $f : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ of r-cyclic type, for some integer $r > 0$. The existence of such Y follows, for every $r > 0$, by Theorem 1.5 and the surjectivity of the period map [Huy99].[5] The morphism $\phi \in \mathrm{Hom}_{\mathcal{G}_{an}^{[n]}}(X,Y)$ in Theorem 1.5 is degree reversing. We get that $e_\lambda^\vee$ is algebraic, by the following elementary lemma.

Let X and Y be IHSMs and let $\phi \in \mathrm{Hom}_{\mathcal{G}_{an}}((X,\epsilon),(Y,\epsilon'))$ be a degree-reversing morphism. Then $\tilde{\phi} := \widetilde{H}(\phi)$ is degree reversing as well (Lemma 3.4). Let $t \in \mathbb{Q}$ be such that $\tilde{\phi}(\beta) = t\alpha$. Let $\lambda \in H^2(X,\mathbb{Q})$ be such that $(\lambda,\lambda) \neq 0$.

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[4] This means that $\widetilde{H}_0 : \mathrm{Hom}_{\mathcal{G}_{an}^{[n]}}(X_1,X_2) \to \mathrm{Hom}_{\mathcal{H}dg}(X_1,X_2)$ is surjective, for any two objects X_1 and X_2 of $\mathcal{G}_{[an]}^{[n]}$.
[5] Choose an isometry $\eta_X : H^2(X,\mathbb{Z}) \to \Lambda$ with a fixed lattice Λ. The period map sends (X,η) to $\eta(H^{2,0}(X)) \in \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{C})$. Choose Y to be an IHSM of $K3^{[n]}$ type with an isometry $\eta_Y : H^2(Y,\mathbb{Z}) \to \Lambda$, such that (X,η_X) and (Y,η_Y) belong to the same connected component of the moduli space of marked IHSM and such that $\eta_Y(H^{2,0}(Y)) = \eta_X(\rho_u(H^{2,0}(X)))$, for a class $u \in H^2(X,\mathbb{Z})$ as in Theorem 1.5. Then $f := \eta_Y^{-1}\eta_X(-\rho_u) : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is a rational Hodge isometry of r-cyclic type preserving the orientations of the positive cones.

Lemma 1.6. *The dual Lefschetz operator is given by $e_\lambda^\vee = (2/t(\lambda,\lambda))\phi^{-1}e_{\tilde{\phi}(\lambda)}\phi$ and is algebraic if λ is, as $e_{\tilde{\phi}(\lambda)}$ and the morphisms ϕ and ϕ^{-1} are.*

The short proof of the lemma is given at the end of §4.

Example 1.7. If X is an IHSM of $K3^{[n]}$ type, $n = 1, 2$, then $\mathrm{Aut}_{\mathcal{G}_{an}^{[n]}}(X)$ contains a degree-reversing involution. When X is a $K3$ surface such an involution is induced by the ideal sheaf of the diagonal in $X \times X$, as the ideal sheaf is the kernel of a derived auto-equivalence. When X is of $K3^{[2]}$ type, a rank-2 twisted reflexive sheaf E was constructed on $X \times X$ in [Mar20]. When $X = S^{[2]}$, for a $K3$ surface S, Addington showed that E is the Fourier–Mukai kernel of an auto-equivalence of $D^b(S^{[2]})$ acting on cohomology as an involution [Add16]. It follows that $[\kappa(E)\sqrt{td_{X\times X}}]_*$ is a degree-reversing involution in $\mathrm{Aut}_{\mathcal{G}_{an}^{[2]}}(X)$, for every X of $K3^{[2]}$ type. Does a degree-reversing automorphism exist in $\mathrm{Aut}_{\mathcal{G}_{an}^{[n]}}(X)$ for every X of $K3^{[n]}$ type for $n \geq 3$? What about other deformation types?

1.3 The Pontryagin product

In §10 we define a *Pontryagin product* $\star$ on the cohomology $H^*(X, \mathbb{Q})$ of every IHSM X with vanishing odd cohomology. The unit with respect to $\star$ is $c_X[pt]/n!$, where $\dim(X) = 2n$, $[pt] \in H^{\mathrm{top}}(X, \mathbb{Q})$ is the class Poincaré dual to the class of a point, and c_X is the Fujiki constant.[6] The product $\star$ maps $H^k(X, \mathbb{Q}) \otimes H^l(X, \mathbb{Q})$ to $H^{k+l-4n}(X, \mathbb{Q})$, so that $\star$ is compatible with the grading for which $H^k(X, \mathbb{Q})$ has degree $4n - k$. Parallel-transport operators induce isomorphisms of the Pontryagin rings (Lemma 10.2).

The Pontryagin product $\star$ on the cohomology $H^*(S, \mathbb{Z})$ of a $K3$ surface S is simple to describe. The unit with respect to $\star$ is the class $[pt] \in H^4(X, \mathbb{Z})$ Poincaré dual to a point. The product $\star$ maps $H^k(S, \mathbb{Z}) \otimes H^l(S, \mathbb{Z})$ to $H^{k+l-4}(S, \mathbb{Z})$ and satisfies $\lambda_1 \star \lambda_2 = (\lambda_1, \lambda_2) \in H^0(S, \mathbb{Z})$, for classes $\lambda_1, \lambda_2 \in H^2(S, \mathbb{Z})$. Let $\rho_u : \widetilde{H}(S, \mathbb{Z}) \to \widetilde{H}(S, \mathbb{Z})$, be the reflection $\rho_u(r, c, s) = (-s, c, -r)$ in the Mukai vector $u := (1, 0, 1)$ of the structure sheaf $\mathcal{O}_S$. Then $-\rho_u$ conjugates the usual product on $H^*(S, \mathbb{Z})$ to the Pontryagin product. Note that $-\rho_u$ corresponds to the action on cohomology of the autoequivalence $\Phi_{\mathcal{I}_\Delta}$ of $D^b(S)$, whose Fourier–Mukai kernel is the ideal sheaf $\mathcal{I}_\Delta$ of the diagonal Δ in $S \times S$.

Morphisms in $\mathcal{G}_{[an]}$, which are associated to equivalences of derived categories with a Fourier–Mukai kernel of positive rank, are degree reversing. This corresponds to a normalization which annihilates the first Chern class of the Fourier–Mukai kernel. We introduce a further normalization, which assures that the compositions of any two such degree-reversing morphisms would map the usual identity $1 \in H^0(X, \mathbb{Q})$ to itself. We speculate in the following that thus normalized, the degree-reversing morphisms conjugate the cup product to the Pontryagin product. We introduce this further normalization in the next paragraph.

Let $\mathbb{Q}^\times$ be the multiplicative group of rational numbers. Let

$$\chi : \mathcal{G}_{an} \to \mathbb{Q}^\times$$

be the groupoid homomorphism sending every object to $\mathbb{Q}^\times$ and sending a morphism ϕ to t, if ϕ is degree preserving and $\widetilde{H}(\phi)(\alpha) = t\alpha$, or if ϕ is degree reversing and $\widetilde{H}(\phi)(\alpha) = (1/t)\beta$. Given a non-zero rational number t, denote by $\mu_t : H^*(X, \mathbb{Q})[2n] \to H^*(X, \mathbb{Q})[2n]$ the graded linear transformation multiplying $H^{2k}(X, \mathbb{Q})[\dim(X)]$ by t^k. Let

$$\mathcal{R}_{an}^{[n]} \tag{1.6}$$

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[6] The *Fujiki constant* c_X is the positive rational number such that the equality $\int_X \lambda^{2n} = ((2n)!/2^n n!)c_X(\lambda, \lambda)^n$ holds for all $\lambda \in H^2(X, \mathbb{Q})$. The Fujiki constant is calculated for all known IHSMs in [Rap08]. If X of $K3^{[n]}$ type, then $c_X = 1$.

be the subgroupoid of $\mathcal{G}_{an}^{[n]}$ with the same objects, but whose morphisms ϕ have the property that $\mu_{\chi(\phi)} \circ \phi$ conjugates the usual product on $H^*(X, \mathbb{Q})$ to itself, if ϕ is degree preserving and to $\star$ if ϕ is degree reversing.

CONJECTURE 1.8. *The subgroupoid $\mathcal{R}_{an}^{[n]}$ is, in fact, the whole of $\mathcal{G}_{an}^{[n]}$. Furthermore, if an object of positive rank F in $D^b(X \times Y)$ is the Fourier–Mukai kernel of an equivalence $\Phi_F : D^b(X) \to D^b(Y)$ of derived categories of projective IHSMs X and Y of $K3^{[n]}$ type, then the rank of F is $n!t^n$, for some rational number t, and setting $\phi := \mu_t \circ [\kappa(F)\sqrt{td_{X \times Y}}]_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ we have*

$$\phi(\gamma_1 \cup \gamma_2) = \phi(\gamma_1) \star \phi(\gamma_2), \tag{1.7}$$

for all $\gamma_1, \gamma_2 \in H^*(X, \mathbb{Q})$.

The stated equality $\text{rank}(F) = n!t^n$, for some rational number t, is known [Bec23, Theorem 4.14]. As evidence for the conjecture, we show that the examples of morphisms of $\mathcal{G}_{an}^{[n]}$ used in the proof of Theorem 1.4 all satisfy Conjecture 1.8. Consequently, Theorem 1.4 holds even after replacing $\mathcal{G}_{an}^{[n]}$ by $\mathcal{R}_{an}^{[n]}$ and we get the following result proven in § 10.

PROPOSITION 1.9 (Proposition 10.7). *The restriction of the functor $\widetilde{H}_0 : \mathcal{G}_{an}^{[n]} \to \mathcal{H}dg^{[n]}$ to the subgroupoid $\mathcal{R}_{an}^{[n]}$ remains full.*

Let X and Y be projective IHSMs of $K3^{[n]}$ type. The proof of Proposition 10.7 implies that the degree-reversing correspondence $\kappa(E)\sqrt{td_{X \times Y}}$ in Theorem 1.5 satisfies Conjecture 1.8. Set $\phi := \mu_t \circ [\kappa(E)\sqrt{td_{X \times Y}}]_*$, as in the conjecture. Let $[\Delta] \in H^*(X^3, \mathbb{Q})$ the class of the small diagonal Δ in X^3. The following corollary is proved in § 10.2.

COROLLARY 1.10. *The Pontryagin product $\star : H^*(Y \times Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ is induced by the algebraic class $\phi^3([\Delta])$ in $H^*(Y^3, \mathbb{Q})$. In particular, Pontryagin product $\gamma\star : H^*(Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ by an algebraic class γ is induced by an algebraic correspondence.*

The algebraicity of the dual Lefschetz operator $e_\lambda^\vee$ in Lemma 1.6 is a special case of the above corollary, since $e_\lambda^\vee$ is a scalar multiple of the operator of Pontryagin product with the algebraic class $\phi^{-1}(\tilde{\phi}(\lambda))$.

1.4 Outline of the proof of Theorem 1.4

The reader familiar with Buskin's proof of the case $n = 1$ of the theorem will note the similarity of our strategy to his. Buskin considers examples of two-dimensional moduli spaces M of vector bundles over a $K3$ surface S and the universal bundle $\mathcal{U}$ over $M \times S$. Here M is a $K3$ surface and so the product $M \times S$ is a hyper-Kähler manifold. Buskin uses Verbitsky's results on hyperholomorphic vector bundles over hyper-Kähler manifolds to deform the pair $(M \times S, \mathcal{U})$ to twisted vector bundles $(X_1 \times X_2, \mathcal{U}')$ over products of not necessarily projective $K3$ surfaces X_1 and X_2. Buskin proves that the groupoid $\mathcal{H}dg^{[1]}$ is generated by parallel-transport operators, which are isomorphisms of Hodge structures, and by the restrictions to $H^2(X_1, \mathbb{Q})$ of the degree-reversing isometries $[\kappa(\mathcal{U}')\sqrt{td_{X_1 \times X_2}}]_* : \widetilde{H}(X_1, \mathbb{Q}) \to \widetilde{H}(X_2, \mathbb{Q})$.

We first construct the functor $\widetilde{H}_0 : \mathcal{G}_{an} \to \mathcal{H}dg$ and establish its properties in §§3 and 4. The construction depends heavily on the recent work of Taelman [Tae23], as well as on the work of the author and independently of Beckmann describing the action of equivalences of derived categories on the LLV lattice [Bec23, Mar21a]. In §7 we lift the equivalence of derived categories $\Phi_{\mathcal{U}} : D^b(M) \to D^b(S)$, with the universal bundle $\mathcal{U}$ as a kernel, to an equivalence $\Phi_{\mathcal{U}^{[n]}} : D^b(M^{[n]}) \to D^b(S^{[n]})$ of derived categories of Hilbert schemes. This is done by conjugating the equivalence of equivariant derived categories on the cartesian products $\Phi_{\mathcal{U}^{\boxtimes n}} : D^b_{\mathfrak{S}_n}(M^n) \to D^b_{\mathfrak{S}_n}(S^n)$, obtained from the cartesian power of $\Phi_{\mathcal{U}}$, with the Bridgeland–King–Reid equivalence $BKR : D^b_{\mathfrak{S}_n}(S^n) \to D^b(S^{[n]})$ and its analogue for M. The Bridgeland–King–Reid equivalence is reviewed in §6. The assignment $S \mapsto S^{[n]}$ and $\Phi_{\mathcal{U}} \mapsto \Phi_{\mathcal{U}^{[n]}}$ extends to define a functor

$$\Theta_n : \mathcal{G}_{an}^{[1]} \to \mathcal{G}_{an}^{[n]} \tag{1.8}$$

constructed in (7.12). The Fourier–Mukai kernel $\mathcal{U}^{[n]}$ is again a locally free sheaf over $M^{[n]} \times S^{[n]}$. Again we use Verbitsky's results on hyperholomorphic vector bundles on hyper-Kähler manifolds to deform the pair $(M^{[n]} \times S^{[n]}, \mathcal{U}^{[n]})$ to pairs $(X_1 \times X_2, E)$ consisting of a twisted vector bundle E over the product $X_1 \times X_2$ of not necessarily projective IHSMs of $K3^{[n]}$ type. The general techniques are developed in §5 and applied to $\mathcal{U}^{[n]}$ in §8.

It remains to prove that we get a full functor already when we restrict the functor $\widetilde{H}_0 : \mathcal{G}_{an}^{[n]} \to \mathcal{H}dg^{[n]}$ to the subgroupoid of $\mathcal{G}_{an}^{[n]}$ generated by parallel-transport operators, which are Hodge isometries, and morphisms in the image of Θ_n. This is done as follows.

Step 1 (generators for the group of rational isometries). Let L be a lattice isometric to the second cohomology of a $K3$ surface. Let Λ be the orthogonal direct sum $L \oplus \mathbb{Z}\delta$, where $(\delta, \delta) = 2 - 2n$. Then Λ is isometric to the second cohomology of an IHSM X of $K3^{[n]}$ type endowed with the BBF pairing [Bea83]. The monodromy group of X acts on $H^2(X, \mathbb{Z})$ via a normal subgroup of $O(H^2(X, \mathbb{Z}))$ and so corresponds to a well-defined subgroup $\mathrm{Mon}(\Lambda)$ of $O(\Lambda)$, by [Mar08, Theorem 1.6 and Lemma 4.10]. We first prove that the group $O(\Lambda_{\mathbb{Q}})$ of rational isometries is generated by $\mathrm{Mon}(\Lambda)$ and $O(L_{\mathbb{Q}})$ (Proposition 2.1). It follows from the Cartan–Dieudonné theorem that every rational isometry $\psi : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$, which preserves the orientation of the positive cone, decomposes as the composition $\psi = \psi_k \circ \cdots \circ \psi_1$, where ψ_i belongs to the double orbit $\mathrm{Mon}(\Lambda)(-\rho_u)\mathrm{Mon}(\Lambda)$, where ρ_u is the reflection in a primitive integral class $u \in L$ with $(u, u) > 0$ (Corollary 2.5). The double orbit $\mathrm{Mon}(\Lambda)(-\rho_u)\mathrm{Mon}(\Lambda)$ depends only on (u, u) and is otherwise independent of the class u, since so does the double orbit $O^+(L)(-\rho_u)O^+(L)$, by [Bus19, Proposition 3.3], and $\mathrm{Mon}(\Lambda)$ contains $O^+(L)$.

Step 2 (reduction to the double orbit of a reflection). Let $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ be a rational Hodge isometry preserving the orientations of the positive cones, where X and Y are of $K3^{[n]}$ type. Choose isometries $\eta_X : H^2(X, \mathbb{Z}) \to \Lambda$ and $\eta_Y : H^2(Y, \mathbb{Z}) \to \Lambda$, so that the marked pairs (X, η_X) and (Y, η_Y) belong to the same connected component $\mathfrak{M}_\Lambda^0$ of the moduli space of marked IHSMs of $K3^{[n]}$ type. Let Ω_Λ be the period domain of marked IHSMs of $K3^{[n]}$ type and let $P : \mathfrak{M}_\Lambda^0 \to \Omega_\Lambda$ be the period map (see (5.1) for the definitions of Ω_Λ and P). Set $\psi := \eta_Y \circ f \circ \eta_X^{-1} \in O(\Lambda_{\mathbb{Q}})$ and choose a decomposition $\psi = \psi_k \circ \cdots \circ \psi_1$ as in Step 1. The group $O^+(\Lambda_{\mathbb{Q}})$ acts on Ω_Λ and $\psi(P(X, \eta_X)) = P(Y, \eta_Y)$. Set $\ell_i := (\psi_i \circ \cdots \circ \psi_1)(P(X, \eta_X))$, $0 \le i \le k$. The surjectivity of the period map implies the existence of marked pairs $(X_i, \eta_i) \in \mathfrak{M}_\Lambda^0$, such that $P(X_i, \eta_i) = \ell_i$, where we choose $(X_0, \eta_0) := (X, \eta_X)$ and $(X_k, \eta_k) := (Y, \eta_Y)$. Set $f_i := \eta_i^{-1}\psi_i \circ \eta_{i-1} : H^2(X_{i-1}, \mathbb{Q}) \to H^2(X_i, \mathbb{Q})$. Then $f = f_k \circ \cdots \circ f_1$. If $k = 0$, so that $\psi = \eta_Y \circ f \circ \eta_X^{-1}$ is the identity, then $f = \eta_Y^{-1}\eta_X$ is a parallel-transport operator and a Hodge isometry, and so f lifts to

a morphism $\phi \in \mathrm{Hom}_{\mathcal{G}_{an}}(X, Y)$ with $\widetilde{H}_0(\phi) = f$, by Verbitsky's Torelli theorem (Theorem 5.2). It remains to prove that each f_i belongs to the image of $\widetilde{H}_0$.

Step 3. The equivalence of derived categories $\Phi_{\mathcal{U}^{[n]}} : D^b(M^{[n]}) \to D^b(S^{[n]})$ described previously yields a morphism in $\mathcal{G}_{an}^{[n]}$ which is mapped via $\widetilde{H}_0$ to a Hodge isometry $H^2(M^{[n]}, \mathbb{Q}) \to H^2(S^{[n]}, \mathbb{Q})$ which, in turn, is simply the extension[7] of the isometry $H^2(M, \mathbb{Q}) \to H^2(S, \mathbb{Q})$ associated to $\Phi_{\mathcal{U}}$ (Corollary 7.3). The resulting Hodge isometry $H^2(M^{[n]}, \mathbb{Q}) \to H^2(S^{[n]}, \mathbb{Q})$ is thus in the double orbit $\mathrm{Mon}(\Lambda)(-\rho_u)\mathrm{Mon}(\Lambda)$, where ρ_u is the reflection in a primitive integral class $u \in L$ with $(u, u) > 0$. The proof of Corollary 7.3 relies on [Mar21a, Theorem 12.2] obtained independently by Beckman [Bec23, Theorem 7.4]. All such double orbits are obtained that way by varying the Mukai vector of the two-dimensional moduli space M, by [Bus19, Proposition 3.3]. We prove that every rational Hodge isometry $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$, with f in a fixed double orbit of a reflection in $u \in L$ and preserving the orientation of the positive cones, is of the form $\widetilde{H}_0([\kappa(E)\sqrt{td_{X \times Y}}]_*)$, for a pair $(X \times Y, E)$, which is a deformation of $(M^{[n]} \times S^{[n]}, \mathcal{U}^{[n]})$ for a suitable choice of M (Corollary 8.5).

2. Generators for the rational isometry group

Let U be the rank-2 lattice with basis $\{e_1, e_2\}$ satisfying $(e_i, e_i) = 0$, $i = 1, 2$, and $(e_1, e_2) = -1$. Let L be an even unimodular lattice of signature $(3, s_-)$, $s_- \geq 3$, which contains a sublattice isometric to the orthogonal direct sum of three copies of U. Let the lattice Λ be the orthogonal direct sum $L \oplus \mathbb{Z}\delta$, where $(\delta, \delta) = -2d$ for some positive integer d. Given a non-degenerate lattice M, denote by $O^+(M)$ the subgroup of isometries of M which preserve the orientation of the positive cone of $M \otimes_{\mathbb{Z}} \mathbb{R}$ (see [Mar11, §4]). Let $\Gamma \subset O^+(\Lambda)$ be the subgroup which acts on the discriminant group Λ^*/Λ by multiplication by ± 1. Set $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$. We regard $O(L_{\mathbb{Q}})$ as a subgroup of $O(\Lambda_{\mathbb{Q}})$ by extending each isometry so that it leaves δ invariant.

PROPOSITION 2.1. *The group $O(\Lambda_{\mathbb{Q}})$ is generated by Γ and $O(L_{\mathbb{Q}})$.*

The following lemmas are needed for the proof of the above proposition.

LEMMA 2.2. *Let $\alpha = \lambda + k\delta$, where λ is a primitive class in L and $k \in \mathbb{Z}$. There exists an isometry $\gamma \in \Gamma$, such that $\gamma(\alpha)$ belongs to L.*

Proof. Let $\widetilde{\Lambda}$ be the orthogonal direct sum of L and U and let $\iota : \Lambda \to \widetilde{\Lambda}$ be the isometric embedding restricting to the identity on L and sending δ to the element of the second direct summand U with coordinate $(1, d)$ in the basis $\{e_1, e_2\}$. Let $v \in \widetilde{\Lambda}$ be the element with coordinates $(1, -d)$ of the second direct summand U, so that $(v, v) = 2d$ and $v^\perp = \iota(\Lambda)$, where $v^\perp$ is the co-rank-1 sublattice of $\widetilde{\Lambda}$ orthogonal to v. The sublattice $\Sigma_1 := \mathrm{span}_{\mathbb{Z}}\{\iota(\alpha), v\}$ of $\widetilde{\Lambda}$ is saturated. Indeed, if $a, b \in \mathbb{Q}$ and $a\iota(\alpha) + bv = a\lambda + (ak + b)e_1 + (adk - bd)e_2$ is an integral class of $\widetilde{\Lambda}$, then a must be an integer, since λ is primitive, hence b must be an integer as well. Let $\beta \in L$ be a primitive class satisfying $(\beta, \beta) = (\alpha, \alpha)$. The sublattice $\Sigma_2 := \mathrm{span}_{\mathbb{Z}}\{\beta, v\}$ is saturated in $\widetilde{\Lambda}$ and is isometric to Σ_1. There exists an isometry $g \in O(\widetilde{\Lambda})$ satisfying $g(v) = v$ and $g(\iota(\alpha)) = \beta$, by results of Nikulin [Nik87, Mar08, Lemma 8.1]. Set $\tilde{\gamma} := g$, if g belongs to $O^+(\widetilde{\Lambda})$, and

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[7] We note that $\widetilde{H}([ch(\mathcal{U}^{[n]})\sqrt{td_{M^{[n]} \times S^{[n]}}}]_*) : \widetilde{H}(M^{[n]}, \mathbb{Q}) \to \widetilde{H}(S^{[n]}, \mathbb{Q})$ is *not* the naive extension of $[ch(\mathcal{U})\sqrt{td_{M \times S}}]_* : \widetilde{H}(M, \mathbb{Q}) \to \widetilde{H}(S, \mathbb{Q})$, by [Tae23, Theorem 9.4] in the $K3^{[2]}$ case and by [Mar21a, Theorem 12.2] and [Bec23, Theorem 7.4] in the $K3^{[n]}$ case, $n \geq 2$ (see Lemma 7.2). Nevertheless, when we replace $ch(\mathcal{U}^{[n]})$ by $\kappa(\mathcal{U}^{[n]})$ the resulting isometry $\widetilde{H}([\kappa(\mathcal{U}^{[n]})\sqrt{td_{M^{[n]} \times S^{[n]}}}]_*) : \widetilde{H}(M^{[n]}, \mathbb{Q}) \to \widetilde{H}(S^{[n]}, \mathbb{Q})$ *is* the naive extension of $[\kappa(\mathcal{U})\sqrt{td_{M \times S}}]_* : \widetilde{H}(M, \mathbb{Q}) \to \widetilde{H}(S, \mathbb{Q})$, by Corollary 7.3.

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$\tilde{\gamma} := -g$ otherwise. Then $\tilde{\gamma} \circ \iota = \iota \circ \gamma$, for a unique element $\gamma \in \Gamma$, by [Mar08, Theorem 1.6 and Lemma 4.2], and $\gamma(\alpha)$ belongs to L. □

Let G be the subgroup of $O(\Lambda_{\mathbb{Q}})$ generated by Γ and $O(L_{\mathbb{Q}})$.

LEMMA 2.3. *Let* $\lambda \in L_{\mathbb{Q}}$ *and* $t \in \mathbb{Q}$ *and assume that* $(\lambda, \lambda) \neq 0$. *There exists an isometry* $g \in G$, *such that* $g(\lambda + t\delta)$ *belongs to* $L_{\mathbb{Q}}$.

Proof. Let $q \in \mathbb{Q}^{\times}$ be such that $q(\lambda + t\delta)$ is integral. Choose an integral and primitive class $\lambda' \in L$ satisfying $(\lambda', \lambda') = q^2(\lambda, \lambda)$. There exists an isometry $f \in O(L_{\mathbb{Q}})$, such that $\lambda' = f(q\lambda)$, by [Ger08, Proposition 2.35]. Then $f(q(\lambda + t\delta)) = \lambda' + qt\delta$ is integral and there exists $h \in \Gamma$, such that $h(f(q(\lambda + t\delta)))$ is a primitive class in L, by Lemma 2.2. The statement follows, since $h \circ f$ belongs to G. □

Proof of Proposition 2.1. Choose $u \in L$ satisfying $(u, u) = 2d + 2$. Then $(u + \delta, u + \delta) = 2$. Let $\rho_{u+\delta}$ be the reflection of Λ in $(u + \delta)^{\perp}$

$$\rho_{u+\delta}(x) = x - (u + \delta, x)(u + \delta).$$

Then $\rho_{u+\delta}(\delta) = (2d)u + (1 + 2d)\delta$. Furthermore, $-\rho_{u+\delta}$ belongs to Γ. There exists $g_1 \in G$, such that $g_1(-\rho_{u+\delta}(\delta))$ belongs to $L_{\mathbb{Q}}$, by Lemma 2.3.

Let ϕ be an isometry in $O(\Lambda_{\mathbb{Q}})$. Write $\phi(\delta) = \lambda + t\delta$. If $(\lambda, \lambda) = 0$ choose $u \in L$, such that $(u, \lambda) = 0$ and $(u, u) = 2d + 2$. Then $\rho_{u+\delta}(\phi(\delta)) = \lambda + (2d)u + (1 + 2d)\delta$ and the self-intersection $(\lambda + (2d)u, \lambda + (2d)u) = 4d^2(2d + 2)$ is non-zero. Now, ϕ belongs to G, if and only if $-\rho_{u+\delta}\phi$ belongs to G. Hence, we may assume that $(\lambda, \lambda) \neq 0$. Then there exists $g_2 \in G$ such that $(g_2 \circ \phi)(\delta)$ belongs to $L_{\mathbb{Q}}$, by Lemma 2.3. There exists $h \in O(L_{\mathbb{Q}})$, such that $h(g_2(\phi(\delta))) = g_1(-\rho_{u+\delta}(\delta))$, by [Ger08, Proposition 2.35]. Hence, $(-\rho_{u+\delta}g_1^{-1}hg_2\phi)(\delta) = \delta$, and so $-\rho_{u+\delta}g_1^{-1}hg_2\phi$ belongs to $O(L_{\mathbb{Q}})$. Hence, ϕ belongs to G. □

LEMMA 2.4. *The group* $O(L_{\mathbb{Q}})$ *is generated by* $O(L)$ *and reflections* ρ_u *in* $u^{\perp}$, *for* $u \in L$ *satisfying* $(u, u) > 0$.

Proof. The group $O(L_{\mathbb{Q}})$ is generated by reflections ρ_u, with $(u, u) \neq 0$, by [Ger08, Proposition 2.36]. The isometry group $O(L)$ acts transitively on the set $\{u : u$ is primitive and $(u, u) = 2d\}$, $d \in \mathbb{Z}$, by [Nik87, Theorem 1.14.4]. Consider the above basis $\{e_1, e_2\}$ of U and observe that $\rho_{e_1 - de_2}\rho_{e_1 + de_2}$ is minus the identity of U. Hence, for every element $u \in L$ with $(u, u) < 0$, there exists an element $w \in L$ with $(w, w) = -(u, u)$, such that $\rho_u\rho_w$ belongs to $O(L)$. Thus, it sufficed to consider reflections ρ_u with $(u, u) > 0$. □

COROLLARY 2.5. *Let* ϕ *be an isometry in* $O^+(\Lambda_{\mathbb{Q}})$. *There exist a positive integer* k, *integral elements* $u_i \in L$ *satisfying* $(u_i, u_i) \geq 2$, $1 \leq i \leq k$, *and elements* $\gamma_i \in \Gamma$, $0 \leq i \leq k$, *such that*

$$\phi = (-1)^k \gamma_k \rho_{u_k} \gamma_{k-1} \rho_{u_{k-1}} \cdots \gamma_1 \rho_{u_1} \gamma_0,$$

where $\rho_{u_i}(x) = x - (2(u_i, x)/(u_i, u_i))u_i$ *is the reflection of* $\Lambda_{\mathbb{Q}}$ *in* $u_i^{\perp}$.

Proof. This follows immediately from Proposition 2.1, Lemma 2.4, the fact that $O^+(L)$ is contained in Γ, and the fact that $-\rho_{u_i}$ belongs to $O^+(\Lambda_{\mathbb{Q}})$ if $(u_i, u_i) > 0$, since the signature of Λ is $(3, 1 + s_-)$ and 3 is odd. □

Remark 2.6. Note that in Corollary 2.5 we can require γ_i to belong to any subgroup Γ_0 of Γ, such that Γ_0 and $O^+(L)$ generate Γ. When L is the $K3$ lattice, the group Γ is the monodromy group of IHSMs of $K3^{[d+1]}$ deformation type [Mar08, Theorem 1.6]. If $L = U^{\oplus 3}$, the monodromy of generalized Kummer varieties of dimension $2d - 2$, for $d \geq 3$, is the following index 2 subgroup

Γ_0 of Γ. Here Γ acts on the discriminant group Λ^*/Λ by a character $\xi : \Gamma \to \{\pm 1\}$, by the definition of Γ. Let $\det : \Gamma \to \{\pm 1\}$ be the determinant character. Then Γ_0 is the kernel of the product character $\det \cdot \xi : \Gamma \to \{\pm 1\}$, by [Mar23, Theorem 1.4]. The character $\det \cdot \xi$ has value -1 on a reflection $\rho_\omega \in O^+(L)$ in a class $w \in L$ with $(w, w) = -2$, and so Γ_0 and $O^+(L)$ generate Γ.

3. The LLV Lie algebra

In §3.1 we recall the LLV Lie algebra $\mathfrak{g}_X$ and its action on the cohomology of a $2n$-dimensional IHSM X. We review the $\mathfrak{g}_X$-equivariant isomorphism between the subring $SH^*(X, \mathbb{Q})$ of $H^*(X, \mathbb{Q})$ generated by $H^2(X, \mathbb{Q})$ and the nth symmetric power of the LLV lattice $\widetilde{H}(X, \mathbb{Q})$. In §3.2 we recall Taelman's definition of the functor $\widetilde{H} : \mathcal{G} \to \widetilde{\mathcal{G}}$, mentioned in (1.4), sending equivalences of derived categories to isometries of LLV lattices. Given a morphism $\phi \in \text{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$, which is degree preserving up to sign, we relate in Lemmas 3.6 and 3.7 the restrictions of ϕ and of $\widetilde{H}(\phi)$ to $H^2(X, \mathbb{Q})$.

A note about orientations: we take pains to carefully keep track of the sign of $\widetilde{H}(\phi)$, for a morphism $\phi \in \text{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$. The sign depends in a subtle way on the orientations ϵ_X, ϵ_Y. We use the functor $\widetilde{H} : \mathcal{G} \to \widetilde{\mathcal{G}}$ in order to define the functor $\widetilde{H}_0 : \mathcal{G}_{an} \to \mathcal{H}dg$ in (1.5) (see also (5.8)). The sign of $\widetilde{H}_0(\phi)$ is independent of the sign of $\widetilde{H}(\phi)$, so the functor $\widetilde{H}_0$ is independent of the orientations. The proof of Theorems 1.1 and 1.4 depends only on $\widetilde{H}_0$. The reader who is only interested in that proof may ignore the orientations of objects in $\mathcal{G}$ and redefine morphisms in $\widetilde{\mathcal{G}}$ as pairs $\{\pm g\}$ of an isometry $g : \widetilde{H}(X, \mathbb{Q}) \to \widetilde{H}(Y, \mathbb{Q})$ and its negative. The resulting coarser version of the functor $\widetilde{H}$ is sufficient to define $\widetilde{H}_0$.

3.1 The subring generated by $H^2(X, \mathbb{Q})$

Let X be a $2n$-dimensional IHSM. Given a class $\lambda \in H^2(X, \mathbb{Q})$ denote by $e_\lambda : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ the endomorphism given by the cup product with λ. The grading operator

$$h_X : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \tag{3.1}$$

acts on $H^k(X, \mathbb{Q})[2n] = H^{2n+k}(X, \mathbb{Q})$, $-2n \leq k \leq 2n$, via multiplication by k. A triple (e, h, f) of endomorphisms of $H^*(X, \mathbb{Q})$ is called an $\mathfrak{sl}_2$-triple, if

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Given e, if f exists, then it is the unique endomorphism completing the pair (e, h) to an $\mathfrak{sl}_2$-triple. The *LLV Lie algebra* $\mathfrak{g}_X \subset \text{End}(H^*(X, \mathbb{Q}))$ is the Lie algebra generated by all $\mathfrak{sl}_2$-triples $(e_\lambda, h, f_\lambda)$, $\lambda \in H^2(X, \mathbb{Q})$.

Denote by $S_{[n]}\widetilde{H}(X, \mathbb{Q})$ the subspace of $\text{Sym}^n(\widetilde{H}(X, \mathbb{Q}))$ generated by nth powers of isotropic elements of $\widetilde{H}(X, \mathbb{Q})$. Verbitsky proved that the subring $SH^*(X, \mathbb{Q})$ is an irreducible $\mathfrak{g}_X$-submodule of $H^*(X, \mathbb{Q})$, appearing with multiplicity one, there exists a Lie algebra isomorphism

$$\dot{\rho} : \mathfrak{so}(\widetilde{H}(X, \mathbb{Q})) \to \mathfrak{g}_X, \tag{3.2}$$

and there exists a unique graded isomorphism of $\mathfrak{so}(\widetilde{H}(X, \mathbb{Q}))$-modules

$$\Psi : SH^*(X, \mathbb{Q})[2n] \to S_{[n]}\widetilde{H}(X, \mathbb{Q}), \tag{3.3}$$

satisfying $\Psi(1) = \alpha^n/n!$, where the $\mathfrak{so}(\widetilde{H}(X, \mathbb{Q}))$-module structure of $SH^*(X, \mathbb{Q})[2n]$ is via $\dot{\rho}$ (see [Tae23, Proposition 3.5]). The pairing on $\widetilde{H}(X, \mathbb{Q})$ induces a pairing $b_{[n]}$ on $S_{[n]}\widetilde{H}(X, \mathbb{Q})$, the

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Mukai pairing on $H^*(X,\mathbb{Q})$ restricts to a pairing b_{SH} on $SH^*(X,\mathbb{Q})$, and $c_X b_{[n]}(\Psi(x),\Psi(y)) = b_{SH}(x,y)$, where c_X is the Fujiki constant [Tae23, Proposition 3.8].

The following is the explicit description of the isomorphism Ψ (see [Tae23, Proposition 3.5]). Given $\lambda \in H^2(X,\mathbb{Q})$, denote by

$$e_\lambda \in \mathfrak{so}(\widetilde{H}(X,\mathbb{Q})) \tag{3.4}$$

also the endomorphism of $\widetilde{H}(X,\mathbb{Q})$ sending α to λ, annihilating β, and sending $\lambda' \in H^2(X,\mathbb{Q})$ to $(\lambda,\lambda')\beta$. Extend e_λ as an endomorphism of $\mathrm{Sym}^n\widetilde{H}(X,\mathbb{Q})$ via the product rule, so that $e_\lambda(x_1\cdots x_n) = \sum_{i=1}^n x_1\cdots e_\lambda(x_i)\cdots x_n$. Then

$$\Psi(\lambda_1\cdots\lambda_k) = e_{\lambda_1}\cdots e_{\lambda_k}(\alpha^n/n!).$$

Let

$$\tilde{h}_X : \widetilde{H}(X,\mathbb{Q}) \to \widetilde{H}(X,\mathbb{Q}) \tag{3.5}$$

be the grading operator multiplying α by -2, β by 2, and $H^2(X,\mathbb{Q})$ by 0.

3.2 The functor $\widetilde{H}$

Let ϕ be a morphism in $\mathrm{Hom}_{\mathcal{G}}((X,\epsilon),(Y,\epsilon'))$, where $\mathcal{G}$ is the groupoid (1.1). Then ϕ is in particular an isometry $\phi : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ with respect to the Mukai pairing and we denote by

$$Ad_\phi : \mathrm{End}(H^*(X,\mathbb{Q})) \to \mathrm{End}(H^*(Y,\mathbb{Q}))$$

the Lie algebra isomorphism given by $Ad_\phi(\xi) = \phi\xi\phi^{-1}$. Then Ad_ϕ restricts to a Lie algebra isomorphism from $\mathfrak{g}_X$ onto $\mathfrak{g}_Y$, by [Tae23, Theorem A]. Hence, ϕ restricts to an isometry $\phi_| : SH^*(X,\mathbb{Q}) \to SH^*(Y,\mathbb{Q})$. Consequently, when X and Y are deformation equivalent, so that $c_X = c_Y$, then $\Psi_Y \circ \phi_| \circ \Psi_X^{-1} : S_{[n]}\widetilde{H}(X,\mathbb{Q}) \to S_{[n]}\widetilde{H}(Y,\mathbb{Q})$ is an isometry, which conjugates $S_{[n]}\mathfrak{so}(\widetilde{H}(X,\mathbb{Q}))$ to $S_{[n]}\mathfrak{so}(\widetilde{H}(Y,\mathbb{Q}))$, where $S_{[n]}\mathfrak{so}(\widetilde{H}(X,\mathbb{Q}))$ is the image of $\mathfrak{so}(\widetilde{H}(X,\mathbb{Q}))$ in $\mathrm{End}(S_{[n]}\widetilde{H}(X,\mathbb{Q}))$.

Given an isomorphism $f : V_1 \to V_2$ of oriented vector spaces, we set $\det(f) = 1$, if f is orientation preserving and $\det(f) = -1$ otherwise.

PROPOSITION 3.1 [Tae23, Proposition 4.4, Theorem 4.10, and Theorem 4.11]. *Let V_1 and V_2 be d-dimensional vector spaces over $\mathbb{Q}$ with non-degenerate bilinear parings and $\phi : S_{[n]}V_1 \to S_{[n]}V_2$ an isometry, such that $\phi(S_{[n]}\mathfrak{so}(V_1))\phi^{-1} = S_{[n]}\mathfrak{so}(V_2)$. If n is even assume that d is odd and choose an orientation for each of V_1 and V_2. Then there exists a unique isometry $\widetilde{H}(\phi) : V_1 \to V_2$, such that the restriction $S_{[n]}(\widetilde{H}(\phi))$ of its n-symmetric power to $S_{[n]}V_1$ satisfies $S_{[n]}(\widetilde{H}(\phi)) = \phi$, if n is odd, and $\det(\widetilde{H}(\phi))S_{[n]}(\widetilde{H}(\phi)) = \phi$, if n is even.*

Remark 3.2. Fix a positive integer n. Consider the two groupoids $\mathcal{V}$ and $S_{[n]}\mathcal{V}$, whose objects are vector spaces over $\mathbb{Q}$, which are odd dimensional if n is even, with non-degenerate symmetric quadratic forms and with the additional choice of orientations, if n is even. The morphisms of $\mathcal{V}$ are isometries. The morphisms in $\mathrm{Hom}_{S_{[n]}\mathcal{V}}(V_1,V_2)$ are isometries $\phi : S_{[n]}V_1 \to S_{[n]}V_2$ satisfying $\phi(S_{[n]}\mathfrak{so}(V_1))\phi^{-1} = S_{[n]}\mathfrak{so}(V_2)$. We have the obvious functor $S_{[n]} : \mathcal{V} \to S_{[n]}\mathcal{V}$, which is the identity on objects, and which sends an isometry from V_1 to V_2 to the restriction of its nth symmetric power to the subrepresentation $S_{[n]}V_1$. Proposition 3.1 gives rise to a functor $\widetilde{H} : S_{[n]}\mathcal{V} \to \mathcal{V}$. If n is odd, the functors $S_{[n]}$ and $\widetilde{H}$ are inverses of each other. If n is even, then $S_{[n]}$ is not faithful, since $S_{[n]}(\phi) = S_{[n]}(-\phi)$. Proposition 3.1 asserts that if n is even, then the functor $F : \mathcal{V} \to S_{[n]}\mathcal{V}$, which is the identity on objects, and which sends $f \in \mathrm{Hom}_{\mathcal{V}}((V_1,\epsilon_1),(V_2,\epsilon_2))$ to $\det(f)S_{[n]}(f)$,
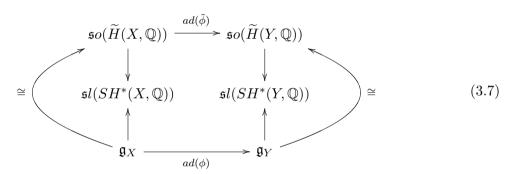
is an equivalence and defines $\widetilde{H}$ as its inverse. We have $S_{[n]}(-f) = -S_{[n]}(f)$, if n is odd, and $F(-f) = -F(f)$, if n is even, and so $\widetilde{H}(-\phi) = -\widetilde{H}(\phi)$. Let $f_i \in O(V_i)$, $i = 1, 2$. We have

$$\widetilde{H}(S_{[n]}(f_2) \circ \phi \circ S_{[n]}(f_1)) = \begin{cases} f_2 \circ \widetilde{H}(\phi) \circ f_1 & \text{if } n \text{ is odd,} \\ \det(f_1)\det(f_2)f_2 \circ \widetilde{H}(\phi) \circ f_1 & \text{if } n \text{ is even.} \end{cases} \tag{3.6}$$

DEFINITION 3.3. The functor $\widetilde{H} : \mathcal{G} \to \widetilde{\mathcal{G}}$, given in (1.4), sends a morphism ϕ in $\mathrm{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$ to $\widetilde{H}(\Psi_Y \circ \phi_| \circ \Psi_X^{-1}) : \widetilde{H}(X, \mathbb{Q}) \to \widetilde{H}(Y, \mathbb{Q})$, where the latter functor $\widetilde{H}$ is from Remark 3.2 (with $V_1 = \widetilde{H}(X, \mathbb{Q})$ with orientation ϵ_X and $V_2 = \widetilde{H}(Y, \mathbb{Q})$ with orientation ϵ_Y) and $\phi_|$ is the restriction of ϕ to $SH^*(X, \mathbb{Q})$.

LEMMA 3.4. *Let X and Y be $2n$-dimensional IHSMs. Let ϕ be a morphism in $\mathrm{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$. Set $\tilde{\phi} := \widetilde{H}(\phi) : \widetilde{H}(X, \mathbb{Q}) \to \widetilde{H}(Y, \mathbb{Q})$. The equality $\tilde{\phi}\tilde{h}_X = (-1)^k \tilde{h}_Y \tilde{\phi}$, for some $k \in \{0, 1\}$, is equivalent to $\phi h_X = (-1)^k h_Y \phi$ for the same k.*

Proof. The proof is modeled after that of [Tae23, Theorem 5.3]. The isomorphism $\phi : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ conjugates the LLV Lie algebra $\mathfrak{g}_X$ to $\mathfrak{g}_Y$, by [Tae23, Theorem A], and it conjugates $SH^*(X, \mathbb{Q})$ to $SH^*(Y, \mathbb{Q})$, by [Tae23, Theorem B]. Hence, $\phi h_X \phi^{-1}$ belongs to $\mathfrak{g}_Y$. Theorems 4.10 and 4.11 in [Tae23] yield the commutative diagram

$$\begin{array}{ccc} \mathfrak{so}(\widetilde{H}(X, \mathbb{Q})) & \xrightarrow{ad(\tilde{\phi})} & \mathfrak{so}(\widetilde{H}(Y, \mathbb{Q})) \\ \downarrow & & \downarrow \\ \mathfrak{sl}(SH^*(X, \mathbb{Q})) & & \mathfrak{sl}(SH^*(Y, \mathbb{Q})) \\ \uparrow & & \uparrow \\ \mathfrak{g}_X & \xrightarrow{ad(\phi)} & \mathfrak{g}_Y \end{array} \tag{3.7}$$

where the left bottom vertical arrow is the restriction to the subrepresentation $SH^*(X, \mathbb{Q})$ of $H^*(X, \mathbb{Q})$ and the right bottom vertical arrow is the analogue. The top left vertical arrow corresponds to $f \mapsto \Psi_X^{-1} \circ (S_{[n]}f) \circ \Psi_X$ and the right is the analogue. All four vertical arrows are injective and each pair in the same column have the same image yielding the curved arrows which are isomorphisms (inverses of $\dot{\rho}$ in (3.2)). The grading operator $h_X \in \mathfrak{g}_X$ restricts to the same element of $\mathfrak{sl}(SH^*(X, \mathbb{Q}))$ as the image of the grading operator $\tilde{h}_X \in \mathfrak{so}(\widetilde{H}(X, \mathbb{Q}))$, since Ψ_X is a graded isomorphism. The same holds for h_Y and $\tilde{h}_Y$. Hence, the equality $\tilde{\phi}\tilde{h}_X\tilde{\phi}^{-1} = (-1)^k \tilde{h}_Y$ is equivalent to the equality of the restrictions of $\phi h_X \phi^{-1}$ and $(-1)^k h_Y$ to elements of $\mathfrak{sl}(SH^*(Y, \mathbb{Q}))$, by the commutativity of the diagram and the injectivity of the top right vertical arrow. The latter is equivalent to the equality $\phi h_X \phi^{-1} = (-1)^k h_Y$, since the restriction homomorphism $\mathfrak{g}_Y \to \mathfrak{sl}(SH^*(Y, \mathbb{Q}))$ is injective. $\square$

The relation between h_X and $\tilde{h}_X$ explains the following result of Verbitsky, Looijenga, and Lunts.

LEMMA 3.5 [LL97, Proposition 4.5(ii)]. *The commutator of h in $\mathfrak{g}_X$ decomposes as a direct sum $\bar{\mathfrak{g}}_X \oplus \mathbb{Q}h_X$, where $\bar{\mathfrak{g}}_X$ is the derived Lie subalgebra of the commutator, and $\bar{\mathfrak{g}}_X$ is isomorphic to $\mathfrak{so}(H^2(X, \mathbb{Q}))$.*

The *Hodge operator*

$$h'_X \in \mathrm{End}(H^*(X, \mathbb{C})) \tag{3.8}$$

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acts on $H^{p,q}(X)$ by multiplication by $q - p$. Verbitsky proved that it is an element of the subalgebra $\bar{\mathfrak{g}}_{X,\mathbb{C}}$ of $\mathfrak{g}_{X,\mathbb{C}}$ (see [Ver99a, Theorem 8.1]; see also [Tae23, Corollary 2.12]).

LEMMA 3.6. *Assume that* $\phi \in \mathrm{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$ *is degree preserving and let* ϕ_2 *be the restriction of* ϕ *to* $H^2(X, \mathbb{Q})$. *Let* $t \in \mathbb{Q}^{\times}$ *be such that* $\widetilde{H}(\phi)(\alpha) = t^{-1}\alpha$. *Then the restriction* $\widetilde{H}(\phi)_0$ *of* $\widetilde{H}(\phi)$ *to* $H^2(X, \mathbb{Q})$ *satisfies*

$$\widetilde{H}(\phi)_0 = \begin{cases} t^{n-1}\phi_2 & \text{if } n \text{ is odd,} \\ \det(\widetilde{H}(\phi))t^{n-1}\phi_2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Assume that $\phi \in \mathrm{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$ is degree preserving. Then so is $\widetilde{H}(\phi)$, by Lemma 3.4. Let λ be a class in $H^2(X, \mathbb{Q})$. We have

$$(S_{[n]}\widetilde{H}(\phi))(\Psi(\lambda)) = (S_{[n]}\widetilde{H}(\phi))(e_\lambda(\alpha^n/n!)) = (S_{[n]}\widetilde{H}(\phi))(\alpha^{n-1}\lambda/(n-1)!)$$

$$= \frac{t^{1-n}}{(n-1)!}\alpha^{n-1}\widetilde{H}(\phi)(\lambda),$$

$$(S_{[n]}\widetilde{H}(\phi))(\Psi(\lambda)) = a\Psi(\phi(\lambda)) = ae_{\phi(\lambda)}(\alpha^n/n!) = a\alpha^{n-1}\phi(\lambda)/(n-1)!,$$

where $a = 1$, if n is odd, and $a = \det(\widetilde{H}(\phi))$, if n is even (Proposition 3.1). The statement follows from the equality of the right-hand sides in the two equations above. $\square$

LEMMA 3.7. *Let* $\phi \in \mathrm{Hom}_{\mathcal{G}}((X, \epsilon_X), (Y, \epsilon_Y))$ *be a degree-reversing morphism, where* X *and* Y *are* $2n$-*dimensional. Then the value* $\widetilde{H}_0(\phi) : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ *at* ϕ *of the functor* $\widetilde{H}_0$ *given in* (1.5) *is a scalar multiple of the composition*

$$H^2(X, \mathbb{Q}) \xrightarrow{\cup c_2(X)^{n-1}} H^{4n-2}(X, \mathbb{Q}) \xrightarrow{\phi} H^2(Y, \mathbb{Q}). \tag{3.9}$$

Proof. Note first that $H^2(X, \mathbb{Q})$ and $H^{4n-2}(X, \mathbb{Q})$ are subspaces of $SH^*(X, \mathbb{Q})$. Let $\phi_|$ be the restriction of ϕ to $SH^*(X, \mathbb{Q})$. Given an isometry $f \in O(\widetilde{H}(X, \mathbb{Q}))$, let

$$\eta_f : SH^*(X, \mathbb{Q}) \to SH^*(X, \mathbb{Q})$$

be given by $\eta_f := \Psi_X^{-1} \circ S_{[n]}(f) \circ \Psi_X$. Then

$$\widetilde{H}(\phi_| \circ \eta_f) = \begin{cases} \widetilde{H}(\phi) \circ f & \text{if } n \text{ is odd,} \\ \det(f)\widetilde{H}(\phi) \circ f & \text{if } n \text{ is even,} \end{cases}$$

by (3.6). The left-hand side is defined to be $\widetilde{H}(\Psi_X \circ (\phi_| \circ \eta_f) \circ \Psi_X^{-1})$ where $\widetilde{H}$ is the functor in Proposition 3.1.

Let $\tau \in O(\widetilde{H}(X, \mathbb{Q}))$ be the element which interchanges α and β and multiplies $H^2(X, \mathbb{Q})$ by -1. Then $\tau \tilde{h}_X = -\tilde{h}_X \tau$ and so η_τ is degree reversing. Hence, $\phi_| \circ \eta_\tau$ is degree preserving and so the restriction $\widetilde{H}(\phi_| \circ \eta_\tau)_0$ of $\widetilde{H}(\phi_| \circ \eta_\tau)$ to $H^2(X, \mathbb{Q})$ is a scalar multiple of $(\phi_| \circ \eta_\tau)_2$, by Lemma 3.6. Furthermore, $\widetilde{H}(\phi_| \circ \eta_\tau)_0 = \widetilde{H}(\phi)_0 \circ \widetilde{H}(\eta_\tau)_0$ and $\widetilde{H}(\eta_\tau)_0 = -id$, so $\widetilde{H}(\phi_| \circ \eta_\tau)_0 = -\widetilde{H}(\phi)_0 = \pm\widetilde{H}_0(\phi)$. We conclude that $\widetilde{H}_0(\phi)$ is a scalar multiple of $(\phi_| \circ \eta_\tau)_2$.

It remains to prove that $\eta_\tau : H^2(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})$ is a scalar multiple of cup product with $c_2(X)^{n-1}$. Let G be the subgroup of $SO(\widetilde{H}(X, \mathbb{Q}))$ leaving each of α and β invariant. Then G commutes with τ and its η-action on $SH^*(X, \mathbb{Q})$ is via degree-preserving isometries with respect to which η_τ is G-equivariant. Now, $H^2(X, \mathbb{Q})$ and $H^{4n-2}(X, \mathbb{Q})$ are dual irreducible G-representations and $\cup c_2(X)^{n-1} : H^2(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})$ is G-equivariant as well and non-vanishing, by [LL97, Theorem 4.7]. Hence, a scalar multiple of the latter is equal to the restriction of η_τ to $H^2(X, \mathbb{Q})$. $\square$

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Let X and Y be projective IHSMs and $\mathcal{P} \in D^b(X \times Y)$ the Fourier–Mukai kernel of an equivalence $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$. Assume that the rank r of $\mathcal{P}$ does not vanish. Let x be a point of X, denote by $\mathcal{P}_x \in D^b(Y)$ the restriction of $\mathcal{P}$ to $\{x\} \times Y$, and set $\lambda := c_1(\mathcal{P}_x) \in H^2(Y, \mathbb{Z})$. Note that the object $\mathcal{P}_x$ of $D^b(Y)$ is the image of the skyscraper sheaf at x via $\Phi_{\mathcal{P}}$. Set $\widetilde{H}(\Phi_{\mathcal{P}}) := \widetilde{H}([ch(\mathcal{P})\sqrt{td_{X \times Y}}]_*)$. The following was proved independently by Beckmann [Bec23, Lemma 4.13(iv)] and the author [Mar21a, Theorem 6.13(3)].

PROPOSITION 3.8. *The isometry $\widetilde{H}(\Phi_{\mathcal{P}}) : \widetilde{H}(X, \mathbb{Q}) \to \widetilde{H}(Y, \mathbb{Q})$ maps β to the isotropic line spanned by $r\alpha + \lambda + s\beta$, where $s = (\lambda, \lambda)/2r$ so that the line is isotropic.*

4. The degree-reversing Hodge isometry of a Fourier–Mukai kernel of non-zero rank

Consider an object E of non-zero rank in the derived category of a product $X \times Y$ of $2n$-dimensional deformation equivalent IHSMs X and Y. Assume that E is the Fourier–Mukai kernel of an equivalence of derived categories. This means that the functor $\Phi_E : D^b(X) \to D^b(Y)$, given by $\Phi_E(\bullet) = R\pi_{Y,*}(L\pi_X^*(\bullet) \otimes E)$, is an equivalence of triangulated categories.

LEMMA 4.1.

(1) *The class $\kappa(E)\sqrt{td_{X \times Y}}$, with $\kappa(E)$ as in Definition 1.3, induces a degree-reversing Hodge isometry*

$$\phi : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}).$$

(2) *The Hodge isometry $\widetilde{H}(\phi) : \widetilde{H}(X, \mathbb{Q}) \to \widetilde{H}(Y, \mathbb{Q})$, associated to ϕ and any choice of orientations of $H^2(X, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$, restricts to a Hodge isometry*

$$\widetilde{H}(\phi)_0 : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q}). \tag{4.1}$$

(3) *The Lie algebras isomorphism $Ad_\phi : \mathfrak{g}_X \to \mathfrak{g}_Y$ is graded and restricts to*

$$Ad_{\widetilde{H}_0(\phi)} : \bar{\mathfrak{g}}_X \to \bar{\mathfrak{g}}_Y.$$

Proof. Let r be the rank of E and let $\lambda_X \in H^2(X, \mathbb{Z})$ and $\lambda_Y \in H^2(Y, \mathbb{Z})$ satisfy $c_1(E) = \pi_X^*(\lambda_X) + \pi_Y^*(\lambda_Y)$. Given $\lambda \in H^2(X, \mathbb{Q})$, let e_λ be the endomorphism of $\widetilde{H}(X, \mathbb{Q})$ given in (3.4). Set $B_\lambda := \exp(e_\lambda)$.

The isometry $\widetilde{H}(\Phi_E)$ maps $\text{span}\{\beta_X\}$ to $\text{span}\{r\alpha_Y + \lambda_Y + ((\lambda_Y, \lambda_Y)/2r)\beta_Y\}$, by Proposition 3.8. We have $B_{-\lambda_X/r}(\beta_X) = \beta_X$ and

$$B_{-\lambda_Y/r}\left(r\alpha_Y + \lambda_Y + \frac{(\lambda_Y, \lambda_Y)}{2r}\beta_Y\right) = r\alpha_Y.$$

The equality $\widetilde{H}(\phi) = B_{-\lambda_Y/r}\widetilde{H}(\Phi_E)B_{-\lambda_X/r}$ implies that $\widetilde{H}(\phi)$ maps $\text{span}\{\beta_X\}$ to $\text{span}\{\alpha_Y\}$. The adjoint $\Phi_E^\dagger$ has Fourier–Mukai kernel $E^*[2n]$. The argument above applied to the adjoint shows that $\widetilde{H}(\phi)^{-1}$ maps $\text{span}\{\beta_Y\}$ to $\text{span}\{\alpha_X\}$. Hence, $\widetilde{H}(\phi)$ maps $\text{span}\{\alpha_X\}$ to $\text{span}\{\beta_Y\}$. It follows that $\widetilde{H}(\phi)$ maps the subspace $H^2(X, \mathbb{Q})$ orthogonal to $\text{span}\{\alpha_X, \beta_X\}$ to the subspace $H^2(Y, \mathbb{Q})$ orthogonal to $\text{span}\{\alpha_Y, \beta_Y\}$. This proves part (2) of the Lemma. We denote by $\widetilde{H}(\phi)_0$ the restriction of $\widetilde{H}(\phi)$ to $H^2(X, \mathbb{Q})$.

The isometry $\widetilde{H}(\phi)$ conjugates the grading operator $\tilde{h}_X$ of $\widetilde{H}(X, \mathbb{Q})$, given in (3.5), to $-\tilde{h}_Y$, where $\tilde{h}_Y$ is the grading operator of $\widetilde{H}(Y, \mathbb{Q})$. Lemma 3.4 implies that ϕ is degree reversing and part (1) is proven.

(3) The subalgebra $\bar{\mathfrak{g}}_X$ is the semi-simple part of the degree-0 subalgebra of the LLV algebra $\mathfrak{g}_X$, by [LL97, Proposition 4.5(ii)], and $Ad_\phi : \mathfrak{g}_X \to \mathfrak{g}_Y$ is a graded Lie algebra isomorphism, by

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part (1), and so Ad_ϕ maps $\bar{\mathfrak{g}}_X$ to the semi-simple part $\bar{\mathfrak{g}}_Y$ of the degree-0 subalgebra of $\mathfrak{g}_Y$. Furthermore, the restriction of Ad_ϕ to $\bar{\mathfrak{g}}_Y$ is equal to $Ad_{\widetilde{H}(\phi)_0}$, by part (2) and the commutativity of the outer square in diagram (3.7). Finally, $Ad_{\widetilde{H}(\phi)_0} = Ad_{\widetilde{H}_0(\phi)}$, as $\widetilde{H}_0(\phi) = \pm\widetilde{H}(\phi)_0$, by definition (1.5). $\qquad\square$

Set $\gamma := \kappa(E)\sqrt{td_{X\times Y}}$ and let γ_i be the graded summand in $H^{2i}(X \times Y, \mathbb{Q})$.

COROLLARY 4.2. *If the odd cohomology of X vanishes, then $\gamma_i = 0$, for odd i. In that case $\kappa(E) = \kappa(E^\vee)$.*

Proof. The class γ_i induces the homomorphism $H^{k-2n}(X,\mathbb{Q})[2n] \to H^{k+2i-6n}(Y,\mathbb{Q})[2n]$. As ϕ is degree reversing, this homomorphism vanishes unless $k + i - 4n = 0$. Hence, it vanishes unless the parities of i and k are the same, and it vanishes if k is odd, by assumption. $\qquad\square$

Proof of Lemma 1.6. Set $\tilde{\phi} := \widetilde{H}(\phi)$. The morphism ϕ conjugates the LLV Lie algebra of X to that of Y, by [Tae23, Theorem A]. The LLV Lie algebras act faithfully on the rational LLV lattices. Hence, it suffices to verify the equality $e_\lambda^\vee = (2/t(\lambda,\lambda))\tilde{\phi}^{-1}e_{\tilde{\phi}(\lambda)}\tilde{\phi}$ for the endomorphisms of $\widetilde{H}(X,\mathbb{Q})$. We have

$$(\tilde{\phi}^{-1}e_{\tilde{\phi}(\lambda)}\tilde{\phi})(\alpha) = \tilde{\phi}^{-1}(e_{\tilde{\phi}(\lambda)}(t^{-1}\beta)) = 0,$$
$$(\tilde{\phi}^{-1}e_{\tilde{\phi}(\lambda)}\tilde{\phi})(\beta) = \tilde{\phi}^{-1}(e_{\tilde{\phi}(\lambda)}(t\alpha)) = t\tilde{\phi}^{-1}(\tilde{\phi}(\lambda)) = t\lambda,$$
$$(\tilde{\phi}^{-1}e_{\tilde{\phi}(\lambda)}\tilde{\phi})(\lambda') = \tilde{\phi}^{-1}((\tilde{\phi}(\lambda),\tilde{\phi}(\lambda'))\beta) = t(\lambda,\lambda')\alpha.$$

Set $\psi := \tilde{\phi}^{-1}e_{\tilde{\phi}(\lambda)}\tilde{\phi}$. We get

$$[e_\lambda,\psi](\alpha) = -\psi(e_\lambda(\alpha)) = -t(\lambda,\lambda)\alpha,$$
$$[e_\lambda,\psi](\beta) = e_\lambda(t\lambda) = t(\lambda,\lambda)\beta,$$
$$[e_\lambda,\psi](\lambda') = t(\lambda,\lambda')\lambda - t(\lambda,\lambda')\lambda = 0.$$

Hence, $[e_\lambda,\psi] = (t(\lambda,\lambda)/2)h$. Clearly, $[h,\psi] = -2\psi$, since ψ has degree -2. Thus, $e_\lambda^\vee = (2/t(\lambda,\lambda))\psi$. $\qquad\square$

5. Hyperholomorphic vector bundles deforming a Fourier–Mukai kernel

Let X and Y be deformation equivalent IHSMs and E a locally free sheaf over $X \times Y$ which is the Fourier–Mukai kernel of an equivalence $\Phi_E : D^b(X) \to D^b(Y)$ of derived categories. Let $\phi : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ be the morphism in $\mathrm{Hom}_{\mathcal{G}_{an}}(X,Y)$ induced by $\kappa(E)\sqrt{td_{X\times Y}}$. Let $\widetilde{H}_0(\phi) : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ be the value on ϕ of the functor $\widetilde{H}_0$, given in (1.5). Let π_X and π_Y be the projections from $X \times Y$. We prove in this section the following theorem.

THEOREM 5.1 (Proposition 5.22 and Lemma 5.7). *Assume that there exists an open subcone of the Kähler cone of X, such that $\widetilde{H}_0(\phi)$ maps each Kähler class ω_1 in this subcone to a Kähler class $\omega_2 := \widetilde{H}_0(\phi)(\omega_1)$ on Y with respect to which E is $[\pi_X^*(\omega_1) + \pi_Y^*(\omega_2)]$-slope stable. Then for every IHSM X' deformation equivalent to X there exists an IHSM Y' and a locally free (possibly twisted) coherent sheaf E' over $X' \times Y'$, such that the pair $(X' \times Y', E')$ is deformation equivalent to $(X \times Y, E)$.*

Note that the assumptions of the theorem are symmetric with respect to X and Y, upon replacing E by E^* and ϕ by ϕ^{-1}, and so the conclusions are symmetric as well. Proposition 5.22 is a slightly stronger version of the theorem, allowing the assumptions in the statement of the theorem to hold instead for a deformation $(X_0 \times Y_0, E_0)$ of the Fourier–Mukai kernel $(X \times Y, E)$.

Theorem 5.1 may be viewed as a global analogue, in the special case of classical and gerby deformations of locally free Fourier–Mukai kernels between IHSMs, of the infinitesimal deformability result of Toda for generalized deformations of Fourier–Mukai kernels [Tod09, Theorem 1.1].

The proof of Theorem 5.1 uses Verbitsky's results on hyperholomorphic vector bundles over hyper-Kähler manifolds and their deformability along twistor paths in moduli [Ver99b]. In §5.1 we recall the notion of a twistor line in the moduli space of marked IHSMs. In §5.2 we define the moduli space $\mathfrak{M}_\psi$ of rational Hodge isometries. Given a lattice Λ and a rational isometry $\psi \in O(\Lambda_\mathbb{Q})$, the moduli space $\mathfrak{M}_\psi$ parametrizes isomorphism classes of quadruples $(X_1, \eta_1, X_2, \eta_2)$ of deformation equivalent marked pairs (X_i, η_i), where $\eta_i : H^2(X_i, \mathbb{Z}) \to \Lambda$ is an isometry, $i = 1, 2$, such that $\eta_2^{-1}\psi\eta_1 : H^2(X_1, \mathbb{Q}) \to H^2(X_2, \mathbb{Q})$ is a Hodge isometry which maps some Kähler class to a Kähler class. In §5.3 we define diagonal twistor lines in $\mathfrak{M}_\psi$. In §5.4 we define the notion of generic twistor paths and prove that every two points in the same connected component of $\mathfrak{M}_\psi$ are connected by such a path. In §5.5 we review Verbitsky's result about hyperholomorphic vector bundles and their deformability along twistor paths in $\mathfrak{M}_\psi$. In §5.6 we associate to an equivalence Φ_E as in Theorem 5.1 a connected component $\mathfrak{M}_\psi^0$ of the moduli space of rational Hodge isometries. In §5.7 we prove Theorem 5.1.

5.1 Twistor lines

A *marking* of an IHSM X is an isometry $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ with a fixed lattice Λ. Two marked pairs (X_1, η_1) and (X_2, η_2) are isomorphic, if there exists an isomorphism $f : X_1 \to X_2$, such that $\eta_2 = \eta_1 \circ f^*$. The moduli space $\mathfrak{M}_\Lambda$ of isomorphism classes of marked IHSMs is a non-Hausdorff complex manifold of dimension rank$(\Lambda) - 2$ (see [Huy99]).

Set $\Omega_\Lambda := \{\ell \in \mathbb{P}(\Lambda_\mathbb{C}) \ : \ (\ell, \ell) = 0 \text{ and } (\lambda, \bar\lambda) > 0 \text{ for all non-zero } \lambda \in \ell\}$. The *period map*

$$P : \mathfrak{M}_\Lambda \to \Omega_\Lambda, \tag{5.1}$$

is given by $P(X, \eta) = \eta(H^{2,0}(X))$. The restriction of the period map to every connected component of $\mathfrak{M}_\Lambda$ is known to be surjective and generically injective, by Verbitsky's Torelli theorem [Huy12, Ver13]. Furthermore, if $\mathrm{Pic}(X)$ is trivial, or cyclic generated by a class of non-negative BBF degree, then (X, η) is the unique point in the fiber of P through the connected component of $\mathfrak{M}_\Lambda$ containing (X, η).

The Torelli theorem solves the case of Theorem 1.1 in which the Hodge isometry f is a parallel-transport operator.

THEOREM 5.2. *Let (X, η) and (X', η') be two marked pairs in the same connected component of $\mathfrak{M}_\Lambda$ satisfying $P(X, \eta) = P(X', \eta')$. There exist an analytic cycle Z in $X \times X'$, such that $[Z]_* : H^*(X', \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is a parallel-transport operator, which restricts to $H^2(X_i', \mathbb{Z})$ as $\eta^{-1} \circ \eta'$.*

Proof. The assumption that (X, η) and (X', η') belong to the same connected component of $\mathfrak{M}_\Lambda$ and have the same period implies that (X, η) and (X', η') are inseparable points of $\mathfrak{M}_\Lambda$, by Verbitsky's Torelli theorem [Huy12, Ver13]. Note also that $\eta' \circ \eta^{-1}$ is a parallel-transport operator, which is a Hodge isometry. The existence of the analytic cycle Z with the claimed properties thus follows from results of Huybrechts in [Huy03b] (see [Mar11, Theorem 3.2]). $\square$

LEMMA 5.3. *Let X and X' be IHSMs and let $f_i : H^*(X, \mathbb{Z}) \to H^*(X', \mathbb{Z})$ be a parallel-transport operator, $i = 1, 2$. Assume that f_1 and f_2 restrict to the same parallel-transport operator $f : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$. Then there exist automorphisms $g \in \mathrm{Aut}(X)$ and $g' \in \mathrm{Aut}(X')$, acting trivially on the second cohomologies, such that $f_2 = g'_* \circ f_1 \circ g_*$.*

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Proof. There exist families $\pi_i : \mathcal{X}_i \to B_i$ over connected analytic spaces B_i, points $b_i, b_i' \in B_i$, and continuous paths γ_i from b_i to b_i' in B_i, such that f_i is the parallel-transport operator along γ_i in the local system $R\pi_{i,*}\mathbb{Z}$, $i = 1, 2$. We may and do assume that B_i is simply connected, possibly after passing to the universal cover and lifting the path γ_i. Choose an isometric trivialization $\eta_i : R^2\pi_{i,*}\mathbb{Z} \to \underline{\Lambda}$, where $\underline{\Lambda}$ is the trivial local system over B_i with a fixed lattice Λ, with $\eta_{1,b_1} = \eta_{2,b_2} : H^2(X, \mathbb{Z}) \to \Lambda$. Set $\eta := \eta_{1,b_1}$. Set $\eta' := \eta \circ f^{-1}$. We get that $\eta_{i,b_i'} = \eta'$, for $i = 1, 2$.

Let $\mathfrak{M}_\Lambda^0$ be the connected component of the moduli space of isomorphism classes of marked IHSMs containing the isomorphism class $[(X, \eta)]$ of (X, η). Let $s_i : B_i \to \mathfrak{M}_\Lambda^0$ be the classifying morphism associated to η_i. Set $q := [(X, \eta)] = s_1(X, \eta_{1,b_1}) = s_2(X, \eta_{2,b_2})$ and $q' := [(X, \eta')] = s_1(X', \eta_{1,b_1'}) = s_2(X', \eta_{2,b_2'})$.

There exists over $\mathfrak{M}_\Lambda^0$ a universal family $\Pi : \mathcal{Y} \to \mathfrak{M}_\Lambda^0$ and isomorphisms $h_i : \mathcal{X}_i \to s_i^{-1}(\mathcal{Y})$ of the two families π_i and $s_i^{-1}(\Pi)$ over B_i, by [Mar21b, Theorem 1.1] and our assumption that B_i is simply connected, for $i = 1, 2$. Furthermore, the local system $R\Pi_*\mathbb{Z}$ over $\mathfrak{M}_\Lambda^0$ is trivial, by [Mar21b, Lemma 2.1]. Hence, the paths $s_i \circ \gamma_i$, $i = 1, 2$, from q to q' in $\mathfrak{M}_\Lambda^0$ induce the same parallel-transport operator from the fiber Y_q of $\mathcal{Y}$ over q to the fiber $Y_{q'}$ of $\mathcal{Y}$ over q'. Let $h_{i,b_i} : X \to Y_q$ and $h_{i,b_i'} : X' \to Y_{q'}$ be the isomorphism associated to h_i, $i = 1, 2$. We conclude that $h_{i,b_i'} \circ f_i \circ h_{i,b_i}^{-1}$ is the parallel-transport along $s_i \circ \gamma_i$, $i = 1, 2$, and so

$$h_{1,b_1'} \circ f_1 \circ h_{1,b_1}^{-1} = h_{2,b_2'} \circ f_2 \circ h_{2,b_2}^{-1}.$$

Then $g := h_{1,b_1}^{-1} \circ h_{2,b_2} \in \mathrm{Aut}(X)$ and $g' := h_{2,b_2'}^{-1} \circ h_{1,b_1'} \in \mathrm{Aut}(X')$ are automorphisms satisfying $f_2 = g_*' \circ f_1 \circ g_*$. $\qquad\square$

Given a positive-definite three-dimensional subspace W of $\Lambda_\mathbb{R}$, set $W_\mathbb{C} := W \otimes_\mathbb{R} \mathbb{C}$. We get the projective conic curve $\mathbb{P}(W_\mathbb{C}) \cap \Omega_\Lambda$, which is called a *twistor line*. Let (X, η) be a marked pair in $\mathfrak{M}_\Lambda$ and $\omega \in H^{1,1}(X, \mathbb{R})$ a Kähler class. The intersection of $W_\mathbb{C} := H^{2,0}(X) \oplus H^{0,2}(X) \oplus \mathbb{C}\omega$ with $H^2(X, \mathbb{R})$ is a positive-definite three-dimensional subspace W and the twistor line $Q_{(X,\eta,\omega)} := \mathbb{P}(\eta(W_\mathbb{C})) \cap \Omega_\Lambda$ passes through $P(X, \eta)$. We endow W with the orientation associated to the basis

$$\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \omega\}, \tag{5.2}$$

for a non-zero class $\sigma \in H^{2,0}(X)$. There exists a family

$$\pi : \mathcal{X} \to Q_{(X,\eta,\omega)} \tag{5.3}$$

of IHSMs with fiber X over the point $P(X, \eta)$, known as the *twistor family associated to ω*, with the following properties [Huy99, paragraph 1.17, p. 76]. The marking η extends to a trivialization of the local system $R^2\pi_*\mathbb{Z}$, as $Q_{(X,\eta,\omega)}$ is simply connected. Denote by η_t the resulting marking of the fiber X_t of π over $t \in Q_{(X,\eta,\omega)}$. Then

$$P(X_t, \eta_t) = t,$$

for each $t \in Q_{(X,\eta,\omega)}$ and the line in $H^2(X_t, \mathbb{R})$, which is contained in $\eta_t^{-1}(\eta(W))$ and orthogonal to $H^{2,0}(X_t)$, is spanned by a Kähler class ω_t. The twistor family depends only on the ray in the Kähler cone of X through ω. Denote by

$$R_t \subset \mathcal{K}_{X_t} \cap \eta_t^{-1}(\eta(W)) \tag{5.4}$$

the ray through ω_t in the Kähler cone $\mathcal{K}_{X_t}$ of X_t. The curve

$$\widetilde{Q}_{(X,\eta,\omega)} := \{(X_t, \eta_t) \; : \; t \in Q_{(X,\eta,\omega)}\}$$

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in $\mathfrak{M}_\Lambda^0$ is mapped isomorphically onto $Q_{(X,\eta,\omega)}$ via the period map. The curve $\widetilde{Q}_{(X,\eta,\omega)}$ is called the *twistor line through* (X,η) *associated to* ω. The twistor line $\widetilde{Q}_{(X,\eta,\omega)}$ is said to be *generic*, if $\mathrm{Pic}(X_t)$ is trivial, for some $t \in Q_{(X,\eta,\omega)}$. This is the case for ω in the complement of a countable union of hyperplanes in the Kähler cone.

5.2 Moduli spaces of rational Hodge isometries

Fix a deformation class of IHSMs with a second integral cohomology lattice isometric to a lattice Λ. Let $\psi \in O(\Lambda_\mathbb{Q})$ be a rational isometry. Let $\mathfrak{M}_\psi$ be the moduli space of isomorphism classes of quadruples (X_1,η_1,X_2,η_2), where $\eta_i : H^2(X_i,\mathbb{Z}) \to \Lambda$ is a marking, such that $\eta_2^{-1}\psi\eta_1 : H^2(X_1,\mathbb{Q}) \to H^2(X_2,\mathbb{Q})$ is a Hodge isometry which maps some Kähler class to a Kähler class. Two quadruples (X_1,η_1,X_2,η_2) and $(X_1',\eta_1',X_2',\eta_2')$ are isomorphic, if the marked pair (X_i,η_i) is isomorphic to (X_i',η_i'), for $i=1,2$. The moduli space $\mathfrak{M}_\psi$ has a natural structure of a non-Hausdorff complex manifold. The proof of the latter statement is identical to that of [Bus19, Proposition 4.9]. The isometry ψ induces an automorphism of Ω_Λ which we denote by ψ as well.

LEMMA 5.4. *If* (X_1,η_1,X_2,η_2) *belongs to* $\mathfrak{M}_\psi$, *then* $P(X_2,\eta_2) = \psi(P(X_1,\eta_1))$.

Proof. The composition $\eta_2^{-1}\psi\eta_1 : H^2(X_1,\mathbb{Q}) \to H^2(X_2,\mathbb{Q})$ is a Hodge isometry. Hence, $P(X_2,\eta_2) := \eta_2(H^{2,0}(X_2)) = [\eta_2 \circ (\eta_2^{-1}\psi\eta_1)](H^{2,0}(X_1)) = \psi(P(X_1,\eta_1))$. $\qquad\square$

Given a quadruple $q := (X_1,\eta_1,X_2,\eta_2)$ in $\mathfrak{M}_\psi$ set

$$\psi_q := \eta_2^{-1}\psi\eta_1 : H^2(X_1,\mathbb{Q}) \to H^2(X_2,\mathbb{Q}). \qquad (5.5)$$

Let ω_1 be a Kähler class on X_1, such that $\omega_2 := \psi_q(\omega_1)$ is a Kähler class on X_2. Note that $\psi : \Omega_\Lambda \to \Omega_\Lambda$ maps the twistor line $Q_{(X_1,\eta_1,\omega_1)}$ to the twistor line $Q_{(X_2,\eta_2,\omega_2)}$. Let $(X_{i,t},\eta_{i,t})$, $t \in Q_{(X_i,\eta_i,\omega_i)}$, be the point of $\widetilde{Q}_{(X_i,\eta_i,\omega_i)}$ over t, for $i=1,2$.

LEMMA 5.5. *The rays* R_t *and* $R_{\psi(t)}$ *in the Kähler cones of* $X_{1,t}$ *and* $X_{2,\psi(t)}$, *given in* (5.4), *satisfy the equality* $\psi(\eta_{1,t}(R_t)) = \eta_{2,\psi(t)}(R_{\psi(t)})$, *for each* $t \in Q_{(X_1,\eta_1,\omega_1)}$. *Consequently, the quadruple* $(X_{1,t},\eta_{1,t},X_{2,\psi(t)},\eta_{2,\psi(t)})$ *belongs to* $\mathfrak{M}_\psi$, *for all* $t \in Q_{(X_1,\eta_1,\omega_1)}$.

Proof. Let $\mathcal{C}_{X_i} := \{\lambda \in H^{1,1}(X_i,\mathbb{R}) \ : \ (\lambda,\lambda) > 0\}$ be the positive cone, $i=1,2$. Each $\mathcal{C}_{X_i}$ has two connected components and we denote by $\mathcal{C}_{X_i}^+$ the one which contains the Kähler cone. The isometry ψ_q maps $\mathcal{C}_{X_1}$ isomorphically onto $\mathcal{C}_{X_2}$ and $\psi_q(\mathcal{K}_{X_1}) \cap K_{X_2}$ is non-empty, hence $\psi_q(\mathcal{C}_{X_1}^+) = \mathcal{C}_{X_2}^+$. Equivalently, $\psi(\eta_{1,t_0}(\mathcal{C}_{X_{t_0}}^+)) = \eta_{2,\psi(t_0)}(\mathcal{C}_{X_{\psi(t_0)}}^+)$, where $t_0 = P(X_1,\eta_1)$. Hence, $\psi(\eta_{1,t}(\mathcal{C}_{X_t}^+)) = \eta_{2,\psi(t)}(\mathcal{C}_{X_{\psi(t)}}^+)$, for all $t \in Q_{(X_1,\eta_1,\omega_1)}$, by continuity and connectedness of $Q_{(X_1,\eta_1,\omega_1)}$. We know that ψ maps the line spanned by $\eta_{1,t}(R_t)$ to the line spanned by $\eta_{2,\psi(t)}(R_{\psi(t)})$, since $\eta_{2,t}^{-1}\psi\eta_{1,t}$ is a Hodge isometry which maps the positive-definite three-dimensional subspace W_1 associated to (X_1,η_1,ω_1) to the positive-definite three-dimensional subspace W_2 associated to (X_2,η_2,ω_2) and $\eta_{1,t}(R_t)$ spans the line in W_1 orthogonal to $\eta_{1,t}(H^{2,0}(X_t))$ and $\eta_{2,t}(R_{\psi(t)})$ spans the line in W_2 orthogonal to $\eta_{2,t}(H^{2,0}(X_{\psi(t)}))$. Now, R_t is the intersection of the line spanned by it with $\mathcal{C}_{X_t}^+$ and the similar statement holds for $R_{\psi(t)}$. Hence, $\psi(\eta_{1,t}(R_t)) = \eta_{2,\psi(t)}(R_{\psi(t)})$. $\qquad\square$

Lemma 5.5 enables us to define the *twistor line in* $\mathfrak{M}_\psi^0$ *through* (X_1,η_1,X_2,η_2) *determined by* $\omega_1 \in \mathcal{K}_{X_1} \cap \psi_q^{-1}(\mathcal{K}_{X_2})$ to be the curve

$$\widetilde{Q}_{(X_1,\eta_1,X_2,\eta_2),\omega_1} := \{(X_{1,t},\eta_{1,t},X_{2,t},\eta_{2,\psi(t)}) \ : \ t \in Q_{(X_1,\eta_1,\omega_1)}\}. \qquad (5.6)$$

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5.3 Diagonal twistor lines

We explain next that the twistor line (5.6) in the above definition is associated to a hyper-Kähler structure on the differential manifold underlying the product $X_1 \times X_2$. This fact is used in the proof of Proposition 5.15. A Kähler class on an IHSM X determines a Ricci-flat hermitian metric g on the manifold M underlying X, by the Calabi–Yau theorem [Bea83]. The metric g admits three complex structures I, J, and K, satisfying the quaternionic relation $IJ = K$, such that X is the complex manifold (M, I). For each point t in the unit sphere $S^2 := \{(a,b,c) : a^2 + b^2 + c^2 = 1\}$, we get the unit purely imaginary quaternion $I_t := aI + bJ + cK$, which is a complex structure on M. The 2-form $\omega_{I_t} := g(I_t(\bullet), (\bullet))$ is a Kähler form. The original class ω is the cohomology class of the 2-form $\omega_I := g(I(\bullet), (\bullet))$. The classes of $\{\omega_I, \omega_J, \omega_K\}$ form a basis for the positive-definite three-dimensional subspace $W \subset H^2(X, \mathbb{R})$ whose complexification is $H^{2,0}(X) \oplus H^{0,2}(X) \oplus \mathbb{C}\omega$. The latter basis is orthogonal with respect to the BBF pairing, and the BBF degrees of the three classes are the same. Conversely, a choice of an orthogonal basis of W with the same orientation consisting of ω and two classes of the same BBF degree as ω determines complex structures J and K and an action of the unit purely imaginary quaternions on the tangent bundle of M via parallel complex structures. The above 2-sphere of complex structures is thus naturally identified with the unit sphere in W.

The standard complex structure on the unit sphere combines with the above construction to define a complex structure on $S^2 \times M$, such that the first projection $\pi : S^2 \times M \to S^2$ is holomorphic and the fiber over t is (M, I_t), giving rise to the twistor family (5.3) (see [HKLR87]). The base of the twistor family is thus naturally identified with the unit sphere in the positive-definite three-dimensional subspace W of $H^2(X, \mathbb{R})$ via the isomorphism sending a class ω' in the unit sphere in W to the unique[8] isotropic line in $\mathbb{P}(W_{\mathbb{C}})$, which is orthogonal to ω' and induces on W the same orientation induced by $H^{2,0}(X)$ in (5.2).

Let M_i be the differential manifold underlying X_i, $i = 1, 2$. Let g_1 be the Ricci-flat hermitian metric on M_1 associated with the Kähler class ω_1 in (5.6) and let g_2 be the metric on M_2 associated with the Kähler class $\omega_2 := \psi_q(\omega_1)$. Let W_i be the positive-definite three-dimensional subspace of $H^2(X_i, \mathbb{R})$, whose complexification is $H^{2,0}(X_i) \oplus H^{0,2}(X_i) \oplus \mathbb{C}w_i$. We have the equality $W_2 = \psi_q(W_1)$, and ψ_q maps the unit sphere in W_1 to that in W_2, as ψ_q is a Hodge isometry. Hence, ψ restricts to an isometry from $\eta_1(W_1)$ onto $\eta_2(W_2)$, which maps the twistor line $Q_{(X_1, \eta_1, \omega_1)}$ isomorphically onto $Q_{(X_2, \eta_2, \omega_2)}$. Denote by

$$Q_{(X_1, \eta_1, X_2, \eta_2), \omega_1} \subset Q_{(X_1, \eta_1, \omega_1)} \times Q_{(X_2, \eta_2, \omega_2)}$$

the graph of the restriction of ψ. Let $\pi_i : \mathcal{X}_i \to Q_{(X_i, \eta_i, \omega_i)}$ be the twistor family through (X_i, η_i) associated to ω_i, $i = 1, 2$. The twistor line (5.6) is the lift of $Q_{(X_1, \eta_1, X_2, \eta_2), \omega_1}$ to a curve in $\mathfrak{M}_\psi^0$ obtained via the extensions of η_i to a trivialization of $R\pi_{i,*}^2 \mathbb{Z}$, $i = 1, 2$. Choose an orthogonal basis of W_1, extending ω_1 and consisting of classes of the same BBF degree, and let I_1, J_1, and K_1 be the resulting basis of the purely imaginary quaternions acting via complex structures on the tangent bundle of M_1. Define the complex structures I_2, J_2, K_2 on M_2 associated to the orthogonal basis of W_2, which is the image via ψ_q of the chosen basis of W_1. Given a point t in the unit sphere in W_i denote by $I_{i,t}$ the corresponding complex structure on M_i such that $g_i(I_{i,t}(\bullet), \bullet)$ is the Kähler form with class t, $i = 1, 2$. We get the 2-sphere of complex structures $I_t := (I_{1,t}, I_{2,\psi(t)})$ on $M_1 \times M_2$ compatible with the Ricci-flat hermitian metric (g_1, g_2) on $M_1 \times M_2$. Then the fiber

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[8] The intersection of the conic Q in $\mathbb{P}(W_{\mathbb{C}})$ of isotropic lines in $W_{\mathbb{C}}$ with the line $\mathbb{P}((\omega')_{\mathbb{C}}^\perp \cap W_{\mathbb{C}})$ orthogonal to ω' consists of two complex conjugate points inducing on W distinct orientations.

product

$$\mathcal{X}_1 \times_{Q_{(X_1,\eta_1,X_2,\eta_2),\omega_1}} \mathcal{X}_2 \to Q_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$$

of the two twistor families is the twistor family associated to the above sphere of complex structures on $M_1 \times M_2$.

5.4 Diagonal twistor paths

Let

$$\Pi_i : \mathfrak{M}_\psi \to \mathfrak{M}_\Lambda$$

be the morphism sending the isomorphism class of (X_1,η_1,X_2,η_2) to that of (X_i,η_i). Note that Π_i maps $\widetilde{Q}_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$ isomorphically onto $\widetilde{Q}_{(X_i,\eta_i,\omega_i)}$, $i = 1, 2$. Fix a connected component $\mathfrak{M}_\Lambda^0$ of $\mathfrak{M}_\Lambda$.

DEFINITION 5.6. (1) A *twistor path in $\mathfrak{M}_\Lambda^0$* from a marked pair (X,η) to a marked pair (X',η') consists of the data of a sequence of marked pairs (X_i,η_i), $0 \le i \le k+1$, with $(X_0,\eta_0) = (X,\eta)$ and $(X_{k+1},\eta_{k+1}) = (X',\eta')$, together with a twistor line $\widetilde{Q}_{(X_i,\eta_i,\omega_i)}$ through (X_i,η_i) passing through (X_{i+1},η_{i+1}) as well, for $0 \le i \le k$. The twistor path is said to be *generic* if $\mathrm{Pic}(X_i)$ is trivial, for $1 \le i \le k$.

(2) A *twistor path in $\mathfrak{M}_\psi^0$* from a quadruple (X_1,η_1,X_2,η_2) to a quadruple (Y_1,e_1,Y_2,e_2) consists of the data of a sequence $(X_{1,i},\eta_{1,i},X_{2,i},\eta_{2,i})$, $0 \le i \le k+1$, with $(X_{1,0},\eta_{1,0},X_{2,0},\eta_{2,0}) = (X_1,\eta_1,X_2,\eta_2)$ and $(X_{1,k+1},\eta_{1,k+1},X_{2,k+1},\eta_{2,k+1}) = (Y_1,e_1,Y_2,e_2)$, together with a twistor line $\widetilde{Q}_{(X_{1,i},\eta_{1,i},X_{2,i},\eta_{2,i}),\omega_i}$ through $(X_{1,i},\eta_{1,i},X_{2,i},\eta_{2,i})$ passing through $(X_{1,i+1},\eta_{1,i+1},X_{2,i+1},\eta_{2,i+1})$ as well, for $0 \le i \le k$. The twistor path is said to be *generic* if $\mathrm{Pic}(X_{1,i})$ is trivial, for $1 \le i \le k$.

LEMMA 5.7. *The morphism $\Pi_i : \mathfrak{M}_\psi^0 \to \mathfrak{M}_\Lambda^0$ is surjective, for $i = 1, 2$.*

Proof. Let $q := (X_1,\eta_1,X_2,\eta_2)$ be a quadruple in $\mathfrak{M}_\psi^0$ and let (Y_1',η_1') be a marked pair in $\mathfrak{M}_\Lambda^0$. Choose a generic Kähler class ω_1 in $\mathcal{K}_{X_1} \cap \psi_q^{-1}(\mathcal{K}_{X_2})$ so that the twistor line $\widetilde{Q}_{(X_1,\eta_1,\omega_1)}$ is generic and let (Y_1,e_1) be a marked pair in $\widetilde{Q}_{(X_1,\eta_1,\omega_1)}$ with a trivial $\mathrm{Pic}(Y_1)$. A pair (Y_2,e_2) in the fiber of the period map through $\mathfrak{M}_\Lambda^0$ over $\psi(P(Y_1,e_1))$ must be the unique marked pair in this fiber, since $\mathrm{Pic}(Y_2)$ is trivial as well. The equalities $\mathcal{K}_{Y_i} = \mathcal{C}_{Y_i}^+$, $i = 1, 2$, imply that $\mathcal{K}_{Y_1} = (e_1^{-1}\psi e_2)(\mathcal{K}_{Y_2})$ and so the quadruple (Y_1,e_1,Y_2,e_2) belongs to the diagonal twistor line $Q_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$ and, hence, to the connected component $\mathfrak{M}_\psi^0$ of $\mathfrak{M}_\psi$. Furthermore, every twistor path $Q_{(Y_1,e_1,\omega_1')}$ through (Y_1,e_1) lifts to a twistor path $Q_{(Y_1,e_1,Y_2,e_2),\omega_1'}$ through (Y_1,e_1,Y_2,e_2), by Lemma 5.5. There exists a generic twistor path from (Y_1,e_1) to (Y_1',η_1'), by [Ver96, Theorems 3.2 and 5.2.e]. Repeating the above argument we get that this twistor path lifts to a twistor path from (Y_1,e_1,Y_2,e_2) to $(Y_1',\eta_1',Y_2',\eta_2')$, for some (Y_2',η_2') in $\mathfrak{M}_\Lambda^0$, by the genericity assumption. Hence, (Y_1',η_1') belongs to the image of Π_1 and Π_1 is surjective. The isomorphism $\mathfrak{M}_\psi \to \mathfrak{M}_{\psi^{-1}}$ sending (X_1,η_1,X_2,η_2) to (X_2,η_2,X_1,η_1) interchanges the roles of Π_1 and Π_2. It follows that Π_2 is surjective as well. $\square$

LEMMA 5.8. *Every two points in $\mathfrak{M}_\psi^0$ are connected by a generic twistor path in $\mathfrak{M}_\psi^0$.*

Proof. Let $q := (X_1,\eta_1,X_2,\eta_2)$ be a quadruple in $\mathfrak{M}_\psi^0$, such that $\mathrm{Pic}(X_1)$ is trivial. Then $\mathrm{Pic}(X_2)$ is trivial as well, since $\psi_q : H^2(X_1,\mathbb{Q}) \to H^2(X_2,\mathbb{Q})$ is a Hodge isometry. Hence, (X_i,η_i) is the unique pair in its connected component in $\mathfrak{M}_\Lambda$ over its period $P(X_i,\eta_i)$, by Verbitsky's Torelli theorem [Huy12, Ver13]. Consequently, q is the unique point in $\mathfrak{M}_\psi^0$ mapping via $P \circ \Pi_i$ to $P(X_i,\eta_i)$, for each of $i = 1, 2$. It follows that every two such quadruples are connected by a generic twistor path in $\mathfrak{M}_\psi^0$, by the analogous theorem for marked pairs and the liftability of such paths to $\mathfrak{M}_\psi^0$ demonstrated in the proof of Lemma 5.7. Now, through each point in $\mathfrak{M}_\psi^0$

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passes a generic twistor line. Hence, every two points in $\mathfrak{M}_\psi^0$ are connected by a generic twistor path. $\qquad\square$

5.5 Deforming vector bundles along diagonal twistor paths

A universal family $f : \mathcal{X} \to \mathfrak{M}_\Lambda^0$ exists, by [Mar21b]. We get the universal families $\Pi_i^*(f) : \Pi_i^*\mathcal{X} \to \mathfrak{M}_\psi^0$, $i = 1, 2$, as well as the universal fiber product

$$\tilde{f} : \Pi_1^*\mathcal{X} \times_{\mathfrak{M}_\psi^0} \Pi_2^*\mathcal{X} \to \mathfrak{M}_\psi^0. \tag{5.7}$$

The local system $Rf_*\mathbb{Z}$ over $\mathfrak{M}_\Lambda^0$ is trivial, by [Mar21b, Lemma 2.1]. Hence, so are the local system $R\tilde{f}_*\mathbb{Z}$ and $R\tilde{f}_*\mathbb{Q}$.

DEFINITION 5.9. Let M be a d-dimensional compact Kähler manifold. Set $H^{ev}(M, \mathbb{Q}) := \oplus_{i=0}^{2d} H^{2i}(M, \mathbb{Q})$. Given a class $\gamma \in H^{ev}(M, \mathbb{Q})$ denote by γ_i the graded summand in $H^{2i}(M, \mathbb{Q})$. We say that γ is a *Hodge class*, if γ_i belongs to $H^{i,i}(M)$, for all $0 \le i \le 2d$.

DEFINITION 5.10. Let $q := (X_1, \eta_1, X_2, \eta_2)$ be a quadruple in $\mathfrak{M}_\psi^0$ and let γ be a Hodge class in $H^{ev}(X_1 \times X_2, \mathbb{Q})$. We say that γ *remains a Hodge class under every deformation in* $\mathfrak{M}_\psi^0$, if for every quadruple $q' := (X_1', \eta_1', X_2', \eta_2')$ in $\mathfrak{M}_\psi^0$ the parallel transport $\gamma' \in H^{ev}(X_1' \times X_2', \mathbb{Q})$ of γ in the local system $R\tilde{f}_*\mathbb{Q}$ along some, hence any, continuous path from q to q' in $\mathfrak{M}_\psi^0$ is a Hodge class.

DEFINITION 5.11. An *Azumaya $\mathcal{O}_X$-algebra of rank r* over a complex manifold X is a sheaf $\mathcal{A}$ of locally free coherent $\mathcal{O}_X$-modules, with a global section $1_\mathcal{A}$, and an $\mathcal{O}_X$-linear associative multiplication $m : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ with identity $1_\mathcal{A}$, admitting an open covering $\{U_\alpha\}$ of X and an isomorphism $\eta_\alpha : \mathcal{A}_{|U_\alpha} \to \mathcal{E}nd(F_\alpha)$ of unital associative algebras for some locally free $\mathcal{O}_{U_\alpha}$-module F_α of rank r over each U_α. We will also refer to such $\mathcal{A}$ as an *Azumaya algebra over X*.

Remark 5.12. The endomorphism sheaf $\mathcal{E}nd(F)$ of a locally free twisted sheaf F over X is an Azumaya algebra over X. Conversely, an Azumaya algebra corresponds to a natural equivalence class of locally free twisted sheaves over X, modulo tensorization by line bundles [Căl00]. The class $\kappa(F)$ depends only on this equivalence class and so may be regarded as a characteristic class of the pair $(\mathcal{A}, m)$ of the locally free sheaf $\mathcal{A}$ underlying the Azumaya algebra over X and its multiplication m (see [Mar20, §2.3]).

DEFINITION 5.13. Let M_1 and M_2 be compact Kähler manifolds. Let $\mathcal{A}_i$ be an Azumaya algebra over M_i. We say that $\mathcal{A}_2$ is *deformation equivalent to $\mathcal{A}_1$*, if there exists a proper family $\pi : \mathcal{M} \to B$ of compact Kähler manifolds over a connected analytic base B, points $b_i \in B$, isomorphisms $f_i : M_i \to \mathcal{M}_{b_i}$ with the fiber of π over b_i, and an Azumaya algebra $\mathcal{A}$ over $\mathcal{M}$, such that $f_i^*\mathcal{A}$ is isomorphic to $\mathcal{A}_i$, for $i = 1, 2$.

Let X be a d-dimensional compact Kähler manifold and let ω be a Kähler class on X. The *ω-slope* of a torsion-free coherent sheaf F on X is $\text{slope}_\omega(F) := \int_X c_1(F)\omega^{d-1}/\text{rank}(F)$.

DEFINITION 5.14 [Mar20, Definition 6.5]. Let E be a torsion free coherent sheaf on X, which is θ-twisted with respect to a class θ in $H^2(X, \mathcal{O}_X^*)$. We say that E is *ω-slope-stable*, if for every non-zero θ-twisted subsheaf F of E of lower rank we have $\text{slope}_\omega(\mathcal{H}om(E, F)) < 0$.

PROPOSITION 5.15. *Let $q := (X_1, \eta_1, X_2, \eta_2)$ be a quadruple in $\mathfrak{M}_\psi^0$ and let F be a locally free sheaf over $X_1 \times X_2$ with the following properties.*

(1) The locally free sheaf F is $\left(\pi_{X_1}^*(\omega) + \pi_{X_2}^*(\psi_q(\omega))\right)$-slope-stable, for every Kähler class ω in some non-empty open subcone $\mathcal{C}_F$ of $\mathcal{K}_{X_1} \cap \psi_q^{-1}(\mathcal{K}_{X_2})$.

(2) The class $c_2(\mathcal{E}nd(F))$ remains of Hodge type $(2,2)$ along every deformation in $\mathfrak{M}_\psi^0$.

For every point $q' := (X_1', \eta_1', X_2', \eta_2')$ in $\mathfrak{M}_\psi^0$ there exists a possibly twisted locally free sheaf F' over $X_1' \times X_2'$, such that the Azumaya algebra $\mathcal{E}nd(F')$ is deformation equivalent to $\mathcal{E}nd(F)$. Furthermore, F' is $\left(\pi_{X_1'}^*(\omega) + \pi_{X_2'}^*(\psi_{q'}(\omega))\right)$-slope-stable with respect to some Kähler class ω in $\mathcal{K}_{X_1'} \cap \psi_{q'}^{-1}(\mathcal{K}_{X_2'})$.

Proof. The slope stability property (1) of F implies that the first Chern class of every direct summand of $\mathcal{E}nd(F)$ vanishes, by [Mar20, Lemma 7.2]. Property (2), that $c_2(\mathcal{E}nd(F))$ remains of Hodge type, implies that the Azumaya algebra $\mathcal{E}nd(F)$ deforms along the twistor family over a generic twistor line $\widetilde{Q}_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$ in $\mathfrak{M}_\psi^0$ associated to a Kähler class ω_1 in $\mathcal{C}_F$, by [Ver99b, Theorem 3.19], which is generalized to Azumaya algebras in [Mar20, Corollary 6.12]. Here we use the fact that the twistor line in $\mathfrak{M}_\psi^0$ is associated to a hyper-Kähler structure, as explained in § 5.3, as well as the fact that the first Chern class of every direct summand of $\mathcal{E}nd(F)$ vanishes. Choose a quadruple $q_0 := (Y_1, e_1, Y_2, e_2)$ in $\widetilde{Q}_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$ with a trivial Picard group $\mathrm{Pic}(Y_1)$ and let $\mathcal{A}$ be the Azumaya algebra over $Y_1 \times Y_2$ resulting by deforming $\mathcal{E}nd(F)$ along $\widetilde{Q}_{(X_1,\eta_1,X_2,\eta_2),\omega_1}$. The Azumaya algebra $\mathcal{A}$ further deforms along every generic twistor path in $\mathfrak{M}_\psi^0$ from q_0 to any point in $\mathfrak{M}_\psi^0$, by [Mar20, Proposition 6.17]. The existence of an Azumaya algebra $\mathcal{A}'$ over $X_1' \times X_2'$ deformation equivalent to $\mathcal{E}nd(F)$ follows from that fact that every point q' in $\mathfrak{M}_\psi^0$ is connected to the point q_0 via a generic twistor path, by Lemma 5.8. The existence of a locally free sheaf F', such that the Azumaya algebras $\mathcal{E}nd(F')$ and $\mathcal{A}'$ are isomorphic, follows from the well-known correspondence between Azumaya algebras and equivalence classes of locally free twisted sheaves modulo tensorization by line bundles [Căl00]. The existence of the Kähler class ω in $\mathcal{K}_{X_1'} \cap \psi_{q'}^{-1}(\mathcal{K}_{X_2'})$ for which F' is $\left(\pi_{X_1}^*(\omega) + \pi_{X_2}^*(\psi_q(\omega))\right)$-slope-stable is automatic, since the twistor deformation F' of the hyperholomorphic sheaf F is slope-stable with respect to any such Kähler class for ω in the ray given in (5.4) parametrized by the point q' of the twistor path, by Verbitsky's theorem (see [Mar20, Corollary 6.12] for this statement in the case of twisted sheaves). $\square$

5.6 The rational Hodge isometry of a Fourier–Mukai kernel of non-zero rank

Let X and Y be the IHSMs in Lemma 4.1. Choose markings $\eta_X : H^2(X, \mathbb{Z}) \to \Lambda$ and $\eta_Y : H^2(Y, \mathbb{Z}) \to \Lambda$ such that (X, η_X) and (Y, η_Y) belong to the same connected component $\mathfrak{M}_\Lambda^0$ of $\mathfrak{M}_\Lambda$.

Convention 5.16. Choose an orientation ϵ_Λ of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Given two pairs (X_1, η_1) and (X_2, η_2) in $\mathfrak{M}_\Lambda$, we endow $H^2(X_i, \mathbb{Q})$ with the orientation ϵ_i, such that η_i is orientation preserving. The homomorphism

$$\widetilde{H} : \mathrm{Hom}_\mathcal{G}(X_1, X_2) \to \mathrm{Hom}(\widetilde{H}(X_1, \mathbb{Q}), \widetilde{H}(X_2, \mathbb{Q})),$$

induced by the functor (1.4), is then always evaluated in terms of the orientations ϵ_1 and ϵ_2. The resulting homomorphism depends on the markings η_1 and η_2, but is independent of the choice of ϵ_Λ.

The unit sphere in any positive-definite three-dimensional subspace of $H^2(X, \mathbb{R})$ is a deformation retract of the positive cone $\mathcal{C}_X' := \{x \in H^2(X, \mathbb{R}) \ ; \ (x, x) > 0\}$ (see [Mar11, § 4]). Hence, the second cohomology of the positive cone is a one-dimensional representation of $O(H^2(X, \mathbb{Q}))$

corresponding to a character $\nu : O(H^2(X, \mathbb{Q})) \to \mu_2 := \{\pm 1\}$. Set

$$O^+(H^2(X, \mathbb{Q})) := \ker(\nu).$$

The orientation given by the basis (5.2) is independent of the choices of a holomorphic 2-form and of a Kähler class and determines a *positive* generator of the cyclic group $H^2(\mathcal{C}'_X, \mathbb{Z})$.

DEFINITION 5.17. The orientation of the positive cone is the positive generator of $H^2(\mathcal{C}'_X, \mathbb{Z})$. An isometry $g : H^2(X_1, \mathbb{Q}) \to H^2(X_2, \mathbb{Q})$ is *orientation preserving*, if $g_* : H^2(\mathcal{C}'_{X_1}, \mathbb{Z}) \to H^2(\mathcal{C}'_{X_2}, \mathbb{Z})$ maps the positive generator to the positive generator.

Set

$$\nu(g) = \begin{cases} 1 & \text{if } g \text{ is orientation preserving,} \\ -1 & \text{if } g \text{ is orientation reveresing.} \end{cases}$$

Note that if g is a Hodge isometry, which maps some Kähler class to a Kähler class, then $\nu(g) = 1$, as it maps a basis of the form (5.2) to such a basis. In particular, the isometry $\eta_2^{-1}\psi\eta_1 : H^2(X_1, \mathbb{Q}) \to H^2(X_2, \mathbb{Q})$ associated to a quadruple $(X_1, \eta_1, X_2, \eta_2)$ in $\mathfrak{M}_\psi$ is orientation preserving.

Remark 5.18. Keep the notation of Convention 5.16. Let $g : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ be a parallel-transport operator and denote by $\bar{g}$ its restriction to $H^2(X_1, \mathbb{Z})$. One checks that if $\bar{g}$ is orientation preserving in the sense of Convention 5.16, so that $\det(\eta_2 \bar{g} \eta_1^{-1}) = 1$, then $\widetilde{H}(g)$ maps α to α, β to β, and restricts to $H^2(X_1, \mathbb{Q})$ as $\bar{g}$, so that $\widetilde{H}_0(g) = \bar{g}$ (see[9] [Mar21a, Remark 5.4]). In particular, this is the case when (X_1, η_1) and (X_2, η_2) belong to the same connected component of $\mathfrak{M}_\Lambda$ and $\bar{g} = \eta_2^{-1}\eta_1$.

Let E be the Fourier–Mukai kernel of non-zero rank of an equivalence $\Phi_E : D^b(X) \to D^b(Y)$ of derived categories of two deformation equivalent IHSMs X and Y. Set $\phi := [\kappa(E)\sqrt{td_{X \times Y}}]_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ and let $\widetilde{H}(\phi)_0$ be the restriction of $\widetilde{H}(\phi)$ to $H^2(X, \mathbb{Q})$. Set

$$\widetilde{H}_0(\phi) := \nu(\widetilde{H}(\phi)_0)\widetilde{H}(\phi)_0. \tag{5.8}$$

We denote $\widetilde{H}_0(\phi)$ also by ψ_E to emphasize its dependence on E:

$$\psi_E := \widetilde{H}_0([\kappa(E)\sqrt{td_{X \times Y}}]_*).$$

Let $\psi : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$, be the isometry given by

$$\psi = \eta_Y \widetilde{H}_0(\phi)\eta_X^{-1}. \tag{5.9}$$

Then ψ belongs to $O^+(\Lambda_{\mathbb{Q}})$, since (X, η_X) and (Y, η_Y) belong to the same connected component of $\mathfrak{M}_\Lambda$ and $\widetilde{H}_0(\phi)$ is orientation preserving (see [Mar11, § 4]).

ASSUMPTION 5.19.

(1) Assume that the Fourier–Mukai kernel E in Lemma 4.1 is represented by a locally free sheaf, which we also denote by E.

---

[9] The statement is true for any parallel-transport operator, when $n := \dim(X_i)/2$ is odd. When n is even we explain in [Mar21a, Remark 5.4] Taelman's sign convention in terms of a functor χ from $\mathcal{G}$ to $\mu_2 := \{\pm 1\}$, such that $\chi(\phi)\widetilde{H}(\phi)$ is orientation preserving, for every $\phi \in \text{Hom}_{\mathcal{G}}(X_1, X_2)$, and $\chi(g) = 1$, for every parallel-transport operator. Thus, when n is even, the restrictions of $\widetilde{H}(g)$ to $H^2(X_1, \mathbb{Z})$ is equal to that of g, if g is orientation preserving, and is equal to that of $-g$, if g is orientation reversing.

(2) Assume that there exist[10] proper and smooth families of Kähler manifolds $p : \mathcal{X} \to B$ and $q : \mathcal{Y} \to B$, over the same simply connected analytic base B, a possibly twisted vector bundle $\mathcal{E}$ over $\mathcal{X} \times_B \mathcal{Y}$, a point $b \in B$, isomorphisms $f : X \to X_b$ and $g : Y \to Y_b$, and an isomorphism of Azumaya algebras $\mathcal{E}nd(E) \cong (f \times g)^* \mathcal{E}nd(\mathcal{E})$, where X_b and Y_b are the fibers of $\mathcal{X}$ and $\mathcal{Y}$ over b, such that the value of the parallel transport ψ_{E_0} of ψ_E in the local system $(R^2 p_* \mathbb{Q})^* \otimes R^2 q_* \mathbb{Q}$ at some point $0 \in B$ maps some Kähler class ω_1 on the fiber X_0 of p to a Kähler class ω_2 over the fiber Y_0 of q.

(3) We assume that E_0 is $[\pi_{X_0}^*(\omega) + \pi_{Y_0}^*(\psi_{E_0}(\omega))]$-slope-stable, for classes ω in a non-empty subcone of the intersection $\mathcal{K}_{X_0} \cap \psi_{E_0}^{-1}(\mathcal{K}_{Y_0})$.

Remark 5.20. Note that the class $\kappa(E_0)\sqrt{td_{X_0 \times Y_0}}$ induces the degree-reversing Hodge isometry (with respect to the Mukai pairings)

$$\phi_0 : H^*(X_0, \mathbb{Q}) \to H^*(Y_0, \mathbb{Q}), \tag{5.10}$$

which is the parallel transport of ϕ in Lemma 4.1, hence a morphism in $\mathrm{Hom}_{\mathcal{G}}(X_0, Y_0)$. Its image $\widetilde{H}(\phi_0)$ restricts to a Hodge isometry $\widetilde{H}(\phi_0)_0 : H^2(X_0, \mathbb{Q}) \to H^2(Y_0, \mathbb{Q})$, which is the parallel transport of $\widetilde{H}(\phi)_0$ in Lemma 4.1. The parallel transport $\psi_{E_0} : H^2(X_0, \mathbb{Q}) \to H^2(Y_0, \mathbb{Q})$ of ψ_E in Assumption 5.19(2) is equal to $\widetilde{H}_0(\phi_0)$.

Let η_{X_0} be the marking of X_0 resulting in the extension of η_X to a trivialization of $R^2 p_* \mathbb{Z}$ for the family $p : \mathcal{X} \to B$ in Assumption 5.19. Define η_{Y_0} similarly. Note that $\psi(\eta_{X_0}(\omega_1)) = \eta_{Y_0}(\omega_2)$ for some Kähler classes ω_1 and ω_2, by Assumption 5.19, and so $(X_0, \eta_{X_0}, Y_0, \eta_{Y_0})$ belongs to $\mathfrak{M}_\psi$. Denote by

$$\mathfrak{M}_\psi^0 \tag{5.11}$$

the connected component of $\mathfrak{M}_\psi$ containing $(X_0, \eta_{X_0}, Y_0, \eta_{Y_0})$.

5.7 Deforming the Fourier–Mukai kernel

DEFINITION 5.21. A quadruple $(X_1, \eta_1, X_2, \eta_2)$ in $\mathfrak{M}_\psi$ is said to *support a compatible vector bundle* if there exists a (possibly twisted) locally free coherent sheaf F over $X_1 \times X_2$ with the following properties.

(1) The class $\kappa(F)\sqrt{td_{X_1 \times X_2}}$ induces a morphism

$$\phi_F : H^*(X_1, \mathbb{Q}) \to H^*(X_2, \mathbb{Q})$$

in $\mathrm{Hom}_{\mathcal{G}_{an}}(X_1, X_2)$, where $\mathcal{G}_{an}$ is the groupoid given in (1.2).
(2) The equality $\psi = \eta_2 \widetilde{H}_0(\phi_F) \eta_1^{-1}$ holds, where $\widetilde{H}_0$ is given in (5.8).

PROPOSITION 5.22. *Every quadruple $(X_1, \eta_1, X_2, \eta_2)$ in the connected component $\mathfrak{M}_\psi^0$ given in (5.11) supports a compatible vector bundle.*

Proof. Step 1. Let $q \in \mathfrak{M}_\psi^0$ be the isomorphism class of $(X_1, \eta_1, X_2, \eta_2)$. Consider the local system $R\tilde{f}_* \mathbb{Q}$ over $\mathfrak{M}_\psi^0$ associated to the universal fiber product (5.7). Let $\gamma_q \in H^*(X_1 \times X_2, \mathbb{Q})$ be a flat deformation of the class

$$\gamma := \kappa(E_0)\sqrt{td_{X_0 \times Y_0}}$$

via a path in $\mathfrak{M}_\psi^0$ from $(X_0, \eta_{X_0}, Y_0, \eta_{Y_0})$ to q. If $\tilde{\eta}_1 : H^*(X_1, \mathbb{Q}) \to H^*(X_0, \mathbb{Q})$ denotes the parallel-transport operator in the local system $\Pi_1^* R f_* \mathbb{Q}$ and $\tilde{\eta}_2 : H^*(X_2, \mathbb{Q}) \to H^*(Y_0, \mathbb{Q})$ denotes the

---

[10] If the isometry ψ_E, given in (5.8), maps some Kähler class ω_1 on X to a Kähler class ω_2 on Y, then one can take the constant families with B a point. This is the case in the $K3$ surface examples used in [Bus19].

parallel-transport operator in the local system $\Pi_2^* R f_* \mathbb{Q}$, both with respect to the chosen path (backward), then $\gamma = (\tilde{\eta}_1 \otimes \tilde{\eta}_2)(\gamma_q)$. The class γ_q is independent of the path chosen, since the local system $R\tilde{f}_* \mathbb{Q}$ over $\mathfrak{M}_\Lambda^0$ is trivial, by [Mar21b, Lemma 2.1]. We prove in this step that γ_q is a Hodge class. In particular, $c_2(\mathcal{E}nd(E_0))$ remains of Hodge type over $X_1 \times X_2$, by the equality $c_2(\mathcal{E}nd(E_0)) = -2 \operatorname{rank}(E_0)\kappa_2(E_0)$. Once it is proven that γ_q is a Hodge class, for all $q \in \mathfrak{M}_\psi^0$, Proposition 5.15 implies the existence of a locally free sheaf F over $X_1 \times X_2$ such that $\mathcal{E}nd(F)$ is deformation equivalent to $\mathcal{E}nd(E_0)$.

The isometry[11] $\phi_0 : H^*(X_0, \mathbb{Q}) \to H^*(Y_0, \mathbb{Q})$, induced by γ, is invariant under the action of $GL(H^*(X_0, \mathbb{Q}))$ on $\operatorname{Hom}(H^*(X_0, \mathbb{Q}), H^*(Y_0, \mathbb{Q}))$, where an element $g \in GL(H^*(X_0, \mathbb{Q}))$ acts by $h \mapsto (\phi_0 g \phi_0^{-1}) h g^{-1}$. The isometry ϕ_0 is related to the class $\gamma := \kappa(E_0)\sqrt{td_{X_0 \times Y_0}}$ via

$$\phi_0 \in \operatorname{Hom}(H^*(X_0), H^*(Y_0)) \cong H^*(X_0)^* \otimes H^*(Y_0) \cong H^*(X_0) \otimes H^*(Y_0) \cong H^*(X_0 \times Y_0) \ni \gamma,$$

where the middle isomorphism is induced by the Poincaré duality isomorphism $PD : H^*(X_0, \mathbb{Q}) \to H^*(X_0, \mathbb{Q})^*$. Assume that g is an isometry with respect to the Poincaré pairing and so $(g^{-1})^* = PD \circ g \circ PD^{-1}$. In this case γ is invariant with respect to the action by $g \otimes (\phi_0 g \phi_0^{-1})$. The Lie algebra $\bar{\mathfrak{g}}_{X_0, \mathbb{R}}$ acts on $H^*(X_0, \mathbb{R})$ by derivations [LL97, Proposition 4.5(ii)], and so it annihilates the cup product and is thus contained in the Lie algebra of the isometry group with respect to the Poincaré pairing. We get that the class γ is annihilated by $(\xi \otimes 1) + (1 \otimes Ad_{\phi_0}(\xi))$, for every element $\xi \in \bar{\mathfrak{g}}_{X_0}$. We conclude that γ is annihilated by the Lie subalgebra

$$\bar{\mathfrak{g}}_{\phi_0} := \{(\xi \otimes 1) + (1 \otimes Ad_{\phi_0}(\xi)) \; : \; \xi \in \bar{\mathfrak{g}}_{X_0}\} \subset \bar{\mathfrak{g}}_{X_0} \times \mathfrak{g}_{Y_0},$$

where the inclusion follows from the equality $Ad_{\phi_0}(\mathfrak{g}_{X_0}) = \mathfrak{g}_{Y_0}$ established in [Tae23, Theorem A]. Note that $\bar{\mathfrak{g}}_{\phi_0}$ is, in fact, the subalgebra

$$\bar{\mathfrak{g}}_{\psi_{E_0}} := \{(\xi \otimes 1) + (1 \otimes Ad_{\psi_{E_0}}(\xi)) \; : \; \xi \in \bar{\mathfrak{g}}_{X_0}\} \subset \bar{\mathfrak{g}}_{X_0} \times \bar{\mathfrak{g}}_{Y_0},$$

by Lemma 4.1(3) and Remark 5.20.

We have the commutative diagram

$$
\begin{array}{ccc}
H^2(X_0, \mathbb{Q}) & \xrightarrow{\eta_1^{-1}\eta_{X_0}} & H^2(X_1, \mathbb{Q}) \\
{\scriptstyle \eta_{Y_0}^{-1}\psi\eta_{X_0} = \psi_{E_0}} \downarrow & & \downarrow {\scriptstyle \psi_q := \eta_2^{-1}\psi\eta_1} \\
H^2(Y_0, \mathbb{Q}) & \xrightarrow[\eta_2^{-1}\eta_{Y_0}]{} & H^2(X_2, \mathbb{Q})
\end{array}
$$

whose horizontal arrows are parallel-transport operators. The Lie algebra $\mathfrak{g}_{X_0}$ deforms flatly to $\mathfrak{g}_{X_1}$, by the topological nature of both, and so the semi-simple part of its degree-0 summand $\bar{\mathfrak{g}}_{X_0}$ deforms flatly to $\bar{\mathfrak{g}}_{X_1}$. The Lie subalgebra $\mathfrak{g}_{\psi_{E_0}}$ deforms flatly to

$$\bar{\mathfrak{g}}_{\psi_q} := \{(\xi \otimes 1) + (1 \otimes Ad_{\psi_q}(\xi)) \; : \; \xi \in \bar{\mathfrak{g}}_{X_1}\} \subset \bar{\mathfrak{g}}_{X_1} \times \bar{\mathfrak{g}}_{X_2},$$

by the commutativity of the above diagram. Indeed,

$$(Ad_{\eta_1^{-1}\eta_{X_0}} \otimes Ad_{\eta_2^{-1}\eta_{Y_0}})\Big((\xi \otimes 1) + (1 \otimes Ad_{\psi_{E_0}}(\xi))\Big) = (Ad_{\eta_1^{-1}\eta_{X_0}}(\xi) \otimes 1) + (1 \otimes Ad_{\eta_2^{-1}\eta_{Y_0}\psi_{E_0}}(\xi))$$

$$= (\xi' \otimes 1) + (1 \otimes Ad_{\psi_q}(\xi')),$$

---

[11] The isomorphism ϕ_0 is an isometry with respect to the Mukai pairings. When the odd cohomology of X vanishes ϕ_0 is also an isometry with respect to the Poincaré pairings, by Corollary 4.2, but we do not assume the vanishing of the odd cohomology here.

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where $\xi' := Ad_{\eta_1^{-1}\eta_{X_0}}(\xi) \in \bar{\mathfrak{g}}_{X_1}$. It follows that γ_q is annihilated by $\bar{\mathfrak{g}}_{\psi_q}$, as we have seen that γ is annihilated by $\bar{\mathfrak{g}}_{\psi_{E_0}}$.

Let h'_{X_i} be the Hodge operator of X_i, $i = 1, 2$, given in (3.8). Then $h'_{X_2} = Ad_{\psi_q}(h'_{X_1})$, since ψ_q is an isomorphism of Hodge structures. Hence, the Hodge operator of $X_1 \times X_2$

$$(h'_{X_1} \otimes 1) + (1 \otimes h'_{X_2}) = (h'_{X_1} \otimes 1) + (1 \otimes Ad_{\psi_q}(h'_{X_1}))$$

belongs to $\bar{\mathfrak{g}}_{\psi_q}$ and, thus, annihilates γ_q. We conclude that γ_q is a Hodge class.

Step 2. Consider first the quadruple $q_0 := (X_0, \eta_{X_0}, Y_0, \eta_{Y_0})$, used in the definition of the connected component $\mathfrak{M}_\psi^0$ in (5.11), and set $F := E_0$. Property (1) in Definition 5.21 follows in this case from Assumption 5.19 and property (2) follows from the definition of ψ in (5.9).

Let $\phi_{\gamma_q} : H^*(X_1, \mathbb{Q}) \to H^*(X_2, \mathbb{Q})$ be the homomorphism induced by γ_q. The equality $\phi_{\gamma_q} = \tilde{\eta}_2^{-1}\phi_0\tilde{\eta}_1$ exhibits ϕ_{γ_q} as a composition of morphisms in the groupoid $\mathcal{G}$, hence in $\operatorname{Hom}_\mathcal{G}(X_1, X_2)$. Choose a twistor path C in $\mathfrak{M}_\psi^0$ from q_0 to q along which the Azumaya algebra $\mathcal{E}nd(E_0)$ deforms via an Azumaya algebra $\mathcal{A}$ over the twistor family $\Pi_1^*\mathcal{X} \times_C \Pi_2^*\mathcal{X} \to C$. Such a path C exists, by Assumption 5.19 and Proposition 5.15. There exists a locally free twisted sheaf $\mathcal{F}$ over $\Pi_1^*\mathcal{X} \times_C \Pi_2^*\mathcal{X}$, such that $\mathcal{E}nd(\mathcal{F})$ is isomorphic to $\mathcal{A}$. $\mathcal{F}$ restricts to a locally free twisted sheaf F_0 over $X_0 \times Y_0$ and a locally free sheaf F over $X_1 \times X_2$, such that the Azumaya algebras $\mathcal{E}nd(F_0)$ is isomorphic to $\mathcal{E}nd(E_0)$ and $\kappa(F)$ is a parallel transport of $\kappa(F_0)$ along a (real) path in C. Hence, γ_q is equal to $\kappa(F)\sqrt{td_{X_1 \times X_2}}$ and property (1) in Definition 5.21 follows. Property (2) in Definition 5.21 is equivalent to the equality $\psi_q = \widetilde{H}_0([\kappa(F)\sqrt{td_{X_1 \times X_2}}]_*)$. The latter equality follows at q via Remark 5.18 from the fact that it holds for q_0. Explicitly,

$$\psi_q = (\eta_2^{-1}\eta_{Y_0})\psi_{E_0}(\eta_{X_0}^{-1}\eta_1) = (\eta_2^{-1}\eta_{Y_0})\widetilde{H}_0([\kappa(E_0)\sqrt{td_{X_0 \times Y_0}}]_*)(\eta_{X_0}^{-1}\eta_1)$$

$$\overset{\text{Remark } 5.18}{=} \widetilde{H}_0\big((\tilde{\eta}_2^{-1}) \circ [\kappa(E_0)\sqrt{td_{X_0 \times Y_0}}]_* \circ \tilde{\eta}_1\big) = \widetilde{H}_0([\kappa(F)\sqrt{td_{X_1 \times X_2}}]_*). \qquad \square$$

6. The BKR equivalence

Let S be a projective surface and denote by $S^{[n]}$ the Hilbert scheme of length n subscheme of S. The symmetric group $\mathfrak{S}_n$ on n letters acts on the cartesian product S^n permuting the factors. Let $Hilb_{\mathfrak{S}_n}(S^n)$ be the Hilbert scheme of length $n!$ subschemes Z of S^n, which are $\mathfrak{S}_n$-invariant, and such that $H^0(Z, \mathcal{O}_Z)$ is the regular representation of $\mathfrak{S}_n$. Then $S^{[n]}$ is isomorphic to the closure in $Hilb_{\mathfrak{S}_n}(S^n)$ of the locus of reduced subschemes which are $\mathfrak{S}_n$-orbits of n-tuples consisting of n distinct points of S, by [Hai01]. The restriction

$$\Gamma_S \subset S^{[n]} \times S^n,$$

to $S^{[n]}$ of the universal subscheme of $Hilb_{\mathfrak{S}_n}(S^n) \times S^n$ is called[12] the *isospectral Hilbert scheme*. The projections $S^{[n]} \overset{q}{\leftarrow} \Gamma_S \overset{b}{\rightarrow} S^n$ have the following properties. The morphism q is flat and $\mathfrak{S}_n$-invariant of degree $n!$, by the proof of [Hai01, Proposition 3.7.4]. The morphism b restricts to an isomorphism over the complement of the union of the diagonals in S^n. The isospectral Hilbert scheme Γ_S is Gorenstein, by [Hai01, Theorem 3.1], and its dualizing sheaf ω_{Γ_S} is isomorphic to the line bundle $q^*(\omega_{S^{[n]}}(\delta))$, by [Hai01, Theorem 3.1, Proposition 3.4.3, comment after Lemma 3.4.2]. Above δ is the divisor class, such that $\mathcal{O}_{S^{[n]}}(-\delta) \cong \big(q_*(\mathcal{O}_{\Gamma_S})\big)^\chi$, where χ is the sign character of $\mathfrak{S}_n$. Note that 2δ is the class of the effective divisor in $S^{[n]}$ of non-reduced subschemes.

---

[12] The isospectral Hilbert scheme Γ_S is defined in [Hai01] as the reduced subscheme associated to the support of the fiber product of $S^{[n]}$ and S^n over the symmetric product $S^{(n)}$. Haiman proves that Γ_S is flat over $S^{[n]}$, and so it is indeed the universal subscheme.

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Let $D^b_{\mathfrak{S}_n}(S^n)$ be the bounded derived category of $\mathfrak{S}_n$-equivariant coherent sheaves on S^n and $D^b_{\mathfrak{S}_n}(S^{[n]} \times S^n)$ its analogue with respect to the permutation action of $\mathfrak{S}_n$ on the factor S^n and the trivial action on the factor $S^{[n]}$. Let ρ be the natural $\mathfrak{S}_n$-linearization of $\mathcal{O}_{\Gamma_S}$. The object $(\mathcal{O}_{\Gamma_S}, \rho)$ of $D^b_{\mathfrak{S}_n}(S^{[n]} \times S^n)$ is the Fourier–Mukai kernel of an equivalence of derived categories

$$BKR : D^b_{\mathfrak{S}_n}(S^n) \to D^b(S^{[n]}),$$

given by the composition $Rq_*^{\mathfrak{S}_n} Lb^*$ of the functors $Lb^* : D^b_{\mathfrak{S}_n}(S^n) \to D^b_{\mathfrak{S}_n}(\Gamma_S)$ and $Rq_*^{\mathfrak{S}_n} : D^b_{\mathfrak{S}_n}(\Gamma_S) \to D^b(S^{[n]})$, by [BKR01].

7. A universal bundle over $M^{[n]} \times S^{[n]}$ from a universal bundle over $M \times S$

In §7.1 we recall the construction of a locally free Fourier–Mukai kernel of an equivalence of the derived categories of the Hilbert schemes $S^{[n]}$ and $M^{[n]}$, where M is a two-dimensional smooth and projective moduli space of vector bundles on a $K3$ surface S. In §7.2 we construct a functor $\widetilde{\Theta}_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]}$, which sends a morphism associated to an equivalence Φ of derived categories of $K3$ surfaces to a morphism associated to the equivalence of the derived categories of their Hilbert schemes, which is the BKR conjugate of the cartesian power Φ^n. In §7.3 we normalize $\widetilde{\Theta}_n$ to obtain the functor $\Theta_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]}$, mentioned in the introduction in (1.8), which maps $\mathcal{G}^{[1]}_{an}$ to $\mathcal{G}^{[n]}_{an}$. The functor Θ_n sends a parallel-transport operator between two $K3$ surfaces S_1 and S_2 to the associated parallel-transport operator between their Hilbert schemes $S_1^{[n]}$ and $S_2^{[n]}$. If $\phi \in \mathrm{Hom}_{\mathcal{G}^{[1]}_{an}}(S_1, S_2)$ is induced by the class $\kappa(\mathcal{U})\sqrt{td_{S_1 \times S_2}}$ of the Fourier–Mukai kernel $\mathcal{U}$ of an equivalence $\Phi_{\mathcal{U}} : D^b(S_1) \to D^b(S_2)$, then $\Theta_n(\phi)$ is induced by the analogous class associated to the BKR conjugate of the nth cartesian power of $\Phi_{\mathcal{U}}$. In Corollary 7.4 we show that $\widetilde{H}(\Theta_n(\phi)) : \widetilde{H}(S_1^{[n]}, \mathbb{Q}) \to \widetilde{H}(S_2^{[n]}, \mathbb{Q})$ is the naive extension of $\widetilde{H}(\phi) : \widetilde{H}(S_1, \mathbb{Q}) \to \widetilde{H}(S_2, \mathbb{Q})$.

7.1 A universal bundle over $M^{[n]} \times S^{[n]}$

Assume that S is a $K3$ surface, $v = (r, \lambda, s) \in \widetilde{H}(S, \mathbb{Z})$ is a primitive and isotropic Mukai vector of rank $r \geq 2$, and H is a v-generic polarization. Then every H-slope-semistable sheaf on S is H-slope-stable and locally free. Let M be the moduli space of H-slope-stable sheaves on S with Mukai vector v. Then M is a $K3$ surface, by [Muk87]. Assume that a universal sheaf $\mathcal{U}$ exists over $M \times S$. Then $\mathcal{U}$ is the Fourier–Mukai kernel of an equivalence

$$\Phi_{\mathcal{U}} : D^b(M) \to D^b(S)$$

(see [Huy06, Proposition 10.25]). We get the equivalence

$$\Phi_{\mathcal{U}}^{\boxtimes n} : D^b_{\mathfrak{S}_n}(M^n) \to D^b_{\mathfrak{S}_n}(S^n)$$

and its conjugate

$$\Phi_{\mathcal{U}}^{[n]} := BKR \circ \Phi_{\mathcal{U}}^{\boxtimes n} \circ BKR^{-1} : D^b(M^{[n]}) \to D^b(S^{[n]}). \tag{7.1}$$

Let $\mathfrak{S}_{n,\Delta}$ be the diagonal subgroup of $\mathfrak{S}_n \times \mathfrak{S}_n$. Let $\mathcal{U}^{\boxtimes n} := \mathcal{U} \boxtimes \cdots \boxtimes \mathcal{U}$ be the exterior tensor power of $\mathcal{U}$ over $M^n \times S^n$. Then $\mathcal{U}^{\boxtimes n}$ admits the permutation $\mathfrak{S}_{n,\Delta}$-linearization $\rho_{\boxtimes}$. Denote by χ the sign character of $\mathfrak{S}_{n,\Delta}$. We get the object $(\mathcal{U}^{\boxtimes n}, \rho_{\boxtimes} \otimes \chi)$ in $D^b_{\mathfrak{S}_{n,\Delta}}(M^n \times S^n)$.

We use the same notation for the morphisms $M^n \xleftarrow{b} \Gamma_M \xrightarrow{q} M^{[n]}$ from the isospectral Hilbert scheme of M. We get the product morphisms

$$M^n \times S^n \xleftarrow{b \times b} \Gamma_M \times \Gamma_S \xrightarrow{q \times q} M^{[n]} \times S^{[n]}.$$

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LEMMA 7.1. *The equivalence* $\Phi_{\mathcal{U}}^{[n]}$ *has the rank* $n! r^n$ *locally free Fourier–Mukai kernel*

$$E := \pi_{M^{[n]}}^*(\mathcal{O}_{M^{[n]}}(\delta)) \otimes (q \times q)_*^{\mathfrak{S}_{n,\Delta}}((b \times b)^*(\mathcal{U}^{\boxtimes n}, \rho_{\boxtimes} \otimes \chi)). \tag{7.2}$$

Proof. The statement is proved in [Mar21a, § 11.1.2, (11.6)] □

7.2 The functor $\widetilde{\Theta}_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]}$

In the remainder of § 7 we will keep the assumption that the rank of the object $\mathcal{U} \in D^b(S \times M)$ is non-zero, but drop the assumption that $\mathcal{U}$ is represented by a coherent sheaf. Let $\mathcal{G}^{[n]}$ be the subgroupoid of the groupoid $\mathcal{G}$ given in (1.1) whose objects are of $K3^{[n]}$ type. We define next the functor $\widetilde{\Theta}_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]}$. Define $\widetilde{\Theta}_n(X, \epsilon) = (X^{[n]}, \epsilon^{[n]})$, where the orientation $\epsilon^{[n]}$ of $H^2(X^{[n]}, \mathbb{Q})$ is such that the class δ extends a basis compatible with the orientation ϵ to one compatible with $\epsilon^{[n]}$.

(1) The functor $\widetilde{\Theta}_n$ maps a parallel-transport operator, associated to two fibers over two points b_0 and b_1 in the base of a family $\mathcal{X} \to B$ of $K3$ surfaces and a path γ from b_0 to b_1, to the parallel-transport operator associated to the same path and the relative Hilbert scheme $\mathcal{X}^{[n]} \to B$. Denote by

$$\theta : H^2(S, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z}) \tag{7.3}$$

the composition of the isomorphism $H^2(S, \mathbb{Z}) \cong H^2(S^{(n)}, \mathbb{Z})$ with the pullback $H^2(S^{(n)}, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z})$ via the Hilbert–Chow morphism.

(2) The functor $\widetilde{\Theta}_n$ sends the cup product with $\exp(\lambda)$, $\lambda \in H^2(X, \mathbb{Q})$, to the cup product with $\exp(\theta(\lambda))$.

(3) The functor $\widetilde{\Theta}_n$ sends the isometry $\phi_{\mathcal{P}} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ induced by the correspondence $\sqrt{td_{X \times Y}} ch(\mathcal{P})$, where $\mathcal{P} \in D^b(X \times Y)$ is the Fourier–Mukai kernel of an equivalence $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$, to the isometry induced by the correspondence $\sqrt{td_{X^{[n]} \times Y^{[n]}}} ch(\mathcal{P}^{[n]})$, where $\mathcal{P}^{[n]}$ is the Fourier–Mukai kernel in (7.2) of the BKR-conjugate of $\Phi_{\mathcal{P}}^{\boxtimes n}$ as in (7.1).

The image of morphisms under the functor $\widetilde{\Theta}_n$ was defined above separately for the three types of morphisms. Let $[\mathcal{O}_{\Gamma_S}]$ be the class of $\mathcal{O}_{\Gamma_S}$, with its natural linearization, in the topological $\mathfrak{S}_n$-equivariant K-ring $K_{\mathfrak{S}_n}(S^n \times S^{[n]})$. We get the correspondence $[\mathcal{O}_{\Gamma_S}]_* : K_{\mathfrak{S}_n}(S^n) \to K(S^{[n]})$ on the level of topological K-rings. The inverse is induced by a class in $K_{\mathfrak{S}_n}(S^n \times S^{[n]})$ as well. The value of $\widetilde{\Theta}_n$ on a morphism in $\mathcal{G}^{[1]}$ of type (3) associated to an equivalence $\Phi_{\mathcal{U}}$ with Fourier–Mukai kernel $\mathcal{U}$ involves first composing the correspondence $[\mathcal{U}^{\boxtimes n}, \rho_{\boxtimes}]_* : K_{\mathfrak{S}_n}(M^n) \to K_{\mathfrak{S}_n}(S^n)$ with $[\mathcal{O}_{\Gamma_S}]_*$ and the inverse of $[\mathcal{O}_{\Gamma_M}]_*$, to obtain the correspondence from $K(M^{[n]})$ to $K(S^{[n]})$, and then using the Chern character isomorphism and its inverse to obtain the cohomological morphism in $\mathrm{Hom}_{\mathcal{G}^{[n]}}(M^{[n]}, S^{[n]})$ (see [BKR01, § 10.1]). The value of $\widetilde{\Theta}_n$ on morphisms of types (1) and (2) may be obtained by first conjugating via the Chern character their cartesian powers to lift these morphisms to correspondences from $K_{\mathfrak{S}_n}(M^n)$ to $K_{\mathfrak{S}_n}(S^n)$. The check that $\widetilde{\Theta}_n$ maps compositions of morphisms to compositions reduces to showing that its value on (1) morphisms that are parallel-transport operators or (2) morphisms that are multiplication by $\exp(\lambda)$, $\lambda \in H^2(X, \mathbb{Q})$, is induced by composing the correspondences from $K_{\mathfrak{S}_n}(M^n)$ to $K_{\mathfrak{S}_n}(S^n)$ with the same correspondences, $[\mathcal{O}_{\Gamma_S}]_*$ and the inverse of $[\mathcal{O}_{\Gamma_M}]_*$, used for morphisms of type (3). For parallel-transport operators this is the case, since the class $[\mathcal{O}_{\Gamma_S}]$ in $K_{\mathfrak{S}_n}(S^n \times S^{[n]})$ is a global class in the corresponding local system of K-rings over the base of every family of $K3$ surfaces, hence the correspondences $[\mathcal{O}_{\Gamma_S}]$ and $[\mathcal{O}_{\Gamma_M}]$ intertwine parallel-transport operators associated to paths in the base of such families. The check for multiplication by $\exp(\lambda)$, $\lambda \in H^2(X, \mathbb{Q})$, reduces to checking for integral classes, as the correspondences are all linear. In this case the

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action corresponds to tensorization by topological line bundles on the level of K-rings. The correspondence $[\mathcal{O}_{\Gamma_S}]_*$ intertwines tensorization in $K_{\mathfrak{S}_n}(S^n)$ by the cartesian power $L^{\boxtimes n}$ of a line bundle on S and by the line bundle $\tilde{L}$ on $S^{[n]}$ with $c_1(\tilde{L}) = \theta(c_1(L))$, since $\theta(c_1(L))$ is defined as the pullback of the class associated to $c_1(L)$ in the symmetric product $S^{(n)}$, and the isospectral Hilbert scheme is a reduced fiber product of S^n and $S^{[n]}$ over $S^{(n)}$. Hence, the projection formula yields that $[\mathcal{O}_{\Gamma_S}]_*$ indeed intertwines the two tensorizations.

7.3 The functor $\Theta_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]}$

Let $\phi : \widetilde{H}(M, \mathbb{Z}) \to \widetilde{H}(S, \mathbb{Z})$ be the isometry induced by the correspondence $ch(\mathcal{U})\sqrt{td_{M \times S}}$. The morphism $\widetilde{\Theta}_n(\phi) : H^*(M^{[n]}, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q})$ is induced by the correspondence

$$ch(E)\sqrt{td_{M^{[n]} \times S^{[n]}}},$$

where E is given in terms of $\mathcal{U}$ in (7.2). We recall next the relationship between ϕ and $\widetilde{H}(\widetilde{\Theta}_n(\phi)) : \widetilde{H}(M^{[n]}, \mathbb{Q}) \to \widetilde{H}(S^{[n]}, \mathbb{Q})$. Let

$$\tilde{\theta} : \widetilde{H}(S, \mathbb{Q}) \to \widetilde{H}(S^{[n]}, \mathbb{Q}) \tag{7.4}$$

be the extension of $\theta : H^2(S, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z})$, given in (7.3), mapping α to α and β to β. Let

$$\tilde{\iota} : O(\widetilde{H}(S, \mathbb{Q})) \to O(\widetilde{H}(S^{[n]}, \mathbb{Q})) \tag{7.5}$$

be the embedding given by $\tilde{\iota}_g(\tilde{\theta}(x)) = \tilde{\theta}(g(x))$ and $\tilde{\iota}_g$ acts as the identity on the one-dimensional subspace orthogonal to the image of $\tilde{\theta}$. Given an isometry $g : \widetilde{H}(M, \mathbb{Q}) \to \widetilde{H}(S, \mathbb{Q})$ we denote by $\tilde{\iota}_g : \widetilde{H}(M^{[n]}, \mathbb{Q}) \to \widetilde{H}(S^{[n]}, \mathbb{Q})$ the isometry satisfying $\tilde{\iota}_g(\tilde{\theta}(x)) = \tilde{\theta}(g(x))$ and mapping $\delta \in H^2(M^{[n]}, \mathbb{Q})$ to $\delta \in H^2(S^{[n]}, \mathbb{Q})$.

Choose markings $\eta_S : H^2(S, \mathbb{Z}) \to \Lambda_{K3}$ and $\eta_M : H^2(M, \mathbb{Z}) \to \Lambda_{K3}$, where Λ_{K3} is the $K3$ lattice, and let Λ be the orthogonal direct sum

$$\Lambda_{K3} \oplus \mathbb{Z}\delta_0, \tag{7.6}$$

where $(\delta_0, \delta_0) = 2 - 2n$, $n \geq 2$. Let

$$\iota : O(\Lambda_{K3}) \to O(\Lambda) \tag{7.7}$$

be the embedding, where ι_g is the extension of g satisfying $\iota_g(\delta_0) = \delta_0$. Choose an orientation of Λ_{K3} and an orientation of Λ, so that δ_0 extends a basis compatible with the orientation of the former to one compatible with that of the latter. Set $\det(\phi) := \det(\eta_S \circ \phi \circ \eta_M^{-1})$. Extend the marking η_S to a marking

$$\eta_{S^{[n]}} : H^2(S^{[n]}, \mathbb{Z}) \to \Lambda \tag{7.8}$$

sending δ to δ_0, where δ is half the class of the divisor of non-reduced subschemes. Define $\eta_{M^{[n]}}$ similarly. Define $\widetilde{H}(\widetilde{\Theta}_n(\phi))$ using Convention 5.16.

LEMMA 7.2. *We have $\widetilde{H}(\widetilde{\Theta}_n(\phi)) = \det(\phi)^{n+1}(B_{-\delta/2} \circ \tilde{\iota}_\phi \circ B_{\delta/2})$.*

Proof. Let $\zeta : H^*(M, \mathbb{Q}) \to H^*(S, \mathbb{Q})$ be a parallel-transport operator such that $\det(\eta_S \circ \bar{\zeta} \circ \eta_M^{-1}) = 1$, where $\bar{\zeta}$ is the restriction of ζ to $H^2(M, \mathbb{Q})$. Then ζ lifts to a parallel-transport operator $\zeta^{[n]} : H^*(M^{[n]}, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q})$, since associated to a family of $K3$ surfaces is a canonical family of relative Douady spaces. Let $\bar{\zeta}^{[n]}$ be the restriction of $\zeta^{[n]}$ to $H^2(M^{[n]}, \mathbb{Q})$. Then $\det(\eta_{S^{[n]}} \circ \bar{\zeta}^{[n]} \circ \eta_{M^{[n]}}^{-1}) = 1$. We get $\widetilde{H}(\zeta^{[n]}) = \tilde{\iota}_\zeta = B_{-\delta/2} \circ \tilde{\iota}_\zeta \circ B_{\delta/2}$, where the first equality is by Remark 5.18. The composition $\phi_1 := \zeta \circ \phi$ is an element of $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S)$. We conclude that the

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equality in the statement of the lemma is equivalent to the equality

$$\widetilde{H}(\widetilde{\Theta}_n(\phi_1)) = \det(\phi_1)^{n+1}(B_{-\delta/2} \circ \tilde{\iota}_{\phi_1} \circ B_{\delta/2}).$$

The latter equality holds for all elements in $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S)$ by [Mar21a, Theorem 12.2] (see also [Bec23, Theorem 7.4] and in case $n = 2$ Taelman's earlier result [Tae23, Theorem 9.4]). $\square$

Let $\varphi : \widetilde{H}(M, \mathbb{Q}) \to \widetilde{H}(S, \mathbb{Q})$ be the isometry induced by $\kappa(\mathcal{U})\sqrt{td_{M \times S}}$. Let

$$\varphi^{[n]} : H^*(M^{[n]}, \mathbb{Q}) \to H^*(S^{[n]}, \mathbb{Q})$$

be the isometry induced by the correspondence $\kappa(E)\sqrt{td_{M^{[n]} \times S^{[n]}}}$. Again, $\varphi^{[n]}$ is a morphism in $\mathrm{Hom}_{\mathcal{G}^{[n]}}(M^{[n]}, S^{[n]})$. As previously, we denote by $\widetilde{H}(\varphi)_0$ the restriction of φ to $H^2(M, \mathbb{Q})$ and set $\psi_{\mathcal{U}} := \widetilde{H}_0(\varphi)$, where $\widetilde{H}_0(\varphi) := \nu(\widetilde{H}(\varphi)_0)\widetilde{H}(\varphi)_0$ as in (5.8). We define $\widetilde{H}(\varphi^{[n]})_0$ as the restriction of $\widetilde{H}(\varphi^{[n]})$ to $H^2(M^{[n]}, \mathbb{Q})$ and set $\psi_E := \widetilde{H}_0(\varphi^{[n]})$, where $\widetilde{H}_0(\varphi^{[n]}) := \nu(\widetilde{H}(\varphi^{[n]})_0)\widetilde{H}(\varphi^{[n]})_0$.

COROLLARY 7.3. *We have*

$$\widetilde{H}(\varphi^{[n]}) = \det(\varphi)^{n+1}\tilde{\iota}_\varphi. \tag{7.9}$$

Consequently, $\psi_E \circ \theta = \theta \circ \psi_{\mathcal{U}}$ and $\psi_E(\delta) = \delta$.

Proof. Let $a_1 \in H^2(M, \mathbb{Z})$ and $a_2 \in H^2(S, \mathbb{Z})$ be the classes satisfying $c_1(\mathcal{U}) = \pi_M^*(a_1) + \pi_S^*(a_2)$. Let $e_1 \in H^2(M^{[n]}, \mathbb{Z})$ and $e_2 \in H^2(S^{[n]}, \mathbb{Z})$ be the classes satisfying $c_1(E) = \pi_{M^{[n]}}^*(e_1) + \pi_{S^{[n]}}^*(e_2)$. Set $R := n!r^n$. We have

$$\begin{aligned}
\widetilde{H}(\varphi^{[n]}) &= B_{-e_2/R} \circ \widetilde{H}(\widetilde{\Theta}_n(\phi)) \circ B_{-e_1/R} \\
&= \det(\phi)^{n+1} B_{-(\delta/2 + e_2/R)} \circ \tilde{\iota}_\phi \circ B_{\delta/2 - e_1/R} \\
&= \det(\phi)^{n+1} B_{(\theta(a_2)/r - \delta/2 - e_2/R)} \circ \iota_\varphi \circ B_{(\theta(a_1)/r + \delta/2 - e_1/R)} \\
&= \det(\phi)^{n+1} B_{\lambda_2} \circ \iota_\varphi \circ B_{\lambda_1},
\end{aligned}$$

where $\lambda_1 = \theta(a_1)/r + \delta/2 - e_1/R$ and $\lambda_2 = \theta(a_2)/r - \delta/2 - e_2/R$. The first equality follows from the definition of $\kappa(E)$, the second from Lemma 7.2, the third from the definition of $\kappa(\mathcal{U})$, and the fourth is obvious. Note that $\det(\phi) = \det(\varphi)$. In order to establish (7.9) it remains to prove that $\lambda_i = 0$, $i = 1, 2$, or, equivalently,

$$e_1 = n!r^n\left(\frac{\theta(a_1)}{r} + \frac{\delta}{2}\right), \tag{7.10}$$

$$e_2 = n!r^n\left(\frac{\theta(a_2)}{r} - \frac{\delta}{2}\right). \tag{7.11}$$

Both φ and $\varphi^{[n]}$ are degree reversing, by Lemma 4.1. Hence, both $\widetilde{H}(\varphi^{[n]})$ and ι_φ are degree reversing, the first by Lemma 3.4. If $\lambda_1 \neq 0$, then $(\lambda_1, \lambda_1') \neq 0$, for some $\lambda_1' \in H^2(M^{[n]}, \mathbb{Z})$ and the coefficient of β in $B_{\lambda_1}(\lambda_1')$ is non-zero, which implies that the coefficient of α in $\iota_\varphi(B_{\lambda_1}(\lambda_1'))$ is non-zero, and hence so is the coefficient of α in $\widetilde{H}(\varphi^{[n]})(\lambda_1')$. This contradicts the fact that $\widetilde{H}(\varphi^{[n]})$ maps $H^2(M^{[n]}, \mathbb{Q})$ to $H^2(S^{[n]}, \mathbb{Q})$. Hence, $\lambda_1 = 0$.

If $\lambda_2 \neq 0$, then $(\lambda_2, \lambda_2') \neq 0$, for some $\lambda_2' \in H^2(S^{[n]}, \mathbb{Z})$ and the coefficient of β in $B_{\lambda_2}(\lambda_2')$ is non-zero. There exists a class $\lambda_1' \in H^2(M^{[n]}, \mathbb{Q})$, such that $\iota_\varphi(\lambda_1') = \lambda_2'$, as ι_φ is an isometry. Hence, the coefficient of β in $\widetilde{H}(\varphi^{[n]})(\lambda_1')$ does not vanish. A contradiction. We conclude that $\lambda_2 = 0$.

Equality (7.9) implies that $\nu(\widetilde{H}(\varphi^{[n]})_0) = \det(\varphi)^{n+1}\nu(\widetilde{H}(\varphi)_0)$. Hence, ψ_E is the extension of $\psi_{\mathcal{U}}$ mapping $\delta \in H^2(M^{[n]}, \mathbb{Q})$ to $\delta \in H^2(S^{[n]}, \mathbb{Q})$. $\square$

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Let $\mathcal{G}^{[n]}$ be the subgroupoid of the groupoid $\mathcal{G}$ given in (1.1) whose objects are of $K3^{[n]}$ type. Let

$$\Theta_n : \mathcal{G}^{[1]} \to \mathcal{G}^{[n]} \tag{7.12}$$

be the following normalization of the functor $\widetilde{\Theta}_n$ defined in §7.2. The functor Θ_n sends an object of $\mathcal{G}^{[1]}$ to the same object that $\widetilde{\Theta}_n$ does. Given a morphism $\phi \in \mathrm{Hom}_{\mathcal{G}^{[1]}}((X, \epsilon), (Y, \epsilon'))$, we define

$$\Theta_n(\phi) = B_{\delta/2} \circ \widetilde{\Theta}_n(\phi) \circ B_{-\delta/2},$$

where we denote by δ half the class of the divisor of non-reduced subschemes in both $X^{[n]}$ and $Y^{[n]}$ and B_λ is the cup product with $\exp(\lambda)$. Note that if ϕ is $B_\lambda \in \mathrm{Aut}(X, \epsilon)$, $\lambda \in H^2(X, \mathbb{Q})$, or if ϕ is a parallel-transport operator, then $\Theta_n(\phi) = \widetilde{\Theta}_n(\phi)$.

COROLLARY 7.4. *The functor Θ_n maps $\mathcal{G}_{an}^{[1]}$ to $\mathcal{G}_{an}^{[n]}$. Furthermore, $\Theta_n(\varphi) = \varphi^{[n]}$, where φ and $\varphi^{[n]}$ are the morphisms in Corollary 7.3.*

Proof. The statement is clear for parallel-transport operators. Hence, it suffices to prove the equality $\Theta_n(\varphi) = \varphi^{[n]}$. Keep the notation of the proof of Corollary 7.3. The second equality below follows from (7.10) and (7.11):

$$\varphi^{[n]} = B_{-(e_2/R)} \circ \widetilde{\Theta}_n(\phi) \circ B_{-(e_1/R)} = B_{-(\theta(a_2/r))} \circ B_{\delta/2} \circ \widetilde{\Theta}_n(\phi) \circ B_{-\delta/2} \circ B_{-(\theta(a_1)/r)}$$

$$= \Theta_n(B_{-a_2/r}) \circ \Theta_n(\phi) \circ \Theta_n(B_{-a_1/r}) = \Theta_n(B_{-a_2/r} \circ \phi \circ B_{-a_1/r}) = \Theta_n(\varphi).$$

The equality $\Theta_n(\varphi) = \varphi^{[n]}$ follows. $\qquad\square$

8. Deforming the universal vector bundle over $M^{[n]} \times S^{[n]}$

We consider in this section a sequence of examples of two-dimensional moduli spaces M of stable vector bundles over a $K3$ surface S, such that the vector bundle E over $M^{[n]} \times S^{[n]}$, given in (7.2) as the Fourier–Mukai kernel of an equivalence of derived categories, can be deformed with $M^{[n]} \times S^{[n]}$ over the whole connected component $\mathfrak{M}_\psi^0$ of a moduli space of rational Hodge isometries introduced in §5.2.

Keep the notation of §7. Assume that $\mathrm{Pic}(S)$ is cyclic, generated by the class $h := c_1(H)$ of an ample line bundle H with $(h, h) = 2rs$, where r and s are positive integers and $\gcd(r, s) = 1$. The Mukai vector $v := (r, h, s)$ is then isotropic. The following properties of the moduli space M of H-stable sheaves with Mukai vector v are proved in [Muk99]. Here M is a $K3$ surface with a cyclic Picard group generated by an ample line bundle $\hat{H}$ with class $\hat{h} := c_1(\hat{H})$ satisfying $(\hat{h}, \hat{h}) = (h, h)$; M parametrizes locally free H-slope-stable sheaves. For each integer k, such that $ks \equiv 1 \pmod{r}$ there exists over $M \times S$ a universal vector bundle $\mathcal{U}$ with $c_1(\mathcal{U}) = \pi_M^*(k\hat{h}) + \pi_S^*(h)$. The universal vector bundle $\mathcal{U}$ restricts to an $\hat{H}$-slope-stable vector bundle on $M \times \{x\}$ of Mukai vector $\hat{v} := (r, k\hat{h}, k^2s)$, for each point $x \in S$, and the classification morphism induces an isomorphism between S and the moduli space $M_{\hat{H}}(\hat{v})$ of $\hat{H}$-stable sheaves on M, by [Muk99, Theorem 1.2].

PROPOSITION 8.1 [Bus19, Proposition 4.19][13]. *The class $-\kappa(\mathcal{U})\sqrt{td_{M \times S}}$ induces a degree-reversing rational Hodge isometry $\phi_\mathcal{U} : \widetilde{H}(M, \mathbb{Q}) \to \widetilde{H}(S, \mathbb{Q})$, which restricts to a rational Hodge isometry $\psi_\mathcal{U} : H^2(M, \mathbb{Q}) \to H^2(S, \mathbb{Q})$ satisfying $\psi_\mathcal{U}(\hat{h}) = h$, such that $\eta_S \circ \psi_\mathcal{U} \circ \eta_M^{-1}$ belongs to the double orbit $O(\Lambda_{K3})\rho_u O(\Lambda_{K3})$ of a reflection ρ_u in the co-rank-1 lattice $u^\perp \subset \Lambda_{K3}$ of a primitive class u with $(u, u) = 2r$.*

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[13] Buskin states it for $\kappa(\mathcal{U}^*)$, but the latter is equal to $\kappa(\mathcal{U})$, by Corollary 4.2.

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We claim that $\mathcal{U}$ is slope-stable with respect to every Kähler class on $M \times S$. It suffices to show it for every ample class, i.e. for $\omega := a\pi_S^*(h) + b\pi_M^*(\hat{h})$, for all $a > 0$ and $b > 0$. Let $\mathcal{W}$ be a saturated non-zero proper subsheaf of $\mathcal{U}$. Then $\mathcal{W}$ is reflexive, hence it is locally free away from a closed subvariety of dimension at most 1. It follows that the restriction of $\mathcal{W}$ to the generic fiber of π_S and of π_M is locally free. Let W_x (respectively, W_m) be such a restriction to the fiber over $x \in S$ (respectively, $m \in M$). Define U_x and U_m similarly in terms of $\mathcal{U}$. Then $c_1(\mathcal{W}) = \pi_M^* c_1(W_x) + \pi_S^* c_1(W_m)$ and $\omega^3 = a^2bh^2\hat{h} + ab^2h\hat{h}^2$. Set $\rho := \mathrm{rank}(\mathcal{W})$. We have

$$\mu_\omega(\mathcal{W}) = \frac{1}{\rho}c_1(\mathcal{W})\omega^3 = a^2b(h,h)\mu_{\hat{h}}(W_m) + ab^2(\hat{h},\hat{h})\mu_h(W_x)$$

$$< a^2b(h,h)\mu_{\hat{h}}(U_m) + ab^2(\hat{h},\hat{h})\mu_h(U_x) = \mu_\omega(\mathcal{U}).$$

Hence, $\mathcal{U}$ is slope-stable with respect to every Kähler class on $M \times S$. The universal vector bundle $\mathcal{U}$ thus satisfies the assumptions of Proposition 5.22 (namely Assumption 5.19 stated for E). Set $\psi := \eta_S \circ \psi_\mathcal{U} \circ \eta_M^{-1}$. Let $\mathfrak{M}_\psi^0$ be the connected component of $\mathfrak{M}_\psi$ through $q_0 := (M, \eta_M, S, \eta_S)$.

Choose a generic twistor line Q in $\mathfrak{M}_\psi^0$ through q_0 and let $(S', \eta_{S'}, M', \eta_{M'})$ be a quadruple in this line with $\mathrm{Pic}(S') = 0$. There exists classes $\zeta_{S'} \in H^2(S', \mathcal{O}_{S'}^*)$ and $\zeta_{M'} \in H^2(M', \mathcal{O}_{M'}^*)$ and a ζ-twisted locally free sheaf $\mathcal{U}'$ over $M' \times S'$, with $\zeta := \pi_{S'}^*(\zeta_{S'})\pi_{M'}^*(\zeta_{M'})$, such that $(M' \times S', \mathcal{E}nd(\mathcal{U}'))$ is deformation equivalent to $(M \times S, \mathcal{E}nd(\mathcal{U}))$, by Proposition 5.22. Moreover, $\mathcal{U}'$ is ω-slope-stable with respect to some Kähler class on $S' \times M'$, by Proposition 5.15. We note that the existence of such $\mathcal{U}'$ follows already from [Bus19, Theorem 5.1] in this case where $n = 1$. It follows that $\mathcal{U}'$ does not have any non-zero proper saturated subsheaf, since $\mathrm{Pic}(M' \times S')$ is trivial. Indeed, for every proper non-zero saturated ζ-twisted subsheaf F of $\mathcal{U}'$ the untwisted sheaf $\mathcal{H}om(\mathcal{U}', F)$ would have slope zero, since $\mathrm{Pic}(M' \times S')$ is trivial, and so F would slope-destabilize $\mathcal{U}'$ (Definition 5.14).

The construction of the sheaf E over $M^{[n]} \times S^{[n]}$, given in (7.2), has a natural relative extension over the chosen twistor line Q in $\mathfrak{M}_\psi^0$ through q_0 resulting in the twisted sheaf E' over $M'^{[n]} \times S'^{[n]}$ given by substituting $\mathcal{U}'$ for $\mathcal{U}$ in (7.2). The check that (7.2) is still meaningful in the twisted setting reduces to checking that $(q \times q)_*^{\mathfrak{S}_n, \Delta}$ is well defined. This is the case, since $\mathcal{U}'^{\boxtimes n}$ is $\zeta^{\boxtimes n}$-twisted, and $(b \times b)^*(\zeta^{\boxtimes n})$ is the pullback via $q \times q$ of a Brauer class on $M'^{[n]} \times S'^{[n]}$ (which is itself the pullback of a Brauer class on $M'^{(n)} \times S'^{(n)}$), as explained in [Mar21a, §10, Proof of Theorem 1.4].

Let $\psi_{\mathcal{U}'} : H^2(M', \mathbb{Q}) \to H^2(S', \mathbb{Q})$ and $\psi_{E'} : H^2(M'^{[n]}, \mathbb{Q}) \to H^2(S'^{[n]}, \mathbb{Q})$ be the parallel transports of $\psi_\mathcal{U}$ and $\psi_\mathcal{E}$ in the relevant local systems over the chosen twistor line Q.

LEMMA 8.2. *The isometry $\psi_{E'}$ maps some Kähler class ω_1 on $M'^{[n]}$ to a Kähler class on $S'^{[n]}$.*

Proof. The isometry $\psi_{\mathcal{U}'}$ maps some Kähler class $\omega_{M'}$ to a Kähler class $\omega_{S'}$, as $(S', \eta_{S'}, M', \eta_{M'})$ is a point of $\mathfrak{M}_\psi$. The equalities $\psi_{E'} \circ \theta = \theta \circ \psi_{\mathcal{U}'}$ and $\psi_{E'}(\delta) = \delta$ follow from Corollary 7.3. The Picard groups of both Douady spaces $M'^{[n]}$ and $S'^{[n]}$ are cyclic generated by the classes δ and their cones of effective curves are thus rays spanned by the class $\delta^\vee := 2(\delta, \bullet)/(n-1)$ in $H^2(M'^{[n]}, \mathbb{Q})^*$ (were we identify $H_2(M'^{[n]}, \mathbb{Q})$ with $H^2(M'^{[n]}, \mathbb{Q})^*$), by [Mar13, Corollary 3.6]. Choose a sufficiently large real number t, such that $(t\theta(\omega_{M'}) - \delta, t\theta(\omega_{M'}) - \delta) > 0$. Then $\omega_1 := t\theta(\omega_{M'}) - \delta$ is a class in the connected component of the positive cone in $H^{1,1}(M'^{[n]}, \mathbb{R})$, which contains the Kähler cone, as it pairs positively with $\theta(\omega_{M'})$, and ω_1 pairs positively with the class of every effective curve. Hence, ω_1 is a Kähler class, by [Huy03b, Corollary 3.4]. Now, $\psi_{E'}(\omega_1) = t\theta(\omega_{S'}) - \delta$ is a Kähler class, by the same reason. $\square$

Let $\eta_{S'^{[n]}}$ be the marking of $S'^{[n]}$ induced by extending the marking $\eta_{S^{[n]}}$ given in (7.8) to a trivialization of the local system over the chosen twistor path Q. Define $\eta_{M'^{[n]}}$ similarly in terms of $\eta_{M^{[n]}}$. Set $\psi^{[n]} := \eta_{S'^{[n]}}^{-1}\psi_{E'}\eta_{M'^{[n]}}$. Let

$$\mathfrak{M}_{\psi^{[n]}}^0 \tag{8.1}$$

be the connected component of $\mathfrak{M}_{\psi^{[n]}}$ containing the quadruple $(M'^{[n]}, \eta_{M'^{[n]}}, S'^{[n]}, \eta_{S'^{[n]}})$.

LEMMA 8.3. *The twisted vector bundle E' is slope-stable with respect to every Kähler class on $M'^{[n]} \times S'^{[n]}$.*

Proof. We have already noted that $\mathcal{U}'$ does not have any non-zero proper saturated subsheaf. Here and in the following, by a subsheaf of a twisted sheaf we mean a twisted subsheaf with respect to the same co-cycle representing the Brauer class. Hence, $\mathcal{U}'^{\boxtimes n}$ does not have any non-zero proper saturated subsheaves, by [Mar21a, Proposition 10.1]. Given $\sigma \in \mathfrak{S}_n$, let $(1 \times \sigma) : M^n \times S^n \to M^n \times S^n$ act as the identity on M^n and via σ on S^n. The vector bundles $\mathcal{U}^{\boxtimes n}$ and $(1 \times \sigma)^*\mathcal{U}^{\boxtimes n}$ are non-isomorphic, for every $\sigma \in \mathfrak{S}_n$, $\sigma \neq 1$. Furthermore, $\mathrm{Hom}(\mathcal{U}^{\boxtimes n}, (1 \times \sigma)^*\mathcal{U}^{\boxtimes n}) = 0$, for such σ. Now the dimensions of $\mathrm{Hom}(\mathcal{U}^{\boxtimes n}, (1 \times \sigma)^*\mathcal{U}^{\boxtimes n})$ and $\mathrm{Hom}(\mathcal{U}'^{\boxtimes n}, (1 \times \sigma)^*\mathcal{U}'^{\boxtimes n})$ are equal, by the invariance of the dimension of sheaf cohomologies of hyperholomorphic sheaves under twistor deformations [Ver99b, §3.5]. Hence,

$$\oplus_{\tau \in \mathfrak{S}_n}(1 \times \tau)^*(\mathcal{U}'^{\boxtimes n})$$

does not have any non-zero proper saturated $\mathfrak{S}_n$-invariant subsheaf, by [Mar21a, Lemma 11.5]. The same is true for $(b \times b)^*\mathcal{U}'^{\boxtimes n}$, as $b \times b$ is a birational morphism.

Consider the following commutative diagram.

$$
\begin{array}{ccc}
(q \times q)^*(q \times q)_*^{\mathfrak{S}_{n,\Delta}}(b \times b)^*\mathcal{U}'^{\boxtimes n} & \xrightarrow{p \circ ev \circ e} & \oplus_{\tau \in \mathfrak{S}_n}(1 \times \tau)^*(b \times b)^*\mathcal{U}'^{\boxtimes n} \\
\downarrow{e} & & \uparrow{p} \\
(q \times q)^*(q \times q)_*(b \times b)^*\mathcal{U}'^{\boxtimes n} & \xrightarrow{ev} & \oplus_{(\sigma,\tau) \in \mathfrak{S}_n \times \mathfrak{S}_n}(\sigma \times \tau)^*(b \times b)^*\mathcal{U}'^{\boxtimes n}
\end{array}
$$

Here e is the natural injective homomorphism from the $\mathfrak{S}_{n,\Delta}$-invariant subsheaf with respect to the linearization $\chi \otimes \rho_{\boxtimes}$ of $\mathcal{U}'^{\boxtimes n}$. The right arrow p is the projection on the subset of direct summands. For each $(\sigma, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ we have the evaluation homomorphism from the bottom left vector bundle to the corresponding direct summand of the bottom right vector bundle due to the equality $(q \times q) \circ (\sigma \times \tau) = (q \times q)$. The homomorphism ev is the resulting diagonal homomorphism. The homomorphism ev restricts to an isomorphism over the complement of the ramification divisor of $q \times q$.

The homomorphism e is $\mathfrak{S}_{n,\Delta}$-equivariant. The homomorphism ev is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant. The homomorphism p is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant, where the latter group acts on the top right vector bundle via the projection $\mathfrak{S}_n \times \mathfrak{S}_n \to \mathfrak{S}_n$ onto the second factor. It follows that the homomorphism $p \circ ev \circ e$ is $\mathfrak{S}_{n,\Delta}$-equivariant, where the latter group acts via the isomorphism $\mathfrak{S}_{n,\Delta} \to \mathfrak{S}_n$, given by $(\sigma, \sigma) \mapsto \sigma$ on the top right vector bundle.

The homomorphism $p \circ ev \circ e$ restricts to an isomorphism over the complement of the ramification divisor of $q \times q$. Hence, the pullback via $q \times q$ of any saturated non-zero proper subsheaf of $(q \times q)_*^{\mathfrak{S}_{n,\Delta}}(b \times b)^*\mathcal{U}'^{\boxtimes n}$ maps via $p \circ ev \circ e$ to an $\mathfrak{S}_n$-invariant non-zero subsheaf of the top right vector bundle, whose saturation is a proper subsheaf. We have seen that such a subsheaf does not exist. It follows that $(q \times q)_*^{\mathfrak{S}_{n,\Delta}}(b \times b)^*\mathcal{U}'^{\boxtimes n}$ does not have any saturated non-zero proper

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subsheaf. Hence, neither does E', as the latter is the tensor product of the former with a line bundle. □

THEOREM 8.4. *Every quadruple* $(X_1, \eta_1, X_2, \eta_2)$ *in the component* $\mathfrak{M}^0_{\psi^{[n]}}$, *given in* (8.1), *supports a compatible vector bundle (Definition 5.21).*

Proof. Lemma 8.2 verifies Assumption 5.19(2) for the locally free sheaf E' supported by the quadruple $(M'^{[n]}, \eta_{M'^{[n]}}, S'^{[n]}, \eta_{S'^{[n]}})$ in the component $\mathfrak{M}^0_{\psi^{[n]}}$. Lemma 8.3 verifies the stability condition in Assumption 5.19(3) for E'. The statement thus follows from Proposition 5.22. □

Let Λ_{K3} be the $K3$ lattice, Λ the lattice given in (7.6), and $\iota : O(\Lambda_{K3}) \to O(\Lambda)$ the inclusion (7.7). Let $u \in \Lambda_{K3}$ be a primitive element with $(u, u) > 0$. Let ρ_u be the reflection in the co-rank-1 sublattice $u^\perp \subset \Lambda_{K3}$. Let $\mathrm{Mon}(\Lambda) \subset O^+(\Lambda)$ be the subgroup which acts on the discriminant group Λ^*/Λ via ± 1.

Let $\mathfrak{M}^0_\Lambda$ be the connected component of the moduli space $\mathfrak{M}_\Lambda$ of marked pairs (X, η), with X of $K3^{[n]}$ type, containing the marked pair $(S^{[n]}, \eta_{S^{[n]}})$, given in (7.8).

COROLLARY 8.5. *Let* $\tilde\psi \in O^+(\Lambda_\mathbb{Q})$ *be an isometry in the double orbit* $\mathrm{Mon}(\Lambda)(-\iota(\rho_u))\mathrm{Mon}(\Lambda)$. *There exists a non-empty connected component* $\mathfrak{M}^0_{\tilde\psi}$ *of* $\mathfrak{M}_{\tilde\psi}$ *parametrizing quadruples* $(X_1, \eta_1, X_2, \eta_2)$, *with* (X_1, η_1) *and* (X_2, η_2) *in* $\mathfrak{M}^0_\Lambda$, *such that each quadruple in* $\mathfrak{M}^0_{\tilde\psi}$ *supports a compatible vector bundle (Definition 5.21).*

Proof. The double orbit $O^+(\Lambda_{K3})(-\rho_u)O^+(\Lambda_{K3})$ depends only on (u, u), by [Bus19, Proposition 3.3]. Hence, the double orbit $\mathrm{Mon}(\Lambda)(-\iota(\rho_u))\mathrm{Mon}(\Lambda)$ depends only on (u, u), and so we may assume that $\tilde\psi = \gamma_1 \iota(\eta_S \psi_\mathcal{U} \eta_M^{-1})\gamma_0$, for some $\gamma_1, \gamma_2 \in \mathrm{Mon}(\Lambda)$, where $\psi_\mathcal{U}$ is the isometry in Proposition 8.1. Set $\psi := \eta_S \psi_\mathcal{U} \eta_M^{-1}$. The isometry $\psi^{[n]}$, given in (8.1), is equal to $\iota(\psi)$, by Corollary 7.3, and so $\tilde\psi = \gamma_2 \psi^{[n]} \gamma_1$. We get the isomorphism $f : \mathfrak{M}_{\psi^{[n]}} \to \mathfrak{M}_{\tilde\psi}$ given by

$$f(X_1, \eta_1, X_2, \eta_2) = (X_1, \gamma_1^{-1}\eta_1, X_2, \gamma_2\eta_2),$$

which maps connected components to connected components, since $\eta_i^{-1}\mathrm{Mon}(\Lambda)\eta_i$ is the monodromy group $\mathrm{Mon}^2(X_i)$ of X_i, $i = 1, 2$, by [Mar08, Theorem 1.6 and Lemma 4.10]. Furthermore, the marked pair $(X_1, \gamma_1^{-1}\eta_1)$ belongs to the connected component $\mathfrak{M}^0_\Lambda$ of (X_1, η_1), since γ_1^{-1} belongs to $\mathrm{Mon}(\Lambda)$ and $\mathrm{Mon}(\Lambda) = \eta_1\mathrm{Mon}^2(X_1)\eta_1^{-1}$. Similarly, $(X_2, \gamma_2\eta_2)$ belongs to the same connected component $\mathfrak{M}^0_\Lambda$ of (X_2, η_2). Clearly, if $q \in \mathfrak{M}_{\psi^{[n]}}$ supports a compatible vector bundle, then $f(q)$ supports (the same) compatible vector bundle. The statement now follows from Theorem 8.4. □

9. Algebraicity of a rational Hodge isometry

We prove Theorems 1.1 and 1.4 in this section. Let $\mathcal{G}_{an}$ be the subgroupoid of $\mathcal{G}$ given in (1.2). Note that morphisms in $\mathrm{Hom}_{\mathcal{G}_{an}}((X, \epsilon), (Y, \epsilon'))$ are induced by analytic correspondences in $X \times Y$ and are degree preserving up to sign.

THEOREM 9.1. *Let* X *and* Y *be IHSMs of* $K3^{[n]}$ *type and let* $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ *be a rational Hodge isometry. For every pair of orientations* ϵ *and* ϵ' *there exists a morphism* ϕ *in* $\mathrm{Hom}_{\mathcal{G}_{an}}((X, \epsilon), (Y, \epsilon'))$, *such that* $\widetilde{H}_0(\phi)$ *is equal to* f *or* $-f$.

Proof. The set $\mathrm{Hom}_{\mathcal{G}_{an}}((X, \epsilon), (Y, \epsilon'))$ is independent of the orientations as is the functor $\widetilde{H}_0$, so the existence of ϕ is independent of the orientations. We may assume that $\nu(f) = 1$, possibly after replacing f by $-f$. Choose markings η_X and η_Y, such that (X, η_X) and (Y, η_Y) belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$. Choose an orientation of Λ and endow $H^2(X, \mathbb{Z})$ and

$H^2(Y, \mathbb{Z})$ with the orientations ϵ and ϵ', so that η_X and η_Y are orientation preserving. We write $\mathrm{Hom}_{\mathcal{G}_{an}}(X, Y)$ instead of $\mathrm{Hom}_{\mathcal{G}_{an}}((X, \epsilon), (Y, \epsilon'))$, as in Convention 5.16, and we will continue to use the convention in the following argument. Set $\psi := \eta_Y \circ f \circ \eta_X^{-1}$. Then ψ belongs to $O^+(\Lambda_{\mathbb{Q}})$ and so it decomposes as

$$\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1,$$

where each ψ_i belongs to the double orbit $\mathrm{Mon}(\Lambda)\iota(-\rho_u)\mathrm{Mon}(\Lambda)$, for some element $u \in \Lambda_{K3}$ with $(u, u) > 0$, by Corollary 2.5. The proof proceeds by induction on k.

Case $k = 0$. In this case $\psi = id$ and $f = \eta_Y^{-1}\eta_X$ is a Hodge isometry. There exists a parallel-transport operator $\phi : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$, which is a morphism in $\mathrm{Hom}_{\mathcal{G}_{an}}(X, Y)$, and such that ϕ restricts to $H^2(X, \mathbb{Q})$ as f, by Theorem 5.2. In this case $\widetilde{H}_0(\phi)$ is also equal to the restriction of ϕ to $H^2(X, \mathbb{Q})$, by Remark 5.18.

Assume that $k \geq 1$ and that the statement holds for $k - 1$. Set $\psi' = \psi_{k-1} \circ \cdots \circ \psi_1$, so that $\psi = \psi_k \psi'$. Set $\ell_0 := P(X, \eta_X)$, $\ell_{k-1} := \psi'(\ell_0)$, and $\ell_k = \psi(\ell_0)$. Note that $\ell_k = P(Y, \eta_Y)$. Let (X_{k-1}, η_{k-1}) be a marked pair in $\mathfrak{M}_\Lambda^0$, such that $P(X_{k-1}, \eta_{k-1}) = \ell_{k-1}$. Such a marked pair exists, by the surjectivity of the period map [Huy99]. Set $f' := \eta_{k-1}^{-1} \circ \psi' \circ \eta_X$. There exists a morphism $\phi' \in \mathrm{Hom}_{\mathcal{G}_{an}}(X, X_{k-1})$, such that $\widetilde{H}_0(\phi') = f'$, by the induction hypothesis. The quadruple $(X_{k-1}, \eta_{k-1}, Y, \eta_Y)$ satisfies the condition that $\eta_Y^{-1}\psi_k\eta_{k-1}$ is a Hodge isometry, since

$$\eta_Y^{-1}\psi_k\eta_{k-1}(H^{2,0}(X_{k-1})) = \eta_Y^{-1}(\psi_k(\ell_{k-1})) = \eta_Y^{-1}(\psi(\ell_0)) = \eta_Y^{-1}(P(Y, \eta_Y)) = H^{2,0}(Y),$$

but it may not belong to $\mathfrak{M}_{\psi_k}$, since $\eta_Y^{-1}\psi_k\eta_{k-1}$ need not map a Kähler class to a Kähler class.

There exists a connected component $\mathfrak{M}_{\psi_k}^0$ of $\mathfrak{M}_{\psi_k}$ in which every quadruple supports a compatible vector bundle, and such that $\Pi_i : \mathfrak{M}_{\psi_k}^0 \to \mathfrak{M}_\Lambda$ has image in the connected component $\mathfrak{M}_\Lambda^0$, for $i = 1, 2$, by Corollary 8.5. Extend the pair (X_{k-1}, η_{k-1}) to a quadruple $q := (X_{k-1}, \eta_{k-1}, Y', \eta_{Y'})$ in $\mathfrak{M}_{\psi_k}^0$. Such a point q exists, by the surjectivity lemma (Lemma 5.7). Let F_q be the compatible vector bundle supported by q. Set $\phi_k' := \left[\kappa(F_q)\sqrt{td_{X_{k-1} \times Y'}}\right]_* : H^*(X_{k-1}, \mathbb{Q}) \to H^*(Y', \mathbb{Q})$. Then $\widetilde{H}_0(\phi_k') = \eta_{Y'}^{-1} \circ \psi_k \circ \eta_{k-1}$, by Definition 5.21 of the compatibility of F_q. We have $P(Y', \eta_{Y'}) = \psi_k(P(X_{k-1}, \eta_{k-1})) = P(Y, \eta_Y)$, where the first equality is by Lemma 5.4. Hence, there exists a morphism $\phi_k \in \mathrm{Hom}_{\mathcal{G}_{an}}(Y', Y)$, such that $\widetilde{H}_0(\phi_k) = \eta_Y^{-1}\eta_{Y'}$, by the case $k = 0$ of the statement. We conclude that $\phi := \phi_k \circ \phi_k' \circ \phi'$ is a morphism in $\mathrm{Hom}_{\mathcal{G}_{an}}(X, Y)$, and $\widetilde{H}_0(\phi) = f$. $\square$

Proof of Theorem 1.4. The theorem is a rephrasing of Theorem 9.1. $\square$

Proof of Theorem 1.1. Let $f : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ be a Hodge isometry. We may assume that $\nu(f) = 1$, possibly after replacing f by $-f$ (and $\tilde{f}$ by $-\tilde{f}$ in the statement of the theorem). There exist orientations ϵ and ϵ' and a morphism $\phi \in \mathrm{Hom}_{\mathcal{G}_{an}}((X, \epsilon), (Y, \epsilon'))$, such that $\widetilde{H}_0(\phi) = f$, by Theorem 9.1. Morphisms of $\mathcal{G}_{an}$ are induced by analytic correspondences, as already noted in the paragraph following (1.2) defining this groupoid. If ϕ is degree preserving, then $\widetilde{H}_0(\phi)$ is a rational scalar multiple of the restriction of ϕ to $H^2(X, \mathbb{Q})$, by Lemma 3.6, and so we choose $\tilde{f}$ to be the appropriate scalar multiple of ϕ. If ϕ is degree reversing, then setting $\tilde{f} = \phi$ we get an analytic correspondence with the desired properties, by Lemma 3.7. $\square$

10. The Pontryagin product on the cohomology of IHSMs of $K3^{[n]}$ type

In § 10.1 we introduce the Pontryagin product on the cohomology of an IHSM with vanishing odd cohomology. The results still apply to the even cohomology if the odd cohomology does not vanish. In § 10.2 we prove Conjecture 1.8 for morphisms in the image of the functor Θ_n:

$\mathcal{G}_{an}^{[1]} \to \mathcal{G}_{an}^{[n]}$, given in (7.12). In particular, morphisms in the image of Θ_n, once normalized as in Conjecture 1.8, conjugate the cup product to itself, if they are degree preserving, and to the Pontryagin product if they are degree reversing.

10.1 The Pontryagin product for an IHSM with vanishing odd cohomology

Let X be an IHSM, let $\mathfrak{g} \subset \mathfrak{sl}(H^*(X, \mathbb{Q}))$ be its LLV algebra, and let $h \in \mathfrak{g}$ be the grading operator given in (3.1). The commutator of h in $\mathfrak{g}$ is $\bar{\mathfrak{g}} \oplus \mathbb{Q} \cdot h$ and composing the isomorphism $\dot{\rho}^{-1} : \mathfrak{g} \cong \mathfrak{so}(\widetilde{H}(X, \mathbb{Q}))$, given in (3.2), with restriction to $H^2(X, \mathbb{Q})$ yields an isomorphism $\bar{\mathfrak{g}} \cong \mathfrak{so}(H^2(X, \mathbb{Q}))$, by Lemma 3.5. Assume that the odd cohomology of X vanishes. If the second Betti number $b_2(X)$ of X is even, assume that the monodromy group of X contains reflections and $\dim(X) = 2n$ for odd n. The latter assumption holds true for all currently known examples of IHSMs with even $b_2(X)$ ($K3$ surfaces and IHSMs of O'Grady 6 deformation type). The $\mathfrak{g}$ action on $H^*(X, \mathbb{C})$ integrates to an action of $SO(\widetilde{H}(X, \mathbb{C}))$ and the commutator of h is $\mathbb{C}^\times \times SO(H^2(X, \mathbb{C}))$, where an element t of the first factor $\mathbb{C}^\times$ maps α to $t\alpha$ and β to $(1/t)\beta$. The image of the second factor $SO(H^2(X, \mathbb{C}))$ in $GL(H^*(X, \mathbb{C}))$ acts on $H^*(X, \mathbb{C})$ via ring automorphisms, by [LL97, Proposition 4.5(ii)]. The coset $\{g \in SO(\widetilde{H}(X, \mathbb{C})) : ghg^{-1} = -h\}$ of the commutator is the product of the component of $O(\mathbb{C}\alpha \oplus \mathbb{C}\beta)$ with determinant -1 with the component of $O(H^2(X, \mathbb{C}))$ with determinant -1. If the monodromy group contains an element R of order 2 acting as a reflection on $H^2(X, \mathbb{C})$, then the factor $O(H^2(X, \mathbb{C}))$ acts on $H^*(X, \mathbb{C})$ via ring automorphisms. Such a monodromy operator R acts on $\widetilde{H}(X, \mathbb{Q})$ via the operator $\widetilde{H}(R)$ stabilizing both α and β and restricting to $H^2(X, \mathbb{Q})$ as the monodromy reflection. A choice of such an element $R \in \mathrm{Mon}(X)$ yields an action of $O(\widetilde{H}(X, \mathbb{C}))$ on $H^*(X, \mathbb{C})$

$$\rho : O(\widetilde{H}(X, \mathbb{C})) \to GL(H^*(X, \mathbb{C})) \tag{10.1}$$

extending the above homomorphism $\rho : SO(\widetilde{H}(X, \mathbb{C})) \to GL(H^*(X, \mathbb{C}))$ and satisfying $\rho(\widetilde{H}(R)) = R$.

Let τ be the element of $O(\widetilde{H}(X, \mathbb{Q}))$ interchanging α and β and acting on $H^2(X, \mathbb{C})$ via multiplication by -1. If $b_2(X)$ is odd, then τ belongs to $SO(\widetilde{H}(X, \mathbb{Q}))$ and ρ_τ is well defined, and if $b_2(X)$ is even we define ρ_τ via (10.1). Define the *Pontryagin product* $\star$ on $H^*(X, \mathbb{Q})$ by

$$\gamma \star \delta := \rho_\tau(\rho_\tau(\gamma) \cup \rho_\tau(\delta)). \tag{10.2}$$

Remark 10.1. The subring $SH^*(X, \mathbb{C})$ of $H^*(X, \mathbb{C})$ generated by $H^2(X, \mathbb{C})$ is invariant under monodromy operators and it is an LLV subrepresentation, hence it is invariant under elements in the image of ρ. The defining (10.2) implies that $SH^*(X, \mathbb{C})$ is a subring with respect to the Pontryagin product.

The values of the representation ρ on the connected component with determinant -1 depends on our choice of the monodromy reflection R (see [Mar08, Lemma 4.13]). Thus, if $b_2(X)$ is even, then ρ_τ depends on the choice of R. Nevertheless, the Pontryagin product does not depend on this choice as we show next (and generalize in Remark 10.5).

LEMMA 10.2.

(1) *The Pontryagin product $\star$ is independent of the choice of the monodromy reflection R.*
(2) *Parallel-transport operators are isomorphisms of the Pontryagin rings.*
(3) *The unit with respect to $\star$ is $\rho_\tau(1)$ and is independent of the choice of R.*

Proof. (1) The statement is clear if $b_2(X)$ is odd. Assume that $b_2(X)$ is even. Let $R' \in \mathrm{Mon}(X)$ be another monodromy reflection and let $\rho' : O(\widetilde{H}(X, \mathbb{C})) \to GL(H^*(X, \mathbb{C}))$ be the resulting representation. Let $f, f' \in SO(\widetilde{H}(X, \mathbb{C}))$ be the elements satisfying $\tau = f\widetilde{H}(R) = f'\widetilde{H}(R')$. Then

$\rho_{f'} = \rho'_{f'}$, as the restrictions of ρ and ρ' to $SO(\widetilde{H}(X,\mathbb{C}))$ are equal. Now, $f' = f\widetilde{H}(R)\widetilde{H}(R') = f\widetilde{H}(RR')$. Thus,

$$\rho'_\tau = \rho_{f'}R' = \rho_{f\widetilde{H}(RR')}R' = \rho_f\rho_{\widetilde{H}(RR')}R' = \rho_f RR\rho_{\widetilde{H}(RR')}R' = \rho_\tau(R\rho_{\widetilde{H}(RR')}R').$$

Set $g := R\rho_{\widetilde{H}(RR')}R'$. Then $\rho'_\tau = \rho_\tau g$, and g is an automorphism of $H^*(X,\mathbb{C})$ with respect to cup product, since each of R, $\rho_{\widetilde{H}(RR')}$, and R' is. Furthermore, $\rho_\tau g\rho_\tau g = (\rho'_\tau)^2 = id$, so $g\rho_\tau g = \rho_\tau$. Now compute

$$\rho'_\tau(\rho'_\tau(\gamma) \cup \rho'_\tau(\delta)) = (\rho_\tau g)(\rho_\tau(g(\gamma)) \cup \rho_\tau(g(\delta))) = \rho_\tau((g\rho_\tau g)(\gamma) \cup (g\rho_\tau g)(\delta))$$
$$= \rho_\tau(\rho_\tau(\gamma) \cup \rho_\tau(\delta)) = \gamma \star \delta.$$

(2) Let $f : H^*(X,\mathbb{C}) \to H^*(Y,\mathbb{C})$ be a parallel-transport operator. We get the commutative diagram

$$\begin{array}{ccc}
SO(\widetilde{H}(X,\mathbb{C})) & \longrightarrow & GL(H^*(X,\mathbb{C})) \\
{\scriptstyle Ad_{\widetilde{H}(f)}}\Big\downarrow & & \Big\downarrow{\scriptstyle Ad_f} \\
SO(\widetilde{H}(Y,\mathbb{C})) & \longrightarrow & GL(H^*(Y,\mathbb{C}))
\end{array}$$

where the horizontal arrows are the integration of the infinitesimal LLV Lie algebras actions. Clearly, $\widetilde{H}(f)$ conjugates τ to τ. Parallel-transport operators conjugate a monodromy reflection to such, and so the representation ρ, given in (10.1), get conjugated via the pair f and $\widetilde{H}(f)$ to $\rho' : O(\widetilde{H}(Y,\mathbb{C})) \to GL(H^*(Y,\mathbb{C}))$, as in part (1), and so f conjugates ρ_τ to ρ'_τ. The statement of part (2) now follows from part (1).

(3) The unit of a ring is unique, and so $\rho_\tau(1)$ is independent of R, by part (1). $\qquad\square$

We calculate the unit explicitly next.

Lemma 10.3. *The unit of $H^*(X,\mathbb{Q})$ with respect to the $\star$ product is $c_X[pt]/n!$, where $\dim(X) = 2n$ and c_X is the Fujiki constant.*

Proof. Consider the $SO(\widetilde{H}(X,\mathbb{Q}))$-equivariant embedding

$$\Psi : SH^*(X,\mathbb{Q}) \to \mathrm{Sym}^n(\widetilde{H}(X,\mathbb{Q})),$$

given in (3.3), where $SO(\widetilde{H}(X,\mathbb{Q}))$ acts on $SH^*(X,\mathbb{Q})$ via the representation ρ integrating the infinitesimal LLV action and via the natural representation on $\mathrm{Sym}^n(\widetilde{H}(X,\mathbb{Q}))$. If $b_2(X)$ is odd, then τ belongs to $SO(\widetilde{H}(X,\mathbb{C}))$. If $b_2(X)$ is even, then X admits monodromy reflections, by assumptions, and Ψ is equivariant with respect to those as well, and so Ψ is equivariant with respect to the ρ_τ action on its domain and the τ action of its codomain. The element 1 belongs to $SH^*(X,\mathbb{C})$ and $\Psi(1) = \alpha^n/n!$, by [Tae23, Proposition 3.5]. Now, $\tau(\alpha) = \beta$ and so $\Psi(\rho_\tau(1)) = \rho_\tau(\Psi(1)) = \beta^n/n! = \Psi(c_X[pt]/n!)$, where the last equality follows from [Tae23, Lemma 3.6]. Hence, $\rho_\tau(1) = c_X[pt]/n!$. $\qquad\square$

Let μ_t be the element of $SO(\mathbb{C}\alpha \oplus \mathbb{C}\beta)$, given by $\mu_t(\alpha) = t^{-1}\alpha$ and $\mu_t(\beta) = t\beta$, and μ_t restricts to $H^2(X,\mathbb{C})$ as the identity. Then ρ_{μ_t} acts on $H^{2k}(X,\mathbb{C})[2n]$ via multiplication by t^k.

Lemma 10.4. *Let g be an element of $O(\widetilde{H}(X,\mathbb{C}))$.*

(1) *If $ghg^{-1} = h$, then $g(\alpha) = t\alpha$, for some $t \in \mathbb{C}^\times$, and $\rho_{\mu_t g}$ is an automorphism of the cohomology ring $H^*(X,\mathbb{C})$ with respect to the usual cup product.*

(2) *If $ghg^{-1} = -h$, then $g(\alpha) = t^{-1}\beta$, for some $t \in \mathbb{C}^\times$, and*

$$\rho_{\mu_t g}(\gamma \cup \delta) = \rho_{\mu_t g}(\gamma) \star \rho_{\mu_t g}(\delta),$$

for all $\gamma, \delta \in H^*(X, \mathbb{C})$.

Proof. (1) If $ghg^{-1} = h$ and $g(\alpha) = t\alpha$, then $\mu_t g$ belongs to the factor $OH^2(X, \mathbb{C})$ of the commutator of h, and so $\rho_{\mu_t g}$ acts by ring automorphisms.

(2) If $ghg^{-1} = -h$ and $g(\alpha) = t^{-1}\beta$, then $f := \tau\mu_t g$ commutes with h and the second equality below follows from part (1):

$$\rho_{\mu_t g}(\gamma \cup \delta) = \rho_{\tau f}(\gamma \cup \delta) = \rho_\tau(\rho_f(\gamma) \cup \rho_f(\delta)) = \rho_\tau(\rho_\tau(\rho_{\tau f}(\gamma)) \cup \rho_\tau(\rho_{\tau f}(\delta)))$$
$$= \rho_{\tau f}(\gamma) \star \rho_{\tau f}(\delta) = \rho_{\mu_t g}(\gamma) \star \rho_{\mu_t g}(\delta). \qquad \square$$

Remark 10.5. Lemma 10.4(2) equivalently states that the same Pontryagin product can be defined in terms of any degree-reversing isometry $g \in O(\widetilde{H}(X, \mathbb{Q}))$, by

$$\gamma \star \delta = \rho_{g^{-1}\mu_{t^{-1}}}(\rho_{\mu_t g}(\gamma) \cup \rho_{\mu_t g}(\delta)).$$

10.2 The Pontryagin product for an IHSM of $K3^{[n]}$ type

Assume from now on that X is of $K3^{[n]}$ type. The following lemma states that the functor Θ_n, given in (7.12), integrates the infinitesimal LLV Lie algebra action when restricted to automorphisms of determinant 1.

LEMMA 10.6. *The restriction of Θ_n to the subgroup $S\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$, of elements of $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$ of determinant 1, is equal to the restriction of the composition of $\tilde{\iota} : SO(\widetilde{H}(S, \mathbb{C})) \to SO(\widetilde{H}(S^{[n]}, \mathbb{C}))$, given in (7.5), with the representation ρ given in (10.1).*

Proof. The infinitesimal LLV action $\dot{\rho}$, given in (3.2), sends $e_{\theta(\lambda)} \in \mathfrak{so}(\widetilde{H}(S^{[n]}, \mathbb{Q}))$ to cup product with $\theta(\lambda)$, by the $\mathfrak{so}(\widetilde{H}(S^{[n]}, \mathbb{Q}))$-equivariance of Ψ in (3.3). Thus, the equality $\Theta_n(\phi) = \rho_{\tilde{\iota}(\phi)}$ holds, when ϕ is cup product with $\exp(e_\lambda)$, $\lambda \in H^2(S, \mathbb{Q})$, by the definition of Θ_n. The map $\Theta_n : \mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon) \to \mathrm{Aut}_{\mathcal{G}^{[1]}}(S^{[n]}, \epsilon^{[n]})$ is induced by an algebraic map, due to its topological nature via Grothendieck–Riemann–Roch, and so extends to the Zariski closure $O(\widetilde{H}(S, \mathbb{C}))$ of $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$. The statement thus reduces to the equality of the differentials of Θ_n and $\rho \circ \tilde{\iota}$. We already established the equality $d\Theta_n(e_\lambda) = \dot{\rho}(d\tilde{\iota}(e_\lambda)) = \dot{\rho}(e_{\theta(\lambda)})$. We denote $\dot{\rho}(e_{\theta(\lambda)})$ by $e_{\theta(\lambda)}$ as well, so that the latter equality becomes

$$d\Theta_n(e_\lambda) = e_{\theta(\lambda)}. \tag{10.3}$$

Choose (M, ϵ') and $\phi \in \mathrm{Hom}_{\mathcal{G}^{[1]}}((S, \epsilon), (M, \epsilon'))$, so that ϕ is degree reversing. We can choose, for example, $(M, \epsilon') = (S, \epsilon)$ and $\phi := [ch(\mathcal{I}_\Delta)\sqrt{td_{S \times S}}]_*$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal. Note that $\phi = \widetilde{H}(\phi)$, as $H^*(S, \mathbb{Q}) = \widetilde{H}(S, \mathbb{Q})$ and similarly for M. Let $t \in \mathbb{Q}$, so that $\phi(\beta) = t\alpha$. Let $\lambda \in H^2(S, \mathbb{Q})$ be an element with $(\lambda, \lambda) \neq 0$. Then

$$\phi^{-1} e_{\phi(\lambda)} \phi = \frac{t(\lambda, \lambda)}{2} e_\lambda^\vee, \tag{10.4}$$

by Lemma 1.6. Here $e_\lambda^\vee$ is the dual Lefschetz operator with respect to the grading operator of the LLV algebra $\mathfrak{g}_S$ of S. The morphism $\widetilde{H}(\Theta_n(\phi))$ is degree reversing as well, by Corollaries 7.3 and 7.4. Hence, so is $\Theta_n(\phi)$, by Lemma 3.4. Say $(\widetilde{H}(\Theta_n(\phi)))(\beta) = t'\alpha$. Then

$$\Theta_n(\phi)^{-1} e_{(\widetilde{H}(\Theta_n(\phi)))(\theta(\lambda))} \Theta_n(\phi) = \frac{t'(\lambda, \lambda)}{2} e_{\theta(\lambda)}^\vee.$$

by Lemma 1.6. Here $e^\vee_{\theta(\lambda)}$ is the dual Lefschetz operator with respect to the grading operator of the LLV algebra $\mathfrak{g}_{S^{[n]}}$ of $S^{[n]}$ given in (3.1). Note that $t' = \det(\phi)^{n+1} t$ and $\widetilde{H}(\Theta_n(\phi)) = \det(\phi)^{n+1} \tilde{\iota}_\phi$, by Corollary 7.3. The second equality below follows (using also that $e_{\tilde{\iota}_\phi(\theta(\lambda))} = e_{\theta(\phi(\lambda))}$):

$$Ad_{\Theta_n(\phi)^{-1}}(d\Theta_n(e_{\phi(\lambda)})) \stackrel{(10.3)}{=} \Theta_n(\phi)^{-1} e_{\theta(\phi(\lambda))} \Theta_n(\phi) = \frac{t(\lambda, \lambda)}{2} e^\vee_{\theta(\lambda)}.$$

Given a morphism $\phi \in \mathrm{Hom}_{\mathcal{G}^{[1]}}((S, \epsilon), (M, \epsilon'))$, we have

$$d\Theta_n \circ Ad_\phi = Ad_{\Theta_n(\phi)} \circ d\Theta_n$$

since $\Theta_n(\phi\psi\phi^{-1}) = \Theta_n(\phi)\Theta_n(\psi)\Theta_n(\phi)^{-1}$. We conclude that

$$d\Theta_n(Ad_{\phi^{-1}}(e_{\phi(\lambda)})) = \frac{t(\lambda, \lambda)}{2} e^\vee_{\theta(\lambda)}.$$

The left-hand side is $d\Theta_n((t(\lambda, \lambda)/2)e^\vee_\lambda)$, by (10.4). Hence, $d\Theta_n(e^\vee_\lambda) = e^\vee_{\theta(\lambda)}$. Now $\dot\rho : \mathfrak{so}(\widetilde{H}(S^{[n]}, \mathbb{Q})) \to \mathfrak{g}_{S^{[n]}}$ maps the grading operator of $\mathfrak{so}(\widetilde{H}(S^{[n]}, \mathbb{Q}))$ to that of $\mathfrak{g}_{S^{[n]}}$ and the differential of $\tilde{\iota}$ sends the grading operator of $\mathfrak{g}_S$ to that of $\mathfrak{so}(\widetilde{H}(S^{[n]}, \mathbb{Q}))$. Hence, the differential of $\rho \circ \tilde{\iota}$ sends $e^\vee_\lambda$ to $e^\vee_{\theta(\lambda)}$ and we get the equality

$$d\Theta_n(e^\vee_\lambda) = d(\rho \circ \tilde{\iota})(e^\vee_\lambda).$$

The statement follows, since $\mathfrak{so}(H^*(S, \mathbb{Q}))$ is generated by the pairs $(e_\lambda, e^\vee_\lambda)$, as λ varies over elements of $H^2(S, \mathbb{Q})$ with $(\lambda, \lambda) \neq 0$. $\square$

Let $\mathcal{R}^{[n]}_{an}$ be the subgroupoid of $\mathcal{G}^{[n]}_{an}$ given in (1.6).

PROPOSITION 10.7. *The restriction of the functor $\widetilde{H}_0$ to the subgroupoid $\mathcal{R}^{[n]}_{an}$ remains full.*

Proof. It suffices to prove that all morphisms of $\mathcal{G}^{[n]}_{an}$, which were used in the proof of Theorem 1.4 to prove the fullness of $\widetilde{H}_0$, are already morphisms of $\mathcal{R}^{[n]}_{an}$. Parallel-transport operators, which are Hodge isometries, are morphisms in $\mathcal{R}^{[n]}_{an}$. The morphisms other than parallel-transport operators where all deformations of those that appear in Corollary 7.4, hence conjugates of those that appear in Corollary 7.4 via parallel-transport operators. It suffices to prove that the morphism $\Theta_n(\varphi)$ of $\mathcal{G}^{[n]}_{an}$ that appears in Corollary 7.4 belongs to $\mathcal{R}^{[n]}_{an}$, in other words, that $\mu_{\chi(\Theta_n(\varphi))}\Theta_n(\varphi)$ conjugates the cup product of $H^*(S^{[n]}, \mathbb{Q})$ to the Pontryagin product.

Let $f : H^*(S, \mathbb{Z}) \to H^*(M, \mathbb{Z})$ be a parallel-transport operator. Then $f \circ \varphi$ belongs to $\mathrm{Aut}_{\mathcal{G}^{[1]}}(M)$, $\mu_{\chi(\Theta_n(f\circ\varphi))} = \mu_{\chi(\Theta_n(\varphi))}$ and $\mu_{\chi(\Theta_n(\varphi))}\Theta_n(\varphi)$ conjugates the cup product of $H^*(M^{[n]}, \mathbb{Q})$ to the Pontryagin product, if and only if $\mu_{\chi(\Theta_n(f\circ\varphi))}\Theta_n(f \circ \varphi)$ does, by Lemma 10.2(2). Hence, it suffices to prove the statement of the Proposition for automorphisms in the image of Θ_n.

The restriction of Θ_n to the subgroup $S\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$ of elements of $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$ of determinant 1 is equal to the restriction of the composition of $\tilde{\iota} : SO(\widetilde{H}(S, \mathbb{C})) \to SO(\widetilde{H}(S^{[n]}, \mathbb{C}))$ with the representation ρ given in (10.1), by Lemma 10.6. The two agree on the whole of $\mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$ if we choose the reflection R in the definition of ρ to be the image of a monodromy reflection of S. Hence, if $\varphi \in \mathrm{Aut}_{\mathcal{G}^{[1]}}(S, \epsilon)$ conjugates h to $-h$, then $\mu_{\chi(\Theta_n(\varphi))}\Theta_n(\varphi) = \rho(\mu_{\chi(\varphi)}\varphi)$ conjugates the cup product to the Pontryagin product and if φ commutes with h, then $\mu_{\chi(\Theta_n(\varphi))}\Theta_n(\varphi)$ is a ring automorphism with respect to the cup product, by Lemmas 10.2 and 10.4. $\square$

Proof of Corollary 1.10. Choose an IHSM X and a twisted vector bundle E over $X \times Y$ as in Theorem 1.5. There exists a rational number t, such that the isomorphism $\phi :=$

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$\mu_t \circ [\kappa(E)\sqrt{td_{X \times Y}}]_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ satisfies (1.7), by the proof of Proposition 10.7. Given classes $\gamma_1, \gamma_2 \in H^*(Y, \mathbb{Q})$, we thus have

$$\phi(\phi^{-1}(\gamma_1) \cup \phi^{-1}(\gamma_2)) = \gamma_1 \star \gamma_2. \tag{10.5}$$

The isomorphism μ_t is algebraic, since the Lefschetz standard conjecture holds for Y, by [CM13], and so the Künneth factors of the diagonal in $H^i(Y, \mathbb{Q}) \otimes H^{4n-i}(Y, \mathbb{Q})$, $0 \le i \le 4n$, are algebraic, by [Kle68, Proposition 1.4.4]. Hence, the isomorphism ϕ is induced by an algebraic correspondence. Now $\phi^{-1} : H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is equal to $[\kappa(E^*)\sqrt{td_{Y \times X}}]_* \circ \mu_{t^{-1}}$ and is algebraic as well. Denote by $(\phi^{-1})^t : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ the isomorphism induce by the class in $H^*(X \times Y, \mathbb{Q})$, which is the transpose of the class in $H^*(Y \times X, \mathbb{Q})$ inducing ϕ^{-1}. Then $(\phi^{-1})^t \times (\phi^{-1})^t \times \phi : H^*(X^3, \mathbb{Q}) \to H^*(Y^3, \mathbb{Q})$ is an algebraic correspondence taking the class $[\Delta]$ of the diagonal to an algebraic class Pont inducing the Pontryagin product. The latter statement is equivalent to the equality

$$\phi\big(\pi_{3,*}\big(\pi_1^*(\phi^{-1}(\gamma_1)) \cup \pi_2^*(\phi^{-1}(\gamma_2)) \cup [\Delta]\big)\big) = \pi_{3,*}\big(\pi_1^*(\gamma_1) \cup \pi_2^*(\gamma_2) \cup \mathrm{Pont}\big),$$

by (10.5). The above equation is verified as follows (let a, b, c be classes in $H^*(X, \mathbb{Q})$):

$$\phi\big(\pi_{3,*}\big(\pi_1^*[\phi^{-1}(\gamma_1) \cup a] \cup \pi_2^*[\phi^{-1}(\gamma_2) \cup b] \cup \pi_3^* c\big)\big)$$
$$= \left(\int_X \phi^{-1}(\gamma_1) \cup a\right)\left(\int_X \phi^{-1}(\gamma_2) \cup b\right)\phi(c)$$
$$= \left(\int_X \gamma_1 \cup (\phi^{-1})^t(a)\right)\left(\int_X \gamma_2 \cup (\phi^{-1})^t(b)\right)\phi(c)$$
$$= \pi_{3,*}\big(\pi_1^*(\gamma_1) \cup \pi_2^*(\gamma_2) \cup \big((\phi^{-1})^t \otimes (\phi^{-1})^t \otimes \phi\big)(\pi_1^*(a) \cup \pi_2^*(b) \cup \pi_3^*(c))\big).$$

Finally, we observe that $[\kappa(E^*)\sqrt{td_{Y \times X}}]_* \circ \mu_{t^{-1}} = \mu_t \circ [\kappa(E^*)\sqrt{td_{Y \times X}}]_*$, as $[\kappa(E^*)\sqrt{td_{Y \times X}}]_*$ is degree reversing, and $\kappa(E^*) = \kappa(E)$, by Corollary 4.2. Hence, $\phi = (\phi^{-1})^t$. $\qquad\square$

Acknowledgements

Conflicts of Interest
None.

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