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Distributivity and Base Trees for $P(\kappa)/{<}\kappa$

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Abstract. **For κ a regular uncountable cardinal, we show that distributivity and base trees for $P(\kappa)/{<}\kappa$ of intermediate height in the cardinal interval $[\omega, \kappa)$ exist in certain models. We also show that base trees of height κ can exist as well as base trees of various heights $\geq \kappa^+$ depending on the spectrum of cardinalities of towers in $P(\kappa)/{<}\kappa$.**

1 Introduction

This paper concerns trees of maximal antichains in $P(\kappa)/{<}\kappa$ for κ a regular cardinal of uncountable cofinality. In one's imagination, trees can grow downward or upward; typically for us here they grow downward when the tree relation has something to do with the subset relation. Every level of these trees is a maximal antichain and is a refinement of the levels above it. A node of the tree may be viewed as both an element of $[\kappa]^\kappa$ and as its equivalence class modulo the ideal $P_\kappa\kappa$.

If a tree of maximal antichains T has the property that there is no maximal antichain refining all levels simultaneously, call this tree a *distributivity tree*. Equivalently, the intersection of the downward closure of the antichains on each level of T is not open dense. It may be that this intersection is nonempty, but by choosing a witness $x \in [\kappa]^\kappa$ to non-density, $T \restriction x$ is a tree of maximal antichains in $P(x)/{<}\kappa$ so that the intersection of the downward closure of the antichains on each level of $T \restriction x$ is empty. Call such a distributivity tree full; so if there is a distributivity tree of some height, there is a full distributivity tree of the same height. Accordingly, the adjective full is assumed and omitted. Trees of maximal antichains are necessarily pruned, in that every node has extensions to every level. A set $x \in [\kappa]^\kappa$ is often identified with its enumerating function $f_x : \kappa \to \kappa$ where $f_x(\alpha)$ is the α^{th} element of x, denoted $x(\alpha)$.

For us, a *path* is a descending sequence through the tree of any length $\alpha \leq \text{ht}(T)$, while a *branch* is a path through the entire tree of length $\text{ht}(T)$. The set of all branches through a tree T is denoted $[T]$. If T is a distributivity tree then the nodes in any branch $b \in [T]$ must form a tower in $P(\kappa)/{<}\kappa$, i.e. a $\subseteq^*$-descending sequence with no $x \in [\kappa]^\kappa$ almost contained in each element of the sequence. Conversely, if T is a tree of maximal antichains where the nodes in every branch $b \in [T]$ form a tower, T is a full distributivity tree. In particular, a tree of maximal antichains with no branches is a full distributivity tree. A special sort of full distributivity tree T is a *base tree* which is one where for every $x \in [\kappa]^\kappa$ there exists $t \in T$ with $t \subseteq^* x$.

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For κ with $\mathrm{cf}(\kappa) > \omega$, there is a distributivity tree T of maximal antichains with $\mathrm{ht}(T) = \omega$. Similarly, for κ with $\mathrm{cf}(\kappa) = \omega$, there exists a distributivity tree of height ω_1. For these κ, there can be no distributivity tree of height ω because all such trees have branches and there are no towers of length $\mathrm{cf}(\kappa)$. This tree then also has no maximal paths of countable length.

These distributivity trees of heights ω and ω_1 (for $\mathrm{cf}(\kappa) > \omega$ and $\mathrm{cf}(\kappa) = \omega$, respectively) were originally constructed combinatorially by Balcar and Vopěnka (see [1]) and can also be built more abstractly with forcing technology (see [7]). κ-Aronszajn trees can be used to build distributivity trees of height κ (Proposition 2.2 below).

It was asked by V. Fischer, M. Koelbing, and W. Wohofsky whether 1) there can exist a distributivity tree for $P(\kappa)/{<}\kappa$ of regular height strictly above κ and 2) whether regular cardinals strictly between ω and κ can be the heights of distributivity trees (see the discussion at the end of Section 8 and Question 9.5 from [6]). In what follows we show that affirmative answers to both questions hold in certain models. The study of the spectrum of heights of distributivity and base trees in various models for $\kappa = \omega$ (and the structure of maximal paths through those trees) has recently seen a resurgence of interest (e.g. [2], [3], [4], and [5]), motivating what follows for the $\kappa > \omega$ case.

2 Partition-type short distributivity trees

Let $\kappa > \omega$ be regular. For a tree of maximal antichains T with $\mathrm{ht}(T) \le \kappa$, by iteratively removing from every node all ordinals not contained in every node above it we may assume that the tree relation for T is the subset relation and not the subset modulo $P_\kappa \kappa$ relation ($\subseteq$ and not $\subseteq^*$). A special case of a tree of maximal antichains T is one where every level of T is of cardinality less than κ. If additionally $\mathrm{ht}(T) < \kappa$, by working within a set of ordinals common to each of the unions over nodes on every level and by taking symmetric differences between elements on every level and removing from every node all ordinals not contained in every node above it, we may assume that each level of T is a partition of κ and the tree relation is $\subseteq$. Call such a tree a *partition-type* tree.

***Definition* 2.1** For a tree T with $\mathrm{ht}(T) = \kappa$, if $|[T]| \ge \lambda \ge \kappa^+$ but $|\mathrm{Lev}_\xi(T)| < \mu$ for every $\xi < \kappa$ for some $\mu \le \lambda$, call T a (μ, κ, λ)-Kurepa tree.

Some parameters are often omitted. For example, if $\mu = \kappa^+$ and $\lambda = \kappa^+$ then a (μ, κ, λ)-Kurepa tree is a tree of cardinality κ with at least κ^+ many branches. This is called a weak κ-Kurepa tree. A $(\omega_1, \omega_1, \omega_2)$-Kurepa tree is the traditional Kurepa tree.

***Proposition* 2.1** *Suppose κ is regular. A partition-type distributivity tree for κ of height $\mu < \kappa$ is necessarily a (κ, μ, κ)-Kurepa tree, while the existence of a (κ, μ, κ)-Kurepa tree implies the existence of a partition-type distributivity tree for κ of height μ.*

Proof First, the intersection along every branch of a partition-type distributivity tree of height μ for κ must be of cardinality less than κ and so because for every $\xi \in \kappa$ there is a unique node t on every level of the tree such that $\xi \in t$, there must be at least κ-many branches through the tree.

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On the other hand, take κ-many branches in a (κ, μ, κ)-Kurepa tree T sufficient to generate the tree and identify them with ordinals in κ. By regularity we can remove every node which doesn't have κ-many of these branches through it and so assume that every node in T has κ-many of these branches through it. Form the partition-type distributivity tree T' for κ where each partition element is the collection of branches (ordinals) inside the downward cones of a node on every level of T. Levels are of size less than κ so the resulting tree is a partition-type tree of maximal antichains for κ of height μ. Furthermore, every branch forms a tower because there is at most a single ordinal contained in the intersection along the branch. ■

Proposition 2.1 shows for example that if the CH holds then as the full binary tree on ω_1 is a weak Kurepa tree, there exists a partition-type distributivity tree of height ω_1 for $\kappa = \omega_2$. More generally, for some κ if $\mu < \kappa$ is minimal with $2^\mu \geq \kappa$ then a partition-type distributivity tree of height μ for κ may be built via a suitable injection from κ to branches through the full binary tree of height μ.

***Proposition* 2.2** *A distributivity tree for κ of height κ where every level is of size less than κ is necessarily a κ-Aronszajn tree and moreover if there exists a κ-Aronszajn tree then there is a distributivity tree for κ of height κ where every level is of size less than κ.*

Proof There are no towers of length κ, so such a distributivity tree cannot have branches. On the other hand, we can identify nodes in an Aronszajn tree with ordinals in κ and note that the downward nodal cones for levels of the tree are (modulo $P_\kappa \kappa$) partitions of κ into $< \kappa$-many pieces. The resulting tree of maximal antichains has no branches and so is a distributivity tree. ■

A tree T of limit height is *pruned* if every node $s \in T$ has at least one compatible node in T on every level of T, i.e. if $\mathrm{ht}(T \upharpoonright s) = \mathrm{ht}(T)$. Say that a tree T is *cofinally splitting* if it is pruned and every node $s \in T$ is *splittable* in T, meaning that s extends to two mutually incompatible nodes t_1, t_2 (i.e. neither $t_1 <_T t_2$ nor $t_2 <_T t_1$). If a pruned tree is not cofinally splitting, non-splittable nodes occur in T. In some cases, T eventually comprises only such nodes: say that T is *eventually nonsplitting* if for some $\alpha < \mathrm{ht}(T)$, every $s \in \mathrm{Lev}_\beta(T)$ for $\beta \in (\alpha, \mathrm{ht}(T))$ is not splittable.

***Observation* 2.3** *Cofinally splitting trees that are too narrow relative to their heights cannot exist: If κ is regular and there exists $\mu < \kappa$ and a pruned tree T of height κ with $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha < \kappa$, then T is not cofinally splitting. In fact, T is eventually nonsplitting.*

Proof It suffices to show that for a pruned tree T of height κ with $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha < \kappa$, $[T] \neq \emptyset$. Because then if T were not eventually nonsplitting, we could witness μ-many distinct branches through the tree all splitting below some level $\xi < \kappa$ by choosing splittable nodes on sufficiently higher and higher levels and looking at branches through the restrictions of the tree to the incompatible nodes witnessing splittability. But then $|\mathrm{Lev}_\xi(T)| \geq \mu$, a contradiction. Choose μ minimal so that $|\mathrm{Lev}_\alpha(T)| < \mu$ for every $\alpha < \kappa$. For every α of cofinality μ in κ (write $\alpha \in \mathrm{cof}(\mu) \cap \kappa$) choose some $s_\alpha \in \mathrm{Lev}_\alpha(T)$. Without loss of generality we may assume that T has no splitting at limit levels, so all splitting between s_α and the other nodes on level α is witnessed below some level $\beta < \alpha$. Let $f(\alpha) = \beta$ and choose a stationary $S \subseteq \mathrm{cof}(\mu) \cap \kappa$ where f is the constant function $f''S = \{\xi\}$. But then on a κ-sized subset $A \subseteq S$, every $s_\alpha \restriction \xi$ is the same $\overline{s}_\xi$ for some particular $\overline{s}_\xi \in \mathrm{Lev}_\xi(T)$. But then for $\alpha < \beta$ in A we have $s_\beta \restriction \alpha = s_\alpha$. So $b_A = \{s \in T : \text{ for some } \alpha \in A, s = s_\alpha \restriction \mathrm{lh}(s)\} \in [T]$. ■

Observation 2.3 shows that a tree of maximal antichains for κ with levels of size less than κ of height $\geq \kappa^+$ is eventually nonsplitting. So for example, there is no distributivity tree of height ω_2 for ω_1 with countable levels (any such tree of maximal antichains eventually stabilizes, so in particular a non-tower branch through the tree exists).

Observation 2.3 and Propositions 2.1 and 2.2 cover all scenarios for the spectrum of distributivity tree heights for κ where the levels are partitions of κ modulo $P_\kappa\kappa$. By collapsing certain large cardinals in specific ways, however, we can build models where for example no weak Kurepa trees exist for ω_1. The analysis above does not resolve whether a distributivity tree for ω_2 of height ω_1 exists in such models; it just implies that any such tree must have levels of size greater than ω_2.

3 Short base trees with wide levels

Let $\mathrm{MAD}(\kappa)$ denote the set of cardinalities of maximal almost disjoint families in $P(\kappa)/{<}\kappa$. Let $\mathfrak{a}_\kappa = \min\{\mathrm{MAD}(\kappa) \cap (\kappa, 2^\kappa]\}$. If MAD families are all of sufficiently large size or in the form of partitions of κ, we can build base trees for κ of heights in the cardinal interval $[\omega, \kappa)$. A preliminary shorthand definition is useful.

***Definition* 3.1** For $x, y \in [\kappa]^\kappa$, say that *x is discontinuous (everywhere) relative to y* if for every $\beta \in \lim(\kappa)$, $x(\beta) > \min(y \setminus \sup\{x(\xi) : \xi \in \beta\})$.

We also say x is almost everywhere discontinuous relative to y (and just discontinuous relative to y if the context is clear) when for some $\gamma < \kappa$ for every limit $\beta \in (\gamma, \kappa)$, $x(\beta) > \min(y \setminus \sup\{x(\xi) : \xi \in \beta\})$. Of particular interest is when $x \subseteq y$. If $x \subseteq y$, x being discontinuous everywhere relative to y is equivalent to saying that $x(\beta) > y(\beta)$ for every $\beta < \kappa$, i.e. for the inverse enumerating functions for x and y we have $f_x^{-1}(\alpha) < f_y^{-1}(\alpha)$ for every $\alpha \in x$. For example, if $y \in [\kappa]^\kappa$ then the set of successor ordinals in its order topology is everywhere discontinuous relative to y. Similarly, if $x \subseteq^* y$ then x being almost everywhere discontinuous relative to y is equivalent to saying that for all β above some ordinal $\gamma < \kappa$, $x(\beta) > y(\beta)$ and for every large enough $\alpha \in x$, $f_x^{-1}(\alpha) < f_y^{-1}(\alpha)$. If x is almost everywhere discontinuous relative to y and $y \subseteq^* z$ then x is almost everywhere discontinuous relative to z. With $\mathrm{cf}(\kappa) > \omega$, there can be no $(\omega + 1)$-length $\subseteq^*$-descending sequence of elements of $[\kappa]^\kappa$ each of which is almost everywhere discontinuous relative to its predecessors, i.e. such an infinite $\subseteq^*$-descending sequence of elements in $[\kappa]^\kappa$ forms a tower.

***Theorem* 3.1** *Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. For $\omega \leq \mu < \kappa$ there exists a base tree of height μ for κ with levels of cardinality 2^κ.*

Proof The tree is built iteratively. We give the general idea then go into more detail. There will be two types of nodes; "root" nodes and "tower" nodes. Every tower node is associated with a tower through the tree passing through that node. Immediately below every root node will be at most one tower node and also 2^κ-many root nodes together forming a maximal antichain with respect to that node. This collection of root nodes is referred to as a root node family. Note that if a root node family in $[z]^\kappa$ for some $z \in [\kappa]^\kappa$ is maximal with respect to being both almost disjoint and everywhere discontinuous relative to z, then in fact the root node family is MAD in $[z]^\kappa$ because any almost disjoint set y can be shrunk to an everywhere discontinuous (relative to y) almost disjoint set by e.g. taking the collection of successor ordinals in its order topology. In the subsequent level, every set x hitting 2^κ-many elements of the root node family is diagonalized against, in the sense that a tower node (and associated tower) is added below which is a subset of x. Immediately below every tower node will be exactly one tower node and also a root node family. In fact, below every tower node is a continuous strictly-descending μ-length tower with empty intersection through T. If $r \subseteq s$ in T and r is a root node then r will be everywhere discontinuous relative to s. Therefore at limit levels of T, only paths containing finitely-many root nodes have nonempty intersection. These paths are exactly the intersections along some continuous μ-tower added previously; by continuity the nodes extending these paths are the appropriate limit elements of those towers. Every every branch in $[T]$ then forms a tower and it remains to be seen that every level of T is a maximal antichain.

Describing the construction in more detail, for a tower node $s \in \mathrm{Lev}_\xi(T)$ its associated tower is denoted $\langle t^s_\alpha : \alpha < \mu\rangle$ with $t^s_\alpha \in \mathrm{Lev}_\alpha(T)$ where t^s_ν is the predecessor of s on level ν written $t^s_\nu = s \restriction \nu$ for every $\nu < \xi$ (and $t^s_\xi = s$). Here $\langle t^s_\alpha : \xi \leq \alpha < \mu\rangle$ is a strictly decreasing continuous μ-length tower consisting entirely of tower nodes with empty intersection through T starting with s. So while the tower $\langle t^s_\alpha : \alpha < \mu\rangle$ may contain root nodes inside the initial portion below level ξ, as above the end segment $\langle t^s_\alpha : \xi \leq \alpha < \mu\rangle$ only contains tower nodes. For ξ limit, $\mathrm{Lev}_\xi(T)$ then comprises the partial continuous intersections along μ-towers added in previous steps. By continuity, such a node is of the form t^s_ξ for some tower node s added previously. Its associated tower is the same, $\langle t_\alpha^{(t^s_\xi)} : \alpha < \mu\rangle = \langle t^s_\alpha : \alpha < \mu\rangle$.

Now we describe how the successor levels $\mathrm{Lev}_{\xi+1}(T)$ are formed. Let $s \in \mathrm{Lev}_\xi(T)$. If s is a tower node (which will for example always be the case if ξ is a limit), then first add the tower element $t^s_{\xi+1} \in \mathrm{Lev}_{\xi+1}(T)$ from $\langle t^s_\alpha : \alpha < \mu\rangle$. Furthermore, assign to $t^s_{\xi+1}$ the same tower as for s, i.e. $\langle t_\alpha^{(t^s_{\xi+1})} : \alpha < \mu\rangle = \langle t^s_\alpha : \alpha < \mu\rangle$. Additionally, let $z = s \setminus t^s_{\xi+1}$ denote the nodal difference between the two successive tower elements and note that because the towers are strictly decreasing, $|z| = \kappa$. Split z into a 2^κ-sized maximal almost disjoint root node family in $[z]^\kappa$ all of whose elements are everywhere discontinuous relative to z. Associate this family with every resulting root node r and denote the family accordingly as R_r.

Next we consider the case where $s \in \mathrm{Lev}_\xi(T)$ is a root node. It is associated with a 2^κ-sized root node family R_s which is MAD in the relevant $z \in [\kappa]^\kappa$ and allows the key diagonalization step to take place. Let $X = \{x \in [z]^\kappa : |\{r \in R_s : |x \cap r| = \kappa\}| = 2^\kappa\}$ be the set of $x \in [z]^\kappa$ hitting 2^κ-many elements of R_s. We want to add at least one tower

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inside every such x below some suitable r. Because $|X| \leq 2^\kappa$ and every $x \in X$ has 2^κ-many options for which r to choose (i.e. which r with $|r \cap x| = \kappa$), this is easy to arrange by ordering X and R_s in type 2^κ and letting the γ^{th} element of X, $X(\gamma)$, be assigned to the δ^{th} element r of R_s with δ minimal such that $|X(\gamma) \cap r| = \kappa$ and none of the $X(\gamma')$ for $\gamma' < \gamma$ has been assigned to r previously.

With the help of this assignment, for every $x \in X$ add a tower node $s' \subseteq (x \cap r)$ with $s' \in \mathrm{Lev}_{\xi+1}(T)$ and $|r \setminus s'| = \kappa$. This $s' = t^{s'}_{\xi+1}$ is associated with the resulting tower $\langle t^{s'}_\alpha : \alpha < \mu \rangle$ where $\langle t^{s'}_\alpha : \xi + 1 \leq \alpha < \mu \rangle$ is then a strictly decreasing continuous sequence with empty intersection inside $x \cap r$. Next, let $z = r \setminus s'$ and split z into a 2^κ-sized root node family and associate every resulting root node with this family. Finally, to every root node r of R_s which did not have an element of x assigned to it, split r directly into a 2^κ-sized root node family and associate all resulting root nodes with that family. This completes the construction of level $\xi + 1$. To summarize, below tower nodes from level ξ we added the next tower node in the sequence along with a root node family inside the resulting nodal difference, while below root nodes from level ξ (depending on how their root node families facilitated diagonalization) we added a root node family and possibly a tower node and associated tower inside the relevant x.

This concludes the construction of T as a tree of antichains of height μ where every branch through T forms a tower. Note that MAD families are added below every node from level ξ on level $\xi + 1$, so to show T is a distributivity tree we only need to see that for $\xi \in \lim(\mu)$, $\mathrm{Lev}_\xi(T)$ is a maximal antichain. Suppose for every $\nu < \xi$, $\mathrm{Lev}_\nu(T)$ is maximal. Let $x \in [\kappa]^\kappa$. If the cardinality of the set of nodes on each level hitting x is less than κ for every ν, then by taking symmetric differences between nodal elements on each level and by removing fewer than κ-many elements from x, we may view $T_\xi \restriction x$ is as a partition-type tree of maximal antichains in $[x]^\kappa$ of height $\xi < \mu < \kappa$. Every ordinal in x is associated with a branch through $T_\xi \restriction x$ by looking at the unique node on each level containing it. However, nonempty branches through $T_\xi \restriction x$ only contain finitely-many root nodes so are determined by some intermediate-level tower node. Therefore κ-many ordinals in x must determine the same branch leading to some t^s_ξ which by continuity must then intersect x in a set of size κ.

If instead x hits at least κ-many elements on some level of T below ξ then because $\mathfrak{a}_\kappa = 2^\kappa$ there is some minimal level $\eta < \xi < \kappa$ where x hits 2^κ-many nodes in $\mathrm{Lev}_\eta(T)$. By the argument above η is not a limit (there are not enough nonempty branches through $T_\eta \restriction x$). So, suppose fewer than κ many nodes on level η but 2^κ-many nodes on level $\eta + 1$ hit x. By construction, nodes on level η are associated with exactly one root node family on level $\eta + 1$, so 2^κ-many of these nodes from level $\eta + 1$ must fall among the same root node family R. But then by construction at level $\eta + 2$ we will have added a tower node $s = t^s_{\eta+2} \subseteq x \cap r$ for some $r \in R$ hitting x, which extends to a μ-length tower through T inside x. In particular, the limit node of this tower at level ξ, t^s_ξ, is a subset of x. Thus, T is a distributivity tree.

Furthermore, for any $x \in [\kappa]^\kappa$ the only time that there might not exist $s \in T$ with $s \subseteq x$ would be if fewer than κ-many nodes hit x on every level of T. But by removing fewer than κ-many elements from x and from overlapping nodes on each level of T, we may assume $T \restriction x$ is a partition-type tree of maximal antichains in $[x]^\kappa$. But then for

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any $\alpha \in x$, α is in a unique nodal element in every $\mathrm{Lev}_\xi(T)$ for $\xi < \mu$, i.e. α is contained in the intersection of a branch through T, which is impossible. So T is a base tree. ■

A construction of a base tree of minimal height $\mu = \omega_1$ for $\mathrm{cf}(\kappa) = \omega$ under additional cardinal arithmetic assumptions is provided in [7].

4 A tall base tree with wide levels

Under the same restrictions on the spectrum of MAD family cardinalities as in Theorem 3.1, we can prove that distributivity trees of height at least κ^+ exist. The following lemma is useful in the construction and is an instance of the phenomenon that trees too simple in structure do not admit complex subtrees.

***Lemma* 4.1** *For $\mu < \kappa$ regular cardinals, the tree $T^\kappa_{<\mu} \subseteq {}^{<\kappa}\kappa$ consisting of sequences with fewer than than μ-many nonzero values does not contain any κ-Aronszajn subtrees.*

Proof A more general structure theorem is true for subtrees of $T^\kappa_{<\mu}$ (Theorem 1.5.8 in [7]), however the lemma can also be proven directly via an argument similar to the proof of Observation 2.3 (Proposition 1.5.20 in [7]). We omit the details. ■

***Theorem* 4.2** *Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. If there is a tower of length κ^+ then there exists a base tree of height κ^+ for κ with levels of cardinality 2^κ.*

Proof Essentially we do the same thing as in Theorem 3.1, except we add κ^+-length $\subseteq^*$-towers below elements in the root node families. We follow the notation of the proof of Theorem 3.1, but in the following when referring to tower nodes we omit the localizing superscripts (s, etc.) as any necessary context will be clear. Only the subscripts indicating tree level are written for simplicity. At successors we do ensure that for every $\xi \in \kappa^+$, $t_{\xi+1} \subsetneq t_\xi$ with $|t_\xi \setminus t_{\xi+1}| = \kappa$. These tower sequences are no longer continuous at limits; indeed we ensure purposeful discontinuity and non-normality at limits so that for every $\beta \in \lim(\kappa^+)$ there exists $x \in [\kappa]^\kappa$ with $|x \setminus t_\beta| = \kappa$ and $x \subseteq^* t_\xi$ for every $\xi < \beta$.

At limit levels $\xi \in \kappa^+$, unlike as in Theorem 3.1 where we took intersections along all paths eventually traveling along a tower added previously forming the single element t_ξ, in the present setting we add this previously-defined tower element and also a root node family. Each of these tower nodes t_ξ is associated with a path $p = \{p_\nu : \nu < \xi\}$ through the tree (along which every node is eventually a tower node). We can also associate to this path a family of 2^κ-many root nodes $\{r_\delta : \delta \in 2^\kappa\}$ which are all almost everywhere discontinuous relative to the nodes in p and for which $\{r_\delta : \delta \in 2^\kappa\} \cup \{t_\xi\}$ is a maximal almost disjoint collection in $A_p = \{x \in [\kappa]^\kappa : x \subseteq^* p_\nu \text{ for every } \nu \in \xi\}$. We call this a path-type root node family.

At successor levels $\xi + 1$, much as in Theorem 3.1 we:

- Add the (successor-type) root node families in the tower nodal differences $t_\xi \setminus t_{\xi+1}$,

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- for every root node family $R = \{r_\delta : \delta \in 2^\kappa\}$ from level ξ, look at the collection of $x \in [\kappa]^\kappa$ hitting 2^κ-many members of R and for every such x ensure that a κ^+-length tower is added below an appropriate r_δ inside $x \cap r_\delta$, with $t_{\xi+1}$ on level $\xi + 1$,
- add a root node family below every r_δ not assigned such an x, and finally
- for every root node r_δ that was assigned such an x, add a root node family inside the nodal difference $r_\delta \setminus t_{\xi+1}$ between that rode node and its first associated tower element added on level $\xi + 1$.

Note that if some x hits 2^κ-many nodes on level $\xi+1$ and hits e.g. fewer than κ-many nodes on level ξ, then necessarily x hits 2^κ-many nodes in a particular root node family; so that at the next step $\xi + 2$ a tower is added inside x.

We have constructed a tree of antichains of height κ^+; we need to see that these are maximal. The successor step is immediate as all nodes are split into MAD collections, so let $\xi \in \lim(\kappa^+)$ and suppose that $\mathrm{Lev}_\nu(T)$ is a maximal antichain for every $\nu < \xi$. Consider first the case where $\mathrm{cf}(\xi) = \mu < \kappa$, and let $x \in [\kappa]^\kappa$. By fixing a continuous $\mathrm{cf}(\xi)$-ladder to ξ of levels, in a slight abuse of notation let $T_\xi \upharpoonright x$ denote the tree of maximal antichains in $[x]^\kappa$ of height μ in the natural way. If $|\mathrm{Lev}_\nu(T_\xi \upharpoonright x)| < \kappa$ for every $\nu < \mu$, then by removing fewer than κ-many elements from x and choosing suitable representatives for each tree node, we may assume that this is a partition-type $\subseteq$-tree of height μ where every root node is everywhere discontinuous relative to all nodes above it. As in the proof to Theorem 3.1 then, every ordinal in x is associated with a branch through this partition system and moreover associated with the final node where that branch no longer follows only its specified sequence of tower nodes. So κ-many ordinals in x, call them x', are contained in the intersection along the same branch $\{t_{f(\nu)} : \nu < \mu\}$ for the ladder $f : \mu \to \xi$ through the tree. Note that $A = \{y \in [\kappa]^\kappa : y \subseteq^* t_\nu \text{ for every } \nu < \xi\} = \{y \in [\kappa]^\kappa : y \subseteq^* t_{f(\nu)} \text{ for every } \nu < \mu\}$, so because at level ξ the appropriate $\{t_\xi\} \cup \{r_\delta : \delta \in 2^\kappa\}$ is maximal with respect to A, some element hits x' and so hits x.

Next, suppose that for some minimal $\nu < \xi$, $|\mathrm{Lev}_\nu(T_\xi \upharpoonright x)| = 2^\kappa$. If ν is of the form $\eta+1$ then necessarily x hits 2^κ-many nodes inside a particular successor-type root node family, so at step $\nu + 1$ a tower will be added inside x. In particular then, some tower node on level ξ is inside x. If ν is a limit and $|\mathrm{Lev}_\eta(T_\nu \upharpoonright x)| < \kappa$ for every $\eta < \nu$, then because there are fewer than $\mathrm{cf}(2^\kappa)$-many branches through $T_\nu \upharpoonright x$ passing through only finitely-many root nodes (a necessary condition for x to hit a node on level ν below a $\subseteq^*$-sequence), x must hit 2^κ-many root nodes within some particular A_p path-type root node family, so that a tower is added inside x at step $\nu + 1$.

Consider next the case where $\mathrm{cf}(\xi) = \kappa$. For simplicity and by using a ladder to ξ, assume $\xi = \kappa$. We need to see that $\mathrm{Lev}_\xi(T)$ is a maximal antichain. If for some minimal $\nu < \xi$, 2^κ-many nodes in $\mathrm{Lev}_\nu(T)$ hit x, then the analysis above shows that whether ν is successor or limit, there is a particular root node family on level ν inside of which x hits 2^κ-many elements. But then at level $\nu + 1$ we added a tower inside x below one of those elements.

On the other hand, if fewer than κ-many nodes in every $\mathrm{Lev}_\nu(T)$ hit x for every $\nu < \xi$ then because there can be no $(\omega + 1)$-length sequences of root nodes, it must be that $\mathrm{T}_\xi \upharpoonright x$ (a κ-tree of maximal antichains in $[x]^\kappa$) is isomorphic to a subtree of $T^\kappa_{<\omega}$. Lemma 4.1 implies that $T_\xi \upharpoonright x$ is not κ-Aronszajn, so there is some $b \in [T_\xi \upharpoonright x]$.

Because there are no towers of length κ, there exists $x' \in A_b$ with $x' \in [x]^\kappa$, where $A_b = \{y \in [\kappa]^\kappa : y \subseteq^* b_\nu$ for every $\nu < \xi\}$. By maximality x' hits at least one element in the associated $\{r_\delta : \delta \in 2^\kappa\} \cup \{t_\xi\}$.

Finally, having constructed T let's see that it is a base tree. Let $x \in [\kappa]^\kappa$. The analysis above shows that in the first level where $T \restriction x$ has 2^κ-many nodes, necessarily x hits 2^κ-many nodes in some successor-type or path-type root node family, so that a tower is added inside x in the subsequent level. However, if $T \restriction x$ has fewer than κ-many nodes on each level then Observation 2.3 shows that for all sufficiently large $\alpha < \kappa^+$, $\mathrm{Lev}_\alpha(T \restriction x)$ only comprises some distinguished μ-many tower nodes $\{t^\xi_\alpha : \xi < \mu\}$ for some $\mu < \kappa$, which is impossible. In detail, if so then $x \subseteq^* \bigcup_{\xi<\mu} t^\xi_\alpha$ for every sufficiently large α. So $\overline{x} \subseteq \bigcup_{\xi<\mu} t^\xi_\alpha$ for some $\overline{x} \in [x]^\kappa$ cofinally often. For every $\beta \in \overline{x}$ and such $\alpha < \kappa^+$ we can find $\xi < \mu$ with $\beta \in t^\xi_\alpha$. Then if $\overline{\overline{x}}$ is a κ-sized set of these β sharing the same ξ, we have $\overline{\overline{x}} \subseteq^* t^\xi_\alpha$ cofinally often. This is a contradiction as the t^ξ_α elements form a tower. ■

In addition to understanding the heights of distributivity (base) trees for e.g. $P(\omega)/{<}\omega$, there has also recently been a desire to understand their nature in terms of degree of path closure. For example, the main result of [3] is that any base tree for $P(\omega)/{<}\omega$ of regular height λ larger than the distributivity number $\mathfrak{h}$ has maximal paths which are not branches.

Another way to see that result (and in fact something stronger) is to note that the degree of closure in one base tree T translates to a preponderance of paths of that length in any other base tree T'. Specifically, if a base tree T has the property that there are no maximal paths of length $\leq \nu$, then the set of paths of length ν is dense in every base tree T'. Start with $s' \in T'$, then find $s_0 \subseteq^* s'$ in T, then $s'_0 \subseteq^* s_0$ below s' in T', etc. forming $s' \supseteq^* s_0 \supseteq^* s'_0 \supseteq^* s_1 \ldots$. At limit stages $\xi < \nu$ we use the closure of T to find some s_ξ almost contained in every element in the chain and refine that to s'_ξ within T' by the base property. This means in particular that any base tree of height λ with $\mathrm{cf}(\lambda) > \mathfrak{h}$ cannot have the property that all $\leq \mathfrak{h}$-length paths can be extended (i.e. many are maximal and so are not branches).

The same argument applies in the $P(\kappa)/{<}\kappa$ setting and it is therefore not surprising that the κ^+-length base tree we built in Theorem 4.2 has a preponderance of maximal ω-length paths, given that if $\mathfrak{a}_\kappa = 2^\kappa$ (which holds under e.g. $2^\kappa = \kappa^+$) a base tree also exists of height ω.

5 A base tree of height κ

We saw in Proposition 2.2 that κ-Aronszajn trees can be used to build distributivity trees for κ of height κ where the levels are partitions of κ modulo $P_\kappa \kappa$. For κ with the tree property, however, these objects do not exist. Nonetheless, if $\mathfrak{a}_\kappa = 2^\kappa$ as in Theorems 3.1 and 4.2, base trees of height κ can be built.

***Theorem* 5.1** *Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. There exists a base tree for κ of height κ with levels of cardinality 2^κ.*

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Proof Fix a strictly increasing sequence of limit ordinals $\langle \delta_\nu : \nu < \kappa \rangle$ cofinal in κ. Essentially we do the same thing as in Theorem 3.1, except instead of just adding one tower for every relevant x in our diagonalization step we add κ-many $\subseteq$-towers indexed by ν for every relevant $x \in [\kappa]^\kappa$ (hitting 2^κ-many elements in the root node family). So in this setting, every such x has κ-many associated r_ν's, and each r_ν has a δ_ν-length tower added below it inside $x \cap r_\nu$. These towers are strictly-descending with empty intersection, continuous at limits, and of length δ_ν. All root nodes are everywhere discontinuous relative to their immediate predecessor and so the only nodes on limit levels ξ can be nonempty tower elements t_ξ formed by the intermediate-length intersections along a continuous δ_ν-length tower added previously. Note that unlike as in Theorem 3.1, at limit stages in this construction many tower paths are maximal (empty intersection) as the δ_ν-length towers expire.

At successors, form level $\xi+1$ of the tree first by splitting every tower nodal difference $t_\xi \setminus t_{\xi+1}$ component into a 2^κ-sized MAD (in $t_\xi \setminus t_{\xi+1}$) family of root nodes everywhere discontinuous relative to t_ξ. Additionally address each of the root node families from level ξ (note that if ξ is a limit no such families exist) as follows. These families will be formed inside tower nodal differences or inside root nodes and are MAD with respect to the relevant $z \in [\kappa]^\kappa$. We add for every $x \in [z]^\kappa$ which hits 2^κ-many elements in the root node family κ-many towers $\{\langle t^\nu_\alpha : \alpha < \delta_\nu \rangle : \nu < \kappa\}$, each below some r from the root node family hitting x, with all ordinals in all tower elements inside $r \cap x$. Add other root node families inside any root nodes not associated with such an x and inside all resulting nodal differences between the r's and the first tower elements. T is a tree of antichains of height κ. We show each level is maximal by induction.

The successor case is clear, so let ξ be a limit in κ and $x \in [\kappa]^\kappa$. Suppose the cardinality of every level of $T_\xi \upharpoonright x$ is less than κ. This is argued as in Theorem 3.1. By removing fewer than κ many elements from x we may assume this is a partition-type tree of maximal antichains of height $\xi < \kappa$, so that to every ordinal in x we may associate a branch through the tree for which that ordinal is contained in the intersection of nodes along the branch. Any branch with nonempty intersection through this tree eventually travels exclusively along some δ_ν-length tower, and there are fewer than κ-many such towers in this scenario, but these towers are continuous so the relevant t_ξ intersects κ-many ordinals in x at level ξ. This argument also shows that the minimal level (if one exists) where x hits 2^κ-many elements of the tree is not a limit. We may then assume that for some $\eta < \xi$, x hits fewer than κ many nodes on level η but 2^κ-many on level $\eta + 1$, so that necessarily x hits 2^κ-many nodes within the same root node family. So in particular towers of length δ_ν for every $\delta_\nu \in (\xi, \kappa)$ are added at level $\eta + 2$ inside x, such that the relevant t_ξ are subsets of x on level ξ.

We showed T is a tree of maximal antichains of height κ. Any maximal path in T either has countable cofinality and contains an ω-subsequence of root nodes or eventually coincides with a δ_ν-tower sequence. Therefore $[T] = \emptyset$, so T is a distributivity tree. Furthermore, the proof to Theorem 4.2 shows T is a base tree: $T \upharpoonright x$ cannot only have levels of size less than κ because then $T \upharpoonright x$ is isomorphic to a subtree of $T^\kappa_{<\omega}$ which has no κ-Aronszajn subtrees (and we know $T \upharpoonright x$ is branchless). But then some minimal level $T \upharpoonright x$ is of cardinality 2^κ and we subsequently add κ-many towers inside x. ∎

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6 Base trees with heights along the tower spectrum

By choosing $\subseteq^*$-towers of multiple lengths (as done with $\subseteq$-towers in Theorem 5.1) below elements of root node families and using the arguments from Theorems 3.1, 5.1, and 4.2 to cover the limit levels ξ for $\mathrm{cf}(\xi) < \kappa$, $\mathrm{cf}(\xi) = \kappa$, and $\mathrm{cf}(\xi) > \kappa$, we can build tall base trees of many different lengths depending on the spectrum of cardinalities of towers in $P(\kappa)/{<}\kappa$.

***Theorem* 6.1** *Suppose $\kappa > \omega$ is regular with $\mathfrak{a}_\kappa = 2^\kappa$. If $\lambda > \kappa$ is either the (regular) length of a tower in $P(\kappa)/{<}\kappa$ or the limit of such cardinalities, then there exists a base tree of height λ for κ with levels of cardinality 2^κ.*

Proof Because this proof is so similar to that of Theorems 3.1, 4.2, and 5.1, we provide only a sketch while highlighting any novel situations. If $\lambda \geq \kappa^+$ is the regular length of a tower, then the proof to Theorem 4.2 can be mimicked with λ in place of κ^+. The argument for why intermediate limit levels ξ of the tree remain maximal depends on whether $\mathrm{cf}(\xi) < \kappa$, $\mathrm{cf}(\xi) = \kappa$, or $\mathrm{cf}(\xi) > \kappa$. The new case to consider is when $\mathrm{cf}(\xi) > \kappa$ which is handled in essentially the same manner as we showed that the tree of height κ^+ in Theorem 4.2 has the base property. Specifically, let $x \in [\kappa]^\kappa$ and first suppose that $|\mathrm{Lev}_\nu(T_\xi \upharpoonright x)| < \kappa$ for every $\nu < \xi$ (by considering a $\mathrm{cf}(\xi)$-ladder to ξ we may assume ξ is a regular cardinal $\geq \kappa^+$). Observation 2.3 shows that $T_\xi \upharpoonright x$ eventually stabilizes so that for some $\mu < \kappa$, for all sufficiently large $\nu < \xi$, $\mathrm{Lev}_\nu(T_\xi \upharpoonright x)$ only comprises μ-many tower nodes $\{t_\nu^\eta : \eta < \mu\}$. So for some $\eta < \mu$, there exists $x' \in [x]^\kappa$ with $x' \subseteq^* t_\nu^\eta$ cofinally often in ξ, i.e. $x' \in A_p = \{y \in [\kappa]^\kappa : y \subseteq^* t_\nu^\eta \text{ for every } \nu < \xi\}$. By maximality then, x' and so x hits at least one element in the associated $\{r_\delta : \delta \in 2^\kappa\} \cup \{t_\xi^\eta\}$. On the other hand if for some minimal level $\nu < \xi$, $|\mathrm{Lev}_\nu(T_\xi \upharpoonright x)| = 2^\kappa$, then whether ν is a successor or limit x must in fact hit 2^κ-many nodes within a particular successor-type or path-type root node family at level ν, so that a λ-tower inside x is added at level $\nu + 1$ passing through level ξ. The novel situation over Theorem 4.2 is where ν is a limit with $\mathrm{cf}(\nu) > \kappa$ and as above because there are only μ-many paths through $T_\nu \upharpoonright x$, x hits 2^κ-many nodes within a particular path-type root node family at level ν. Similarly, if $\mathrm{cf}(\nu) = \kappa$ then there are at most $\kappa^{<\omega} = \kappa < \mathrm{cf}(2^\kappa)$-many path-type root node families and if $\mathrm{cf}(\nu) < \kappa$ there are fewer than κ-many, so in all cases x hits 2^κ-many nodes within a particular path-type root node family and a tower is added inside x at level $\nu+1$ passing through level ξ. The resulting tree T of regular height $\lambda \geq \kappa^+$ is shown to be a base tree as in Theorem 4.2: For any set $x \in [\kappa]^\kappa$ the above shows that if any level of $T \upharpoonright x$ has size 2^κ then a tower is added inside x, while if all levels have size less than κ then a κ-sized subset of x is almost contained in members of a tower sequence cofinally often, which is a contradiction.

If λ is the limit of lengths of towers of regular lengths $\geq \kappa^+$ in $P(\kappa)/{<}\kappa$, then the proof to Theorem 5.1 can be roughly mimicked with this sequence of cardinalities in place of the $\langle \delta_\nu \rangle$ sequence from Theorem 5.1, with the caveat that the towers added below every relevant $x \in [\kappa]^\kappa$ are now $\subseteq^*$-towers as in Theorem 4.2 and not $\subseteq$-towers. Note that $\lambda \leq 2^\kappa$ in all cases. To show maximality of all intermediate levels $\xi < \lambda$, as usual and as above branches through any induced tree $T_\xi \upharpoonright x$ with nonempty pseudointersection are formed and the process proceeds. The method for identifying such a branch depends on the cofinality of ξ. Similarly, it is argued depending on the cofinality of λ that $T \upharpoonright x$ cannot have levels of size $< \kappa$ everywhere, and so the resulting distributivity tree must in fact be a base tree. ■

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