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Algebraicity of L-values attached to quaternionic modular forms

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Abstract. In this paper, we prove the algebraicity of some L-values attached to quaternionic modular forms. We follow the rather well-established path of the doubling method. Our main contribution is that we include the case where the corresponding symmetric space is of non-tube type. We make various aspects very explicit, such as the doubling embedding, coset decomposition, and the definition of algebraicity of modular forms via CM-points.

1 Introduction

Special values of L-functions attached to automorphic forms have a long history in modern number theory. Their importance is difficult to overestimate, and for this reason, they have been the subject of intense study in recent decades. There is no doubt that it is important to study L-values of automorphic forms whose underlying symmetric space does not have hermitian structure (for example, automorphic forms for GL_n); however, in this paper, we will be dealing with a kind of automorphic form where the corresponding symmetric space has a hermitian structure. To go a bit further, we now introduce some notation.

Let $\mathbb{D}$ be a division algebra over $\mathbb{Q}$, and let V be a free left $\mathbb{D}$-module of finite rank n. Denote $\mathrm{End}(V,\mathbb{D})$ be the ring of $\mathbb{D}$-linear endomorphism of V and $\mathrm{GL}(V,\mathbb{D}) = \mathrm{End}(V,\mathbb{D})^\times$. For a nondegenerate hermitian (or skew-hermitian) form $\langle\, ,\rangle : V\times V \to \mathbb{D}$, we define a generalized unitary group

$$G := G_n := \{g \in \mathrm{GL}(V,\mathbb{D}) : \langle gx, gy\rangle = \langle x, y\rangle\}.$$

One can define automorphic forms associated with such group as in [2]. These can be seen as functions on a symmetric space $G(\mathbb{R})/K$, where K is a maximal compact subgroup of $G(\mathbb{R})$. In addition, when the associated symmetric space can be given a hermitian structure, one can define holomorphic automorphic forms which is what we refer as modular forms in this paper. The symmetric spaces $G(\mathbb{R})/K$ have been classified [12, Chapter X] (see also [17]). For positive integers n and m, we denote by $\mathbb{C}^n_m$ the set of $n\times m$ matrices with entries in $\mathbb{C}$. There are four infinite families of irreducible hermitian symmetric spaces of non-compact type as follows:

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(A) $\{z \in \mathbb{C}^n_m : zz^* < 1_n\}$,
(B) $\{z \in \mathbb{C}^n : z^*z < 1 + |{}^tzz/2|^2 < 2\}$,
(C) $\{z \in \mathbb{C}^n_n : {}^tz = z, z^*z < 1_n\}$,
(D) $\{z \in \mathbb{C}^n_n : {}^tz = -z, z^*z < 1_n\}$.

The spaces above are the so-called bounded realizations of the symmetric spaces, and one can with the use of the Cayley transform show that they are biholomorphic to unbounded domains. For example, when $\mathbb{D} = \mathbb{Q}$ and $\langle\, ,\, \rangle$ is skew-hermitian, then G is the symplectic group and we have the notion of Siegel modular forms defined over symmetric spaces of type C. The unbounded realization is the classical Siegel upper space. When $\mathbb{D}$ is an imaginary quadratic field, G is the unitary group and we have the notion of Hermitian modular forms defined over symmetric spaces of type A. For these two types of domains (and their groups), there has been an intensive study on the algebraic properties of their attached special L-values. We will not cite here the vast literature that has grown in the past few decades, so we will only mention here the book [32], the more recent article [21] and the references therein for a more complete account of the Siegel case, and the work of Harris [11] in the Hermitian modular forms case.

The focus of this paper is on the domains of type D above. This domain arises when we select $\mathbb{D}$ to be a definite quaternion algebra and the form $\langle\, ,\, \rangle$ skew-hermitian (see the next section for details). There are already some works for these modular forms (for example, [3, 16, 35, 37]), but it is fair to say that these modular forms are not as intensively studied as the Siegel or Hermitian ones. Even more importantly, most, if not all, of the works are restricted to the case when the dimension of V is even. The importance of this restriction is related to the unbounded realization of the corresponding symmetric domain. In particular, when n is even, the unbounded domain is biholomorphic to a tube domain, or what is usually called a domain of Siegel Type I. When n is odd, the domain is not any more of tube type (a similar aspect is seen also for Hermitian modular forms in the non-split case $U(n, m)$ with $n \neq m$). The significance of this distinction will become clear later in this paper, since the non-tube case is considerably more technical. For example, as we will see, the notion of an algebraic modular form cannot be the usual one (algebraic Fourier coefficients) or the doubling embedding which is needed in the doubling method is considerably more complicated to write explicitly in the unbounded realization.

As we have indicated, we will be studying the special values of L-functions by using the doubling method of Garrett, Shimura, Piatetski-Shapiro, and Rallis. Without going here into details but referring later to the paper, the key idea is to obtain an integral representation relating the L-function to the pullback of a Siegel-type Eisenstein series. Then the analytic and algebraic properties of the L-function can be studied from those of the Eisenstein series. The latter is well understood thanks to the rather explicitly known Fourier expansion.

Our starting point is a cuspidal Hecke eigenform $\mathbf{f} \in \mathcal{S}_k(K_1(\mathfrak{n}))$. We then consider two copies of our group G_n with an embedding $G_n \times G_n \to G_N$ with $N = 2n$, and hence G_N splits (see Section 4 for notation). For P_N the Siegel parabolic subgroup of G_N, we describe in Proposition 4.3 the double coset $P_N \backslash G_N / G_n \times G_n$. Then a Siegel-type Eisenstein series over G_N can be decomposed into several orbits, and except for one "main orbit," all orbits vanish when considering an inner product (in one variable) with the cusp form $\mathbf{f}$. This allows us to prove the following formula (see Section 4 and, in particular, Theorem 4.7 for details and further notation):

$$\int_{G_n(\mathbb{Q})\backslash G_n(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}(g\times h, s)\mathbf{f}(h)\mathbf{d}h = c_k(s)D(s,\mathbf{f},\chi)\mathbf{f}(g),$$

where χ is a Dirichlet character, $c_k(s)$ is an explicit function on s, and $D(s,\mathbf{f},\chi)$ is a Dirichlet series which is related (see equation (3.1)) to the twisted standard L-function $L(s,\mathbf{f},\chi)$.

In Section 5, we review the definition of algebraic modular forms and differential operators. It is well known how to define algebraic modular forms on hermitian symmetric space. There are several different definitions, and we will mainly follow the one via CM-point as in [32]. Using the Maass–Shimura differential operators, we discuss the notion of a nearly holomorphic modular form in our setting. These differential operators for all four types of symmetric spaces mentioned above have been studied in [27, 29]. We will summarize the result there and apply it to the Siegel-type Eisenstein series mentioned above. Based on this and thanks to the well-understood Fourier expansion of Siegel-type Eisenstein series, we will prove our main algebraic result for L-functions by the same method as in [32]. Our main result is Theorem 6.3 which gives the following.

Theorem 1.1 *Let $\mathfrak{n}$ be an ideal in $\mathbb{Z}$, and assume that all finite places v with $v \nmid \mathfrak{n}$ are split in $\mathbb{B}$. Let $\mathbf{f}\in\mathcal{S}_k(K_1(\mathfrak{n}),\overline{\mathbb{Q}})$ be an algebraic cuspidal Hecke eigenform, and let χ be a Dirichlet character whose conductor divides the ideal $\mathfrak{n}$. Assume that $k>2n-1$, and let $\mu\in\mathbb{Z}$ such that $2n-1<\mu\le k$. Then*

$$\frac{L(\mu,\mathbf{f},\chi)}{\pi^{n(k+\mu)-\frac{3}{2}n(n-1)}\langle \mathbf{f},\mathbf{f}\rangle}\in\overline{\mathbb{Q}}.$$

Remark 1.2 We note here that the condition on the conductor of the Dirichlet character is not restrictive. Indeed, for $\mathbf{f}\in\mathcal{S}_k(K_1(\mathfrak{n}),\overline{\mathbb{Q}})$ and χ of conductor $\mathfrak{m}$, we can select $\mathfrak{n}'=\mathfrak{n}\mathfrak{m}$ instead of $\mathfrak{n}$ since $\mathcal{S}_k(K_1(\mathfrak{n}),\overline{\mathbb{Q}})\subset\mathcal{S}_k(K_1(\mathfrak{n}'),\overline{\mathbb{Q}})$.

Most of our arguments to prove the above are straightforward generalization of [32] from the unitary and symplectic setting to our setting. Our main contribution is making some of the not always obvious generalizations as explicit as possible, such as the diagonal embedding, especially in the non-tube case (see Section 2.3), and the coset decomposition $P_N\backslash G_N/G_n\times G_n$ (see Section 4). Another contribution of the present paper is in the definition of algebraic modular forms (see Section 5), especially in the non-tube case, where we follow a rather more explicit approach by using the theory of CM-points developed by Shimura [32] rather than simply referring to the more advanced and general theory of Harris [9, 10], Deligne [5], and Milne [20] on automorphic vector bundles of Shimura varieties. Finally, we should add that our computations are done mainly using the adelic language (in comparison to the more classical in [32]), which is also inspired by [21].

2 Groups and symmetric spaces

In this and the next section, we introduce the notion of a quaternionic modular form and discuss some main properties. Such modular forms have been already studied (see, for example, [3, 16]), but we extend the discussion to include also the case of non-split groups. For most of our notation here, we follow the one introduced in the books [31, 32], where the case of Siegel and Hermitian modular forms is considered.

2.1 Quaternionic unitary groups

We start by fixing some notation. For more details on quaternion algebras, the reader is referred to [36]. In this work, a quaternion algebra will mean a central simple algebra of dimension 4 over $\mathbb{Q}$. After selecting a basis, we can write it in the form

$$\mathbb{B} = \mathbb{Q} \oplus \mathbb{Q}\zeta \oplus \mathbb{Q}\xi \oplus \mathbb{Q}\zeta\xi,$$

where

$$\zeta^2 = \alpha, \xi^2 = \beta, \zeta\xi = -\xi\zeta,$$

with α, β nonzero square-free integers. We assume in this paper that $\mathbb{B}$ is definite, i.e., $\alpha, \beta < 0$. The main involution of $\mathbb{B}$ is given by

$$\bar{\cdot} : \mathbb{B} \to \mathbb{B} : a + b\zeta + c\xi + d\zeta\xi \mapsto \overline{a + b\zeta + c\xi + d\zeta\xi} = a - b\zeta - c\xi - d\zeta\xi.$$

We warn the reader that we may, by abusing the notation, denote $\bar{\cdot}$ various involution of algebras (for example, complex conjugation on quadratic imaginary fields), but it will be always clear from the context what is meant. The trace and the norm are defined by $\mathrm{tr}(x) = x + \bar{x}, N(x) = x\bar{x}$ for $x \in \mathbb{B}$. As usual, we write $M_n(\mathbb{B})$ for the set of $n \times n$ matrices with entries in $\mathbb{B}$. We also use the notation $\mathbb{B}_n^m$ for the set of $m \times n$ matrices with entries in $\mathbb{B}$. For $X \in M_n(\mathbb{B})$, we write $X^* = {}^t\bar{X}, \hat{X} = (X^*)^{-1}$ for the conjugate transpose and its inverse (if makes sense).

Identify ζ, ξ with $\sqrt{\alpha}, \sqrt{\beta} \in \overline{\mathbb{Q}}$, and let $\mathbb{K} = \mathbb{Q}(\xi)$. We define the embedding

$$\mathfrak{i} : \mathbb{B} \to M_2(\mathbb{K}), a + b\zeta + c\xi + d\zeta\xi \mapsto \begin{bmatrix} a + c\xi & \alpha(b - d\xi) \\ b + d\xi & a - c\xi \end{bmatrix}.$$

One easily checks that for $x \in \mathbb{B}$,

$$\mathfrak{i}(x)^* = I^{-1}\mathfrak{i}(x^*)I, \quad I := \begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix},$$

$${}^t\mathfrak{i}(x) = J^{-1}\mathfrak{i}(x^*)J, \quad J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and $\mathfrak{i}$ induces an isomorphism

$$\mathfrak{i} : \mathbb{B} \xrightarrow{\sim} \{x \in M_2(\mathbb{K}) : \bar{x}IJ = IJx\}.$$

We extend this map to an embedding $\mathfrak{i} : M_n(\mathbb{B}) \to M_{2n}(\mathbb{K})$ by sending $x = (x_{ij})$ to $(\mathfrak{i}(x_{ij}))$. Denote $I'_n = \mathrm{diag}[I, \ldots, I], J'_n = \mathrm{diag}[J, \ldots, J]$ with n copies. Then, for $x \in M_n(\mathbb{B})$,

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$$\mathfrak{i}(x)^* = I_n'^{-1}\mathfrak{i}(x^*)I_n', \quad {}^t\mathfrak{i}(x) = J_n'^{-1}\mathfrak{i}(x^*)J_n',$$

and $\mathfrak{i}$ induces an isomorphism

$$\mathfrak{i} : M_n(\mathbb{B}) \xrightarrow{\sim} \{x \in M_{2n}(\mathbb{K}) : \overline{x}I_n'J_n' = I_n'J_n'x\}.$$

For a matrix with entries in quaternion algebra, the determinant det and the trace tr will mean the reduced norm and the reduced trace. That is taking the determinant and the trace for its image under $\mathfrak{i}$. It is well known that the definition of reduced norm and trace is indeed independent of the choice of such embedding and the field $\mathbb{K}$. Denote

$$\mathrm{GL}_n(\mathbb{B}) = \{g \in M_n(\mathbb{B}) : \det(g) \neq 0\}, \mathrm{SL}_n(\mathbb{B}) = \{g \in M_n(\mathbb{B}) : \det(g) = 1\}.$$

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. By a place v, we mean either a finite place corresponding to a prime or the archimedean place ∞. The set of finite places is denoted by $\mathbf{h}$. We write $\mathbb{A} = \mathbb{A}_\mathbf{h}\mathbb{R}$ with finite adeles $\mathbb{A}_\mathbf{h}$ and $x = x_\mathbf{h}x_\infty$ with $x \in \mathbb{A}, x_\mathbf{h} \in \mathbb{A}_\mathbf{h}, x_\infty \in \mathbb{R}$. Fix embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ and set $\mathbb{B}_v = \mathbb{B} \otimes_\mathbb{Q} \mathbb{Q}_v$. The previous definition of trace, norm, and determinant naturally extends locally or adelically. Fix a maximal order $\mathcal{O}$ of $\mathbb{B}$ and set $\mathcal{O}_v = \mathcal{O} \otimes_\mathbb{Z} \mathbb{Z}_v$. For a place v, we say v splits if $\mathbb{B}_v \cong M_2(\mathbb{Q}_v)$. If this is the case, we fix an isomorphism $\mathfrak{i}_v : \mathbb{B}_v \xrightarrow{\sim} M_2(\mathbb{Q}_v)$ and assume $\mathfrak{i}_v(\mathcal{O}_v) = M_2(\mathbb{Z}_v)$ for finite place. $\mathbb{B}$ is called *indefinite* if B_v is split for $v = \infty$ and *definite* otherwise. In particular, $\mathbb{B}$ is definite if $\alpha, \beta < 0$, and indefinite otherwise. That is, for the infinite place, by our assumption, $\mathbb{B}_\infty$ is the Hamilton quaternion

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{ij}, \quad \mathbf{i}^2 = \mathbf{j}^2 = -1, \mathbf{ij} = -\mathbf{ji},$$

and the map $\mathfrak{i}$ above induces an isomorphism

$$\mathfrak{i} : M_n(\mathbb{H}) \xrightarrow{\sim} \{x \in M_{2n}(\mathbb{C}) : \overline{x}J_n' = J_n'x\}.$$

In this paper, we consider the following algebraic groups:

$$G = G_n(\mathbb{Q}) = \{g \in \mathrm{SL}_n(\mathbb{B}) : g^*\phi g = \phi\}, \phi = \begin{bmatrix} 0 & 0 & -1_m \\ 0 & \zeta \cdot 1_r & 0 \\ 1_m & 0 & 0 \end{bmatrix}.$$

Here, $n = 2m + r$ and we assume that $m > 1$ is the global Witt index of the group G. Such a group is usually called a *quaternionic unitary group*, and has $\mathbb{Q}$-rank equal to m. For a split place v, $G(\mathbb{Q}_v)$ is isomorphic to an orthogonal group. That is,

$$G(\mathbb{Q}_v) \cong \left\{ g \in \mathrm{SL}_{2n}(\mathbb{Q}_v) : {}^tg \begin{bmatrix} 0 & 0 & 1_{m_v} \\ 0 & \theta & 0 \\ 1_{m_v} & 0 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 0 & 1_{m_v} \\ 0 & \theta & 0 \\ 1_{m_v} & 0 & 0 \end{bmatrix} \right\}, \tag{2.1}$$

with some anisotropic matrix $\theta = {}^t\theta \in \mathrm{SL}_{r_v}(\mathbb{Q}_v)$ (that is, the corresponding quadratic form does not represent zero), and $2n = 2m_v + r_v$ for some positive integers m_v, r_v with $m_v \geq 2m$ and $r_v \leq 2r$. In particular, $G(\mathbb{Q}_v)$ is totally isotropic if $r = 0$ or (using [33, Lemma 1.7]) if $\alpha \in \mathbb{Q}_v^{\times 2}$. In these two cases, we have $m_v = n, r_v = 0$, and

$$G(\mathbb{Q}_v) \cong \left\{ g \in \mathrm{SL}_{2n}(\mathbb{Q}_v) : {}^tg \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} \right\}.$$

We remark here that the condition on m being the Witt index of our group G implies that $r \leq 3$. Indeed, using a result in [13], we know that for $r \geq 4$, $\zeta \cdot 1_r$ is isotropic if and only if it is locally isotropic for all finite places v and infinity. But the latter (i.e., locally isotropic for all finite places and infinity) is always the case for $r \geq 4$. Indeed, for v split, this follows from [33, Theorem 7.6] and for v nonsplit (including ∞) from [34].

We will discuss the local archimedean group $G(\mathbb{R})$ and the associated symmetric spaces in the next subsection.

We fix an integral two-sided ideal $\mathfrak{n} = (\mathcal{N})$ of $\mathcal{O}$ generated by $\mathcal{N} = \prod_v p_v^{n_v} \in \mathbb{Z}$. We define an open compact subgroup $K_1(\mathfrak{n}) \subset G(\mathbb{A}_\mathbf{h})$ by $K_1(\mathfrak{n}) = \prod_v K_v$, where

$$K_v = \left\{ \gamma = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \in G(\mathcal{O}_v) : \gamma \equiv \begin{bmatrix} 1_m & * & * \\ 0 & 1_r & * \\ 0 & 0 & 1_m \end{bmatrix} \bmod p_v^{n_v} \right\}.$$

It is well known (see, for example, [22, p. 251]) that we have a finite decomposition

$$G(\mathbb{A}) = \bigcup_j G(\mathbb{Q}) t_j K_1(\mathfrak{n}) G(\mathbb{R}).$$

Moreover, thanks to the weak approximation which is valid for our group (see [22, Proposition 7.11]), we can take t_j such that $(t_j)_v = 1$ for $v|\mathfrak{n}$ (compare with [31, Lemma 8.12]). For finite places v not in the support of $\mathfrak{n}$, the Iwasawa decomposition is valid, and hence we can take t_j to be upper triangular. Let $\Gamma_1^j = t_j K_1(\mathfrak{n}) t_j^{-1} \cap G(\mathbb{Q})$. We can take $t_0 = 1$ so that

$$\Gamma_1^0 = \Gamma_1(\mathcal{N}) = \left\{ \gamma = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \in G(\mathcal{O}) : \gamma \equiv \begin{bmatrix} 1_m & * & * \\ 0 & 1_r & * \\ 0 & 0 & 1_m \end{bmatrix} \bmod \mathcal{N} \right\}.$$

2.2 Symmetric spaces

2.2.1 Abstract symmetric spaces

To motivate our definition of symmetric spaces, we start with a rather general and abstract setting before giving explicit realizations of our symmetric spaces. Let $\mathfrak{i}$ be any embedding $M_n(\mathbb{H}) \to M_{2n}(\mathbb{C})$. Then, by the Skolem–Noether theorem, there exists $\alpha \in M_{2n}(\mathbb{C})$ with $\alpha\alpha^* = 1$ such that ${}^t\mathfrak{i}(x) = \alpha \mathfrak{i}(x^*)\alpha^{-1}$. Let $\Phi \in \mathrm{GL}_n(\mathbb{B})$ be a skew-hermitian form similar to ϕ above, that is, $\Phi = \gamma^* \phi \gamma$ for some $\gamma \in \mathrm{GL}_n(\mathbb{B})$. Then the group $G(\mathbb{R})$ is isomorphic to

$$\mathcal{G} = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^* H g = H, {}^t g K g = K\},$$

with $H = \mathfrak{i}(\Phi), K = \alpha^{-1}\mathfrak{i}(\Phi)$. We call it a realization of $G(\mathbb{R})$. Suppose that we are given two such data $(\mathfrak{i}_1, \Phi_1, H_1, K_1, \mathcal{G}_1)$ and $(\mathfrak{i}_2, \Phi_2, H_2, K_2, \mathcal{G}_2)$ with $\Phi_1 = S^* \Phi_2 S$. Again, by Skolem–Noether, there exists β with $\beta\beta^* = 1$ such that $\mathfrak{i}_1(x) = \beta^{-1}\mathfrak{i}_2(x)\beta$. Put $R = \mathfrak{i}_2(S)\beta$, then $H_1 = R^* H_2 R, K_1 = {}^t R K_2 R$. Therefore, $g \mapsto RgR^{-1}$ gives isomorphism $\mathcal{G}_1 \cong \mathcal{G}_2$.

Following [23], we will define the associated symmetric space via its Borel embedding into its compact dual symmetric space. In our case, we have that the semisimple compact dual of our group is the group $\mathrm{SO}(2n)$ (see [12, p. 330]), and the corre-

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sponding dual symmetric space is $\mathrm{SO}(2n)/\mathrm{U}(n)$. This space may be identified (see, for example, [28, p. 6]) with the space $V = L/GL_n(\mathbb{C})$ where

$$L = \{U \in \mathbb{C}_n^{2n} : \ {}^tUKU = 0\}.$$

We set

$$\Omega = \{U \in \mathbb{C}_n^{2n} : -iU^*HU > 0, {}^tUKU = 0\} \subset L,$$

with the action of $\mathrm{GL}_n(\mathbb{C})$ by right multiplication and $\mathcal{G}$ by left multiplication. The symmetric space $\mathcal{H}$ is defined as

$$\mathcal{H} = \{z \in \mathbb{C}_n^{2n} : U(z) \in \Omega\}, \quad U(z) := \begin{bmatrix} z \\ u_0 \end{bmatrix},$$

for some fixed suitable u_0, which we make explicit later. The following lemma is a direct consequence of our definition for $\mathcal{H}$.

Lemma 2.1 *There is a bijection $\mathcal{H} \times \mathrm{GL}_n(\mathbb{C}) \to \Omega$ given by $z \times \lambda = U(z)\lambda$.*

Note that $\mathcal{G}$ acts on Ω by left multiplication. By the above lemma, it follows that for any element $\alpha \in \mathcal{G}$, we can find a $z' \in \mathcal{H}$ and an $\lambda(\alpha, z) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\alpha U(z) = U(z')\lambda(\alpha, z).$$

We then define the action of $G(\mathbb{R})$ on $\mathcal{H}$ by $\alpha.z := \alpha z := z'$ and $\lambda(\alpha, z)$ satisfies the cocycle relation

$$\lambda(\alpha_1\alpha_2, z) = \lambda(\alpha_1, \alpha_2 z)\lambda(\alpha_2, z) \text{ for } \alpha_1, \alpha_2 \in \mathcal{G}, z \in \mathcal{H}.$$

We set $j(\alpha, z) := \det(\lambda(\alpha, z)) \in \mathbb{C}^\times$. We call $\lambda(\alpha, z)$ or $j(\alpha, z)$ automorphy factors. More explicitly, write $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\alpha U(z) = \begin{bmatrix} az + bu_0 \\ cz + du_0 \end{bmatrix} = \begin{bmatrix} (az + bu_0)(cz + du_0)^{-1}u_0 \\ u_0 \end{bmatrix} u_0^{-1}(cz + du_0).$$

That is, $\alpha z = (az + bu_0)(cz + du_0)^{-1}u_0$,and $\lambda(\alpha, z) = u_0^{-1}(cz + du_0)$.

For $z_1, z_2 \in \mathcal{H}$, we set

$$\eta(z_1, z_2) := iU(z_1)^*HU(z_2),$$
$$\delta(z_1, z_2) := \det(\eta(z_1, z_2)) \text{ and } \eta(z) := \eta(z, z), \delta(z) := \delta(z, z).$$

We now note that

$$U(z_1)^*HU(z_2) = \lambda(\alpha, z_1)^*U(\alpha z_1)^*HU(\alpha z_2)\lambda(\alpha, z_2)$$

and

$$iU(\alpha z_1)^*HU(\alpha z_2) = \begin{bmatrix} \eta(\alpha z_1, \alpha z_2) & * \\ * & * \end{bmatrix}, iU(z_1)^*HU(z_2) = \begin{bmatrix} \eta(z_1, z_2) & * \\ * & * \end{bmatrix}.$$

In particular, we obtain that

$$\lambda(\alpha, z_1)^*\eta(\alpha z_1, \alpha z_2)\lambda(\alpha, z_2) = \eta(z_1, z_2),$$

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and after taking the determinant, we have

$$\overline{j(\alpha, z_1)}\delta(\alpha z_1, \alpha z_2) j(\alpha, z_2) = \delta(z_1, z_2).$$

In particular,

$$\lambda(\alpha, z)^* \eta(\alpha z)\lambda(\alpha, z) = \eta(z), \quad \delta(\alpha z) = |j(\alpha, z)|^{-2}\delta(z).$$

We now discuss the relation between different realizations of the symmetric space $\mathcal{H}$. Given H_1, K_1 and H_2, K_2 as above, we have seen at the beginning of this subsection that we can find an R such that $H_1 = R^* H_2 R, K_1 = {}^t R K_2 R$. We then have an isomorphism $\Omega_1 \cong \Omega_2$ given by $U \mapsto RU$ which induces isomorphism $\rho : \mathcal{H}_1 \cong \mathcal{H}_2$. Indeed, for $z_1 \in \mathcal{H}_1$, there exists some $z_2 \in \mathcal{H}_2, \mu(z_1) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$R \begin{bmatrix} z_1 \\ u_{01} \end{bmatrix} = \begin{bmatrix} z_2 \\ u_{02} \end{bmatrix} \mu(z_1), \tag{2.2}$$

and the isomorphism can be given by $\rho(z_1) = z_2$.

In the following lemma, we write ρ also for the isomorphism $\mathcal{G}_1 \to \mathcal{G}_2$ given by $\rho(g_1) := R g_1 R^{-1}$.

Lemma 2.2 *Let $\rho : \mathcal{G}_1 \to \mathcal{G}_2, \rho : \mathcal{H}_1 \to \mathcal{H}_2$ given as above. Then:*

(1) $\rho(\alpha z) = \rho(\alpha)\rho(z)$ *with* $\alpha \in \mathcal{G}_1, z \in \mathcal{H}_1$.

(2) $\lambda(\rho(\alpha), \rho(z)) = \mu(\alpha z)\lambda(\alpha, z)\mu(z)^{-1}$.

(3) $\eta(\rho(z_1), \rho(z_2)) = \widetilde{\mu(z_1)}\eta(z_1, z_2)\mu(z_2)^{-1}$ *for* $z_1, z_2 \in \mathcal{H}_1$.

Proof (1) It suffices to prove that $\begin{bmatrix} \rho(\alpha z) \\ u_{02} \end{bmatrix} = \begin{bmatrix} \rho(\alpha)\rho(z) \\ u_{02} \end{bmatrix}$. By definition of the isomorphism and action,

$$\begin{aligned} \begin{bmatrix} \rho(\alpha z) \\ u_{02} \end{bmatrix} &= R \begin{bmatrix} \alpha z \\ u_{01} \end{bmatrix} \mu(\alpha z)^{-1} = R\alpha \begin{bmatrix} z \\ u_{01} \end{bmatrix} \lambda(\alpha, z)^{-1}\mu(\alpha z)^{-1} \\ &= \rho(\alpha) \begin{bmatrix} \rho(z) \\ u_{02} \end{bmatrix} \mu(z)\lambda(\alpha, z)^{-1}\mu(\alpha z)^{-1} \\ &= \begin{bmatrix} \rho(\alpha)\rho(z) \\ u_{02} \end{bmatrix} \lambda(\rho(\alpha), \rho(z))\mu(z)\lambda(\alpha, z)^{-1}\mu(\alpha z)^{-1}. \end{aligned}$$

We must have $\lambda(\rho(\alpha), \rho(z))\mu(z)\lambda(\alpha, z)^{-1}\mu(\alpha z)^{-1} = 1$, and our desired result follows which we also obtain (2). (3) can be computed similarly by definition of η. ■

2.2.2 The symmetric space $\mathfrak{Z}$

We now apply the above considerations to some explicit realizations of $G(\mathbb{R})$. We first define a symmetric space $\mathfrak{Z}$ which can be directly obtained from $G(\mathbb{R})$. This realization is useful in the computations of the doubling map and Lemma 4.6.

Note that the map $\mathfrak{i}$ defined above induces the following isomorphism on $\mathbb{Q}$-groups:

$$\mathfrak{i} : G \xrightarrow{\sim} \mathfrak{G} = \{g \in \mathrm{GL}_{2n}(\mathbb{K}) : g^*\Phi g = \Phi, {}^t g\Psi g = \Psi\},$$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & -1_{2m} \\ 0 & 0 & -1_r & 0 \\ 0 & 1_r & 0 & 0 \\ 1_{2m} & 0 & 0 & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & 0 & 0 & J'_m I'_m \\ 0 & -\alpha^{-1} & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ -J'_m I'_m & 0 & 0 & 0 \end{bmatrix}.$$

This induces the following isomorphism on $\mathbb{R}$-groups:

$$\mathfrak{i} : G(\mathbb{R}) \xrightarrow{\sim} G_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^*\phi_\infty g = \phi_\infty, {}^t g\psi_\infty g = \psi_\infty\},$$

$$\phi_\infty = \begin{bmatrix} 0 & 0 & 0 & -1_{2m} \\ 0 & 0 & -1_r & 0 \\ 0 & 1_r & 0 & 0 \\ 1_{2m} & 0 & 0 & 0 \end{bmatrix}, \psi_\infty = \begin{bmatrix} 0 & 0 & 0 & J'_m \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & 1_r & 0 \\ -J'_m & 0 & 0 & 0 \end{bmatrix}.$$

Let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$. As in the last subsection,

$$\Omega = \{U \in \mathbb{C}^{2n}_n : -iU^*\phi_\infty U > 0, {}^t U\psi_\infty U = 0\},$$

and define the symmetric space by

$$\mathfrak{Z} = \mathfrak{Z}_n = \mathfrak{Z}_{m,r} = \{z \in \mathbb{C}^n_n : U(z) \in \Omega\}, U(z) = \begin{bmatrix} z \\ u_0 \end{bmatrix}, u_0 = \begin{bmatrix} 0 & 1_r \\ 1_{2m} & 0 \end{bmatrix}.$$

Explicitly,

$$\mathfrak{Z} = \left\{ z = (u, v, w) := \begin{bmatrix} u & v \\ w\,{}^t v J'_m & w \end{bmatrix} : \begin{matrix} u \in \mathbb{C}^{2m}_{2m}, v \in \mathbb{C}^{2m}_r, w \in \mathbb{C}_{r^r}, i(z^* - z) > 0, \\ {}^t ww + 1 = 0, uJ'_m + v\,{}^t v - J'_m\,{}^t U = 0. \end{matrix} \right\}.$$

The action of G_∞ on $\mathfrak{Z}$ is given by

$$gz = (az + bu_0)(cz + du_0)^{-1}u_0, \lambda(g, z) = u_0^{-1}(cz + du_0), g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty.$$

For $z_1, z_2 \in \mathfrak{Z}$, we set $\eta(z_1, z_2) = i(z_1^* - z_2), \delta(z_1, z_2) = \det(\eta(z_1, z_2))$ and $\eta(z) = \eta(z, z), \delta(z) = \delta(z, z)$. We will take $z_0 = i \cdot 1_n$ to be the origin of $\mathfrak{Z}$ and K_∞ the subgroup of G_∞ fixing z_0. Then $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K_\infty \cong U(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^*g = 1_n\}$ and our symmetric space $\mathfrak{Z} \cong G_\infty/K_\infty$. We note that we are using the same notation K_∞ for maximal subgroup of G_∞ and its preimage in $G(\mathbb{R})$.

2.2.3 The symmetric spaces $\mathfrak{H}$ and $\mathfrak{B}$

We now give another two useful realizations. They are much simpler than the symmetric space $\mathfrak{Z}$ and are useful in studying CM-points in Section 5.1. However, as the isomorphism between G_∞ above and G'_∞ below is rather complicated, the action of the $\mathbb{Q}$-group G on the symmetric space is difficult to compute.

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Note that G is isomorphic to

$$G' = \{g \in \mathrm{GL}_n(\mathbb{B}) : g^* \phi' g = \phi'\}, \phi' = \begin{bmatrix} \zeta \cdot 1_m & 0 & 0 \\ 0 & \zeta \cdot 1_r & 0 \\ 0 & 0 & -\zeta \cdot 1_m \end{bmatrix}.$$

By changing rows and columns, the map $\mathfrak{i}$ induces isomorphism

$$\mathfrak{i} : G' \xrightarrow{\sim} \mathfrak{G}' = \{g \in \mathrm{GL}_{2n}(\mathbb{K}) : g^* \Phi' g = \Phi', {}^t g \Psi' g = \Psi'\},$$

$$\Phi' = J_n, \Psi' = \mathrm{diag}[1, 1, 1, -\alpha, -\alpha, -\alpha],$$

and

$$\mathfrak{i} : G'(\mathbb{R}) \cong G'_\infty := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^* J_n g = J_n, {}^t g g = 1_{2n}\}.$$

Take $u_0 = 1$, and the symmetric space associated with this group is

$$\mathfrak{H} = \mathfrak{H}_n = \{z \in \mathbb{C}_n^n : {}^t z z + 1 = 0, i(z^* - z) > 0\}.$$

This is an unbounded realization of type-D domain in [17]. The action of G'_∞ on $\mathfrak{H}$ and the automorphy factor is given by

$$gz = (az + b)(cz + d)^{-1}, \lambda(g, z) = cz + d, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For $z_1, z_2 \in \mathfrak{H}$, we set $\eta(z_1, z_2) = i(z_1^* - z_2)$. We take $z_0 = i_n := i \cdot 1_n$ to be the origin of $\mathfrak{H}$ and K'_∞ the subgroup of G'_∞ fixing z_0. Since $\eta(gz_0) = \eta(z_0) = 2$ for $g \in K_\infty$, $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K'_\infty \cong U(n)$ and thus $\mathfrak{H} \cong G'_\infty / K'_\infty$.

Let $T' = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, and sending $g \mapsto T'^{-1} g T'$, we have isomorphism

$$\mathfrak{i}_\infty" : G'_\infty \xrightarrow{\sim} G_\infty" = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^* \phi_\infty" g = \phi_\infty", {}^t g \psi_\infty" g = \psi_\infty"\},$$

with

$$\phi_\infty" = \begin{bmatrix} i_n & 0 \\ 0 & -i_n \end{bmatrix}, \psi_\infty" = \begin{bmatrix} 0 & -i_n \\ -i_n & 0 \end{bmatrix}.$$

Take $u_0 = 1$, and the symmetric space associated with this group is defined as

$$\mathfrak{B} = \mathfrak{B}_n = \{z \in \mathbb{C}_n^n : {}^t z = -z, zz^* < 1_n\}.$$

This is a bounded domain of type $\mathfrak{R}_{\mathrm{III}}$ in [14]. The action of $G_\infty"$ on $\mathfrak{B}$ and the automorphy factor is given by

$$gz = (az + b)(cz + d)^{-1}, \lambda(g, z) = cz + d, g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For $z_1, z_2 \in \mathfrak{H}$, we set $\eta(z_1, z_2) = i(z_1^* z_2 - 1)$. We take $z_0 = 0$ to be the origin of $\mathfrak{B}$ and $K_\infty"$ the subgroup of $G_\infty"$ fixing z_0. Since $\eta(gz_0) = \eta(z_0) = -i$ for $g \in K_\infty$, $g \mapsto \lambda(g, z_0)$ gives an isomorphism $K_\infty" \cong U(n)$ and thus $\mathfrak{H} \cong G_\infty" / K_\infty"$. The relation between $\mathfrak{H}$ and $\mathfrak{B}$ can be given explicitly by the Cayley transform

$$\mathfrak{H} \xrightarrow{\sim} \mathfrak{B} : z \mapsto (z-i)(z+i)^{-1}.$$

Let $z_1, z_2 \in \mathfrak{B}_n, \alpha \in G(\mathbb{R})$ as above, and $dz = (dz_{hk})$ be a matrix of the same shape as $z \in \mathbb{C}_n^n$ whose entries are 1-forms dz_{hk}. Comparing

$$\begin{bmatrix} z_1 & 1 \\ 1 & -\bar{z}_1 \end{bmatrix}^* \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix} \begin{bmatrix} z_2 & 1 \\ 1 & -\bar{z}_2 \end{bmatrix} = \begin{bmatrix} z_1^* z_2 - 1 & z_1^* + \bar{z}_2 \\ z_2 + {}^t z_1 & 1 - {}^t z_1 \bar{z}_2 \end{bmatrix} = \begin{bmatrix} z_1^* z_2 - 1 & z_1^* - z_2^* \\ z_2 - z_1 & 1 - {}^t z_1 \bar{z}_2 \end{bmatrix},$$

$$\begin{bmatrix} \alpha z_1 & 1 \\ 1 & -\overline{\alpha z_1} \end{bmatrix}^* \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix} \begin{bmatrix} \alpha z_2 & 1 \\ 1 & -\overline{\alpha z_2} \end{bmatrix} = \begin{bmatrix} (\alpha z_1)^*(\alpha z_2) - 1 & (\alpha z_1)^* - (\alpha z_2)^* \\ \alpha z_2 - \alpha z_1 & 1 - {}^t(\alpha z_1)(\overline{\alpha z_2}) \end{bmatrix},$$

and using the fact (which can be obtained from the property of $U(z)$)

$$\alpha \begin{bmatrix} z & 1 \\ 1 & -\bar{z} \end{bmatrix} = \begin{bmatrix} \alpha z & 1 \\ 1 & -\overline{\alpha z} \end{bmatrix} \begin{bmatrix} \lambda(\alpha, z) & 0 \\ 0 & \overline{\lambda(\alpha, z)} \end{bmatrix},$$

we have

$$\alpha z_2 - \alpha z_1 = {}^t\lambda(\alpha, z_1)^{-1}(z_2 - z_1)\lambda(\alpha, z_2)^{-1}.$$

Therefore,

$$d(\alpha z) = {}^t\lambda(\alpha, z)^{-1} \cdot dz \cdot \lambda(\alpha, z)^{-1}.$$

Since the jacobian of the map $z \mapsto \alpha z$ is $j(\alpha, z)^{-n+1}$, the differential form

$$\mathbf{d}z = \delta(z)^{-n+1} \prod_{h \le k} [(i/2) dz_{hk} \wedge d\bar{z}_{hk}]$$

is an invariant measure. If we have another realization $\mathcal{H}$ (e.g., $\mathfrak{Z}, \mathfrak{H}$) with identification $\rho : \mathcal{H} \to \mathcal{B}$, we then define $\mathbf{d}z := \mathbf{d}(\rho(z))$ with $z \in \mathcal{H}$ to be the differential form on $\mathcal{H}$. Clearly, this is also an invariant measure.

2.3 Doubling embedding

We keep the notation as before and consider two groups

$$G_{n_1} = \{g \in \mathrm{GL}_{n_1}(\mathbb{B}) : g^* \phi_1 g = \phi_1\}, \phi_1 = \begin{bmatrix} 0 & 0 & -1_{m_1} \\ 0 & \zeta \cdot 1_r & 0 \\ 1_{m_1} & 0 & 0 \end{bmatrix},$$

$$G_{n_2} = \{g \in \mathrm{GL}_{n_2}(\mathbb{B}) : g^* \phi_2 g = \phi_2\}, \phi_2 = \begin{bmatrix} 0 & 0 & -1_{m_2} \\ 0 & \zeta \cdot 1_r & 0 \\ 1_{m_2} & 0 & 0 \end{bmatrix},$$

with $n_1 = 2m_1 + r, n_2 = 2m_2 + r$. We always assume that $m_1 \geq m_2 > 0$. We set $N = n_1 + n_2 = 2m_1 + 2m_2 + 2r$, and consider the map

$$G_{n_1} \times G_{n_2} \to G^\omega = \{g \in \mathrm{GL}_N(\mathbb{B}) : g^* \omega g = \omega\}, \omega = \begin{bmatrix} \phi_1 & 0 \\ 0 & -\phi_2 \end{bmatrix},$$

by sending $g_1 \times g_2 \mapsto \mathrm{diag}[g_1, g_2]$. Note that $R^* \omega R = J_{N/2} := \begin{bmatrix} 0 & -1_{N/2} \\ 1_{N/2} & 0 \end{bmatrix}$, with

$$R = \begin{bmatrix} 1_{m_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & -\zeta^{-1} & 0 \\ 0 & 0 & 0 & 1_{m_2} & 0 & 0 \\ 0 & 0 & -1_{m_1} & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & -\zeta^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1_{m_2} \end{bmatrix}.$$

Composing the above map with $g \mapsto R^{-1} g R$, we obtain an embedding

$$\rho : G_{n_1} \times G_{n_2} \to G^\omega \to G_N = \{g \in \mathrm{GL}_N(\mathbb{B}) : g^* J_{N/2} g = J_{N/2}\}.$$

We can thus view $G_{n_1} \times G_{n_2}$ as a subgroup of G_N. To ease notation, we write $G_{n_1} \times G_{n_2}$ as its image in G_N under ρ and $\beta \times \gamma$ for $\beta \in G_{n_1}, \gamma \in G_{n_2}$ as an element in G_N under the embedding ρ.

Now we consider a special case of this embedding, namely the case where $m_1 = m_2 = m, n_1 = n_2 = n$. To ease the notation, we always omit the subscript "n" and keep the subscript $N = 2n$. The embedding ρ then induces

$$\rho : G_\infty \times G_\infty \to G_{N\infty}, g_1 \times g_2 \mapsto R^{-1} \mathrm{diag}[g_1, g_2] R,$$

with

$$R = R_\infty = \begin{bmatrix} 1_{2m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{2m} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_{2m} & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1_{2m} \end{bmatrix},$$

where the entries with $\pm 1, \pm 1/2$ should be understood as $\pm I_r, \pm \frac{1}{2} I_r$. Let $\mathfrak{Z}, \mathfrak{Z}_N$ be symmetric spaces associated with $G_\infty, G_{N\infty}$. We are now going to define an associated embedding of symmetric space $\iota : \mathfrak{Z} \times \mathfrak{Z} \to \mathfrak{Z}_N$. We first define the embedding

$$\Omega_1 \times \Omega_2 \to \Omega_N, U_1 \times U_2 \mapsto R^{-1} \begin{bmatrix} U_1 & 0 \\ 0 & \mathfrak{J} \overline{U_2} \end{bmatrix}, \mathfrak{J} = \begin{bmatrix} J'_m & 0 & 0 & 0 \\ 0 & 0 & -1_r & 0 \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & J'_m \end{bmatrix}.$$

Let $z_1 = (u_1, v_1, w_1), z_2 = (u_2, v_2, w_2) \in \mathfrak{Z}$. The image of $U(z_1) \times U(z_2) \in \Omega_1 \times \Omega_2$ under this map is

$$\begin{bmatrix} u_1 & v_1 & 0 & 0 \\ w_1{}^t v_1 J'_m & w_1 & 0 & 1 \\ 0 & 1 & -\overline{w}_2 v_2^* J'_m & -\overline{w}_2 \\ 0 & 0 & -J'_m \overline{u}_2 & -J'_m \overline{v}_2 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\overline{w}_2 v_2^* J'_m}{2} & \frac{\overline{w}_2}{2} \\ \frac{-w_1{}^t v_1 J'_m}{2} & \frac{-w_1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & J'_m & 0 \end{bmatrix} =: \begin{bmatrix} A(z_1, z_2) \\ B(z_1, z_2) \end{bmatrix}.$$

We then define the embedding of symmetric space as

$$\iota : \mathfrak{Z} \times \mathfrak{Z} \to \mathfrak{Z}_N, \quad z_1 \times z_2 \mapsto A(z_1, z_2) B(z_1, z_2)^{-1} S,$$

where $S := \mathrm{diag}[1, 1/2, 1/2, 1]$.

Explicitly, if we write $w_0^{-1} := 1 + w_1 \overline{w}_2$, $w'_0 := 1 - w_1 \overline{w}_2$, then

$$\iota(z_1, z_2) =$$

$$\begin{bmatrix} u_1 - v_1 \overline{w}_2 w_0 w_1{}^t v_1 J'_m & v_1 w_1^{-1} w_0 w_1 & -v_1 \overline{w}_2 w_0 & -v_1 w_1^{-1} w_0 w_1 \overline{w}_2 v_2^* \\ 2 w_0 w_1{}^t v_1 J'_m & 2 w_0 w_1 & w'_0 w_0 & -2 w_0 w_1 \overline{w}_2 v_2^* \\ -2 \overline{w}_2 w_0 w_1{}^t v_1 J'_m & w_1^{-1} w'_0 w_0 w_1 & -2 \overline{w}_2 w_0 & -2 w_1^{-1} w_0 w_1 \overline{w}_2 v_2^* \\ -J'_m \overline{v}_2 w_0 w_1{}^t v_1 J'_m & -J'_m \overline{v}_2 w_0 w_1 & -J'_m \overline{v}_2 w_0 & -J'_m \overline{u}_2 J'^{-1}_m + J'_m \overline{v_2} w_0 w_1 \overline{w}_2 v_2^* \end{bmatrix}.$$

Here, we note that we have "normalized" our embedding by S so that ι maps the origin of $\mathfrak{Z} \times \mathfrak{Z}$ to the "origin" of $\mathfrak{Z}_N$. That is, $\iota(z_0 \times z_0) = i \cdot 1_N =: Z_0$, where z_0 and Z_0 are the origins of $\mathfrak{Z}$ and $\mathfrak{Z}_N$, respectively.

For example, in the case where $r = 0$, then the embedding is quite simple, namely $\iota(z_1, z_2) = \mathrm{diag}[u_1, -u_2^*]$, where in the case of $r = 1$, it is given by (note that in this case $w_1 = w_2 = i$)

$$\iota(z_1, z_2) = \begin{bmatrix} u_1 - \frac{1}{2} v_1{}^t v_1 J'_m & \frac{1}{2} v_1 & -\frac{i}{2} v_1 & \frac{i}{2} v_1 v_2^* \\ i{}^t v_1 J'_m & i & 0 & -v_2^* \\ -{}^t v_1 J'_m & 0 & i & i v_2^* \\ -\frac{i}{2} J'_m \overline{v}_2{}^t v_1 J'_m & -\frac{i}{2} J'_m \overline{v}_2 & -\frac{1}{2} J'_m \overline{v}_2 & -u_2^* - \frac{1}{2} J'_m \overline{v}_2 v_2^* \end{bmatrix}.$$

We now show that the embedding of the symmetric spaces is compatible with the embedding of the groups.

Proposition 2.3 *For $g_1, g_2 \in G_\infty$, $z_1, z_2 \in \mathfrak{Z}$, we have:*

(1) $\iota(g_1 z_1, g_2 z_2) = \rho(g_1, g_2) \iota(z_1, z_2)$.

(2) $j(\rho(g_1, g_2), \iota(z_1, z_2)) \det(B(z_1, z_2)) = j(g_1, z_1) \overline{j(g_2, z_2)} \det(B(g_1 z_1, g_2 z_2))$.

(3) $\delta(\iota(z_1, z_2)) = |\det(B(z_1, z_2))|^{-2} \delta(z_1) \delta(z_2)$.

Proof By our definition of embedding and the action,

$$\begin{bmatrix} \iota(g_1z_2, g_2z_2) \\ 1 \end{bmatrix} = R^{-1}\begin{bmatrix} U(g_1z_1) & 0 \\ 0 & \overline{\mathfrak{J}U(g_2z_2)} \end{bmatrix} B(g_1z_1, g_2z_2)^{-1}S$$

$$= R^{-1}\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} RR^{-1}\begin{bmatrix} U(z_1) & 0 \\ 0 & \overline{\mathfrak{J}U(z_2)} \end{bmatrix}\begin{bmatrix} \lambda(g_1,z_1) & 0 \\ 0 & \overline{\lambda(g_2,z_2)} \end{bmatrix}^{-1} B(g_1z_1, g_2z_2)^{-1}S$$

$$= \begin{bmatrix} \rho(g_1,g_2)\iota(z_1,z_2) \\ 1 \end{bmatrix}\lambda(\rho(g_1,g_2),\iota(z_1,z_2))\times$$

$$S^{-1}B(z_1,z_2)\begin{bmatrix} \lambda(g_1,z_1) & 0 \\ 0 & \overline{\lambda(g_2,z_2)} \end{bmatrix}^{-1} B(g_1z_1, g_2z_2)^{-1}S.$$

We must have

$$\lambda(\rho(g_1,g_2),\iota(z_1,z_2))S^{-1}B(z_1,z_2)\begin{bmatrix} \lambda(g_1,z_1) & 0 \\ 0 & \overline{\lambda(g_2,z_2)} \end{bmatrix}^{-1} B(g_1z_1, g_2z_2)^{-1}S = 1,$$

and the desired result follows. Taking the determinant, we also obtain (2).

Suppose $z_1 = g_1z_0, z_2 = g_2z_0$ for $g_1 \in G_{1\infty}, g_2 \in G_{2\infty}$ with z_0 the origin of $\mathfrak{Z}_1$ or $\mathfrak{Z}_2$. Then

$$\delta(\iota(z_1,z_2)) = \delta(\rho(g_1,g_2)\iota(z_0,z_0)) = |j(\rho(g_1,g_2),\iota(z_0,z_0))|^{-2}\delta(\iota(z_0,z_0))$$

$$= |j(g_1,z_0)j(g_2,z_0)\det(B(z_0,z_0)^{-1}B(g_1z_0,g_2z_0))|^{-2}\delta(z_0)\delta(z_0)$$

$$= |\det(B(z_1,z_2))|^{-2}\delta(z_1)\delta(z_2).$$

■

3 Quaternionic modular forms, Hecke operators, and *L*-functions

In this section, we introduce the notion of a modular form of scalar weight and define the Hecke operators in our setting. We then define the associated standard L-function. We keep writing G for G_n with $n = 2m + r$ as above.

3.1 Modular forms and Fourier–Jacobi expansion

Fix an integral ideal $\mathfrak{n}$ as in the previous section.

Definition 3.1 A holomorphic function $f : \mathfrak{Z} \to \mathbb{C}$ is called a *quaternionic modular form* for a congruence subgroup Γ and weight $k \in \mathbb{N}$ if for all $\gamma \in \Gamma$,

$$f(\gamma z) = j(\gamma, z)^k f(z).$$

We note here that since we are assuming $m \geq 2$, we do not need any condition at the cusps due to Koecher's principle (see [16, Lemma 1.5]).

Denote the space of such functions by $M_k(\Gamma)$. Here, we are using the realization $(G_\infty, \mathfrak{Z})$ for our symmetric space. In fact, the definition is independent of the choice of realizations in the following sense. If we choose another realization $\mathcal{H}$ (e.g., $\mathfrak{H}, \mathfrak{B}$) with identification $\rho : \mathcal{H} \to \mathfrak{Z}$, then with notation as in equation (2.2), to a function

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$f:\mathfrak{Z}\to\mathbb{C}$, we associate a function g on $\mathcal{H}$ by setting $g(z)=\det(\mu(z))^{-k}f(\rho(z))$. Then $f:\mathfrak{Z}\to\mathbb{C}$ is a modular form if and only if $g:\mathcal{H}\to\mathbb{C}$ is a modular form.

We write $S:=S(\mathbb{Q}):=\{X\in M_m(\mathbb{B}):X^*=X\}$ for the (additive) algebraic group of hermitian matrices. We use S^+ (resp. S_+) denote the subgroup of S consisting of positive-definite (resp. positive) elements. For a fractional ideal $\mathfrak{a}\subset\mathbb{B}$, we set $S(\mathfrak{a})=S\cap M_m(\mathfrak{a})$. Denote $e_\infty(z):=\exp(2\pi iz)$ for $z\in\mathbb{C}$ and $\lambda=\frac{1}{2}\mathrm{tr}$. For $f\in M_k(\Gamma)$ and $\gamma\in G$, there is a Fourier–Jacobi expansion of the form

$$(f|_k\gamma)(z)=\sum_{\tau\in S_+}c(\tau,\gamma,f;v,w)e(\lambda(\mathfrak{i}(\tau)u)),z=(u,v,w)\in\mathfrak{Z}.$$

In particular, for $\gamma=1$, we simply write

$$f(z)=\sum_{\tau\in S_+}c(\tau;v,w)e(\lambda(\mathfrak{i}(\tau)u)).$$

We call f a cusp form if $c(\tau,\gamma,f;v,w)=0$ for every $\gamma\in G$ and every τ such that $\det(h)=0$. The space of cusp form is denoted by $S_k(\Gamma)$.

Given a function $\mathbf{f}:G(\mathbb{A})\to\mathbb{C}$, we can, by abusing the notation, also view it as a function $\mathbf{f}:G(\mathbb{A}_\mathbf{h})\times\mathfrak{Z}\to\mathbb{C}$ by setting $\mathbf{f}(g_\mathbf{h},z):=j(g_z,z_0)^k\mathbf{f}(g_\mathbf{h}g_z)$ with $z=g_z\cdot z_0$.

Definition 3.2 A function $\mathbf{f}:G(\mathbb{A})\to\mathbb{C}$ is called a *quaternionic modular form* of weight k, level $\mathfrak{n}$ if:

(1) viewed as a function $\mathbf{f}:G(\mathbb{A}_\mathbf{h})\times\mathfrak{Z}\to\mathbb{C}$, $\mathbf{f}(g_\mathbf{h},z)$ is holomorphic in z,

(2) for $\alpha\in G(\mathbb{Q})$, $k_\infty\in K_\infty$, and $k\in K_1(\mathfrak{n})$,

$$\mathbf{f}(\alpha gk_\infty k)=j(k_\infty,z_0)^{-k}\mathbf{f}(g),$$

or equivalently,

(2') viewed as a function $\mathbf{f}:G(\mathbb{A}_\mathbf{h})\times\mathfrak{Z}\to\mathbb{C}$, for $\alpha\in G(\mathbb{Q})$, $k\in K_1(\mathfrak{n})$, we have

$$\mathbf{f}(\alpha g_\mathbf{h}k,\alpha z)=j(\alpha,z)^k\mathbf{f}(g_\mathbf{h},z).$$

We will denote the space of such functions by $\mathcal{M}_k(K_1(\mathfrak{n}))$.

We call $\mathbf{f}\in\mathcal{M}_k(K_1(\mathfrak{n}))$ a *cusp form* if

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})}\mathbf{f}(ug)\mathbf{d}u=0,$$

for all unipotent radicals U of all proper parabolic subgroups of G. The space of cusp forms will be denoted by $\mathcal{S}_k(K_1(\mathfrak{n}))$.

It is well known that the above two definitions are related by

$$\mathcal{M}_k(K_1(\mathfrak{n}))\cong\bigoplus_j M_k(\Gamma_1^j(N)),\qquad \mathcal{S}_k(K_1(\mathfrak{n}))\cong\bigoplus_j S_k(\Gamma_1^j(N)).$$

Write $\mathbf{f}\leftrightarrow(f_0,f_1,\dots,f_h)$ for the correspondence under above maps. Here, $f_j(z)=\mathbf{f}(t_j,z)=j(g_z,i)^k\mathbf{f}(t_jz)$ with $z=g_z\cdot i$.

When $n=2m$, $r=0$, then the Fourier–Jacobi expansion becomes the usual Fourier expansion. For $x\in\mathbb{Q}_v$, $v\in\mathbf{h}$, define $e_v(x)=e_\infty(-y)$ with $y\in\bigcup_{n=1}^\infty p^{-n}\mathbb{Z}$ such that $x-y\in\mathbb{Z}_v$. Set $e_\mathbb{A}(x)=e_\infty(x_\infty)\prod_{v\in\mathbf{h}}e_v(x_v)$. Let $\mathbf{f}\in\mathcal{M}_k(K_1(\mathfrak{n}))$. For $g\in G(\mathbb{A})$, we write it as $g=\gamma t_jkp_\infty k_\infty$ with $\gamma\in G(\mathbb{Q})$, $k\in K_1(\mathfrak{n}),k_\infty\in K_\infty$. Take t_j of the

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form $\begin{bmatrix} q_j & \sigma_j \hat{q}_j \\ 0 & \hat{q}_j \end{bmatrix}$ with $q_j \in \mathrm{GL}_m(\mathbb{B}_{\mathbf{h}})$, $\sigma_j \in S(\mathbb{A}_{\mathbf{h}})$ and $p_\infty = \begin{bmatrix} q_\infty & \sigma_\infty \hat{q}_\infty \\ 0 & \hat{q}_\infty \end{bmatrix}$ with $q_\infty \in \mathrm{GL}_m(\mathbb{B}_\infty)$, $\sigma_\infty \in S(\mathbb{R})$. Set $q = q_j q_\infty$, $\sigma = \sigma_j \sigma_\infty$, then $\mathbf{f}$ has a Fourier expansion of the form

$$\mathbf{f}(g) = j(k_\infty, z_0)^{-k} \sum_{\tau \in S} \det(q_\infty)^{-k} c(\tau, q; \mathbf{f}) e_\infty(\lambda(q^* \tau q) z_0) e_{\mathbb{A}}(\lambda(\tau\sigma)).$$

We call $c(\tau, q; \mathbf{f})$ the Fourier coefficients of $\mathbf{f}$.

For two modular forms $f, h \in M_k(\Gamma)$, we define the Petersson inner product by

$$\langle f, h \rangle = \int_{\Gamma \backslash \mathfrak{Z}} f(z) \overline{h(z)} \delta(z)^k \mathbf{d}z,$$

whenever the integral converges. For example, this is well defined when one of f, g is a cusp form. Adelically, for $\mathbf{f}, \mathbf{h} \in \mathcal{M}_k(K_1(\mathfrak{n}))$, we define

$$\langle \mathbf{f}, \mathbf{h} \rangle = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_1(\mathfrak{n}) K_\infty} \mathbf{f}(g) \overline{\mathbf{h}(g)} \mathbf{d}g.$$

Here, $\mathbf{d}g$, an invariant differential of $G(\mathbb{A})$, is given as follows: $\mathbf{d}g = \mathbf{d}g_{\mathbf{h}} \mathbf{d}g_\infty$, where $\mathbf{d}g_h$ is the canonical measure on $G(\mathbb{A}_{\mathbf{h}})$ normalized such that the volume of $K_1(\mathfrak{n})$ is 1 and $\mathbf{d}g_\infty = \mathbf{d}(g_\infty z_0)$ with $\mathbf{d}z$ an invariant differential of $\mathfrak{Z}$.

Viewing $\mathbf{f}, \mathbf{h}$ as functions $\mathbf{f}, \mathbf{g} : G(\mathbb{A}_{\mathbf{h}}) \times \mathfrak{Z} \to \mathbb{C}$, we have

$$\langle \mathbf{f}, \mathbf{h} \rangle = \int_{G(\mathbb{Q}) \backslash (G(\mathbb{A}_{\mathbf{h}}) / K_1(\mathfrak{n}) \times \mathfrak{Z})} \mathbf{f}(g, z) \overline{\mathbf{h}(g, z)} \delta(z)^k \mathbf{d}g_{\mathbf{h}} \mathbf{d}z.$$

Again, these integrals are well defined if one of $\mathbf{f}, \mathbf{h}$ is a cusp form. If $\mathbf{f} \leftrightarrow (f_j), \mathbf{h} \leftrightarrow (h_j)$, then

$$\langle \mathbf{f}, \mathbf{h} \rangle = \sum_j \langle f_j, h_j \rangle.$$

3.2 Hecke operators and *L*-functions

In the rest of the paper, we make the assumption that all finite places v with $v \nmid \mathfrak{n}$ are split in $\mathbb{B}$. We define the groups

$$E = \prod_{v \in \mathbf{h}} \mathrm{GL}_m(\mathcal{O}_v), \quad M = \{x \in \mathrm{GL}_m(\mathbb{B})_{\mathbf{h}} : x_v \in M_m(\mathcal{O}_v)\}.$$

Let $\mathfrak{X} = \prod_v \mathfrak{X}_v$ be a subgroup of $G(\mathbb{A}_{\mathbf{h}})$ with $\mathfrak{X}_v = K_v$ if $v | \mathfrak{n}$ and $\mathfrak{X}_v = G_v$ if otherwise. Define the Hecke algebra $\mathbb{T} = \mathbb{T}(K_1(\mathfrak{n}), \mathfrak{X})$ be the $\mathbb{Q}$-algebra generated by double coset $[K_1(\mathfrak{n}) \xi K_1(\mathfrak{n})]$ with $\xi \in \mathfrak{X}$. Given $\mathbf{f} \in \mathcal{M}_k(K_1(\mathfrak{n}))$, the Hecke operator $[K_1(\mathfrak{n}) \xi K_1(\mathfrak{n})]$ acts on $\mathbf{f}$ by

$$(\mathbf{f} | [K_1(\mathfrak{n}) \xi K_1(\mathfrak{n})])(g) = \sum_{y \in Y} \mathbf{f}(g y^{-1}),$$

where Y is a finite subset of $G_{\mathbf{h}}$ such that

$$K_1(\mathfrak{n}) \xi K_1(\mathfrak{n}) = \bigcup_{y \in Y} K_1(\mathfrak{n}) y.$$

We say that $\mathbf{f} \in \mathcal{S}_k(K_1(\mathfrak{n}))$ is an eigenform if there exist some numbers $\lambda_{\mathbf{f}}(\xi) \subset \mathbb{C}$ called *eigenvalues* such that

$$\mathbf{f}|[K_1(\mathfrak{n})\xi K_1(\mathfrak{n})] = \lambda_{\mathbf{f}}(\xi)\mathbf{f} \text{ for all } \xi \in \mathfrak{X}.$$

We use the notation $l(\xi) := \det(r)$ if $\xi \in G(\mathcal{O}_{\mathbf{h}})\mathrm{diag}[\hat{r}, 1, r]G(\mathcal{O}_{\mathbf{h}})$ with $r \in M$. For a Hecke character χ, we define the series

$$D(s, \mathbf{f}, \chi) = \sum_{\xi \in K_1(\mathfrak{n}) \backslash \mathfrak{X} / K_1(\mathfrak{n})} \lambda_{\mathbf{f}}(\xi)\chi^*(l(\xi))l(\xi)^{-s}, \ \mathrm{Re}(s) \gg 0.$$

Here, χ^* is the associated Dirichlet character of χ. We further define the L-function by

$$L(s, \mathbf{f}, \chi) = \Lambda_{\mathfrak{n}}(s, \chi)D(s, \mathbf{f}, \chi), \qquad \Lambda_{\mathfrak{n}}(s, \chi) = \prod_{i=0}^{n-1} L_{\mathfrak{n}}(2s - 2i, \chi^2). \tag{3.1}$$

Here, the subscript $\mathfrak{n}$ means the Euler factors at $v|\mathfrak{n}$ are removed.

Define $\mathbb{T}_v = \mathbb{T}(K_v, \mathfrak{X}_v)$ be the local counterpart of $\mathbb{T}$ so $\mathbb{T} = \otimes'_v \mathbb{T}_v$. Obviously, $\mathbb{T}_v$ is trivial if $v|\mathfrak{n}$. For $v \nmid \mathfrak{n}$, by our assumptions, we can identify the local group G_v with local orthogonal group, as in equation (2.1). Such local Hecke algebra is discussed in [33] where a Satake map is constructed

$$\omega : \mathbb{T}_v \to \mathbb{Q}[t_1, ..., t_{m_v}, t_1^{-1}, \ldots, t_{m_v}^{-1}],$$

where m_v is the local Witt index of $G(\mathbb{Q}_v)$ and $2n = 2m_v + r_v$ as in equation (2.1). Given an eigenform $\mathbf{f}$, the map $\xi \mapsto \lambda_{\mathbf{f}}(\xi)$ induces homomorphism $\mathbb{T}_v \to \mathbb{C}$ which are parameterized by Satake parameters

$$\alpha_{1,v}^{\pm 1}, \ldots, \alpha_{m_v,v}^{\pm 1}.$$

The L-function then has an Euler product expression

$$L(s, \mathbf{f}, \chi) = \prod_{p \nmid \mathfrak{n}} L_p(s, \mathbf{f}, \chi),$$

with $L_p(s, \mathbf{f}, \chi)$ given by

$$\prod_{i=1}^{\frac{r_v}{2}} (1 - \chi(p)p^{2i+2m_v-2-2s})^{-1} \prod_{i=1}^{m_v} \left((1 - \alpha_{i,p}\chi(p)p^{m_v+r_v-2-s})(1 - \alpha_{i,p}^{-1}\chi(p)p^{m_v-s})\right)^{-1}.$$

In particular, if $r = 0$ or $\alpha \in \mathbb{Q}_v^{\times 2}$, then $G(\mathbb{Q}_v)$ is totally isotropic (i.e., $m_v = n, r_v = 0$) so that

$$L_p(s, \mathbf{f}, \chi) = \prod_{i=1}^{n} \left((1 - \alpha_{i,p}\chi(p)p^{n-2-s})(1 - \alpha_{i,p}^{-1}\chi(p)p^{n-s})\right)^{-1}.$$

Here, we write p for the prime corresponding to some place v in the notation above. Finally, it is known that $L(s, \mathbf{f}, \chi)$ is absolutely convergent for $Re(s) > 2n - 1$ (see [33, Proposition 17.4]).

Remark 3.1 We first note that for p not dividing $\mathfrak{n}$, the local L-factors defined above are given in [31, Theorem 16.16] or [33, Proposition 17.14]. These agree with the Euler factors of the Langlands L-function defined by the (standard) embedding

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${}^L G_o = \mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ (with a suitable normalization on s). Here, ${}^L G_o$ denotes the connected component of the L-group of G, which can be identified with $SO_{2n}(\mathbb{C})$ (see, for example, [37, Appendix B]).

In this paper, our starting point is an eigenform $\mathbf{f}$ for the Hecke algebra related to $K_1(\mathfrak{n})$ (i.e., the analog of $\Gamma_1(N)$ in the classical GL_2 setting). As it is noted above, this Hecke algebra is locally trivial for primes dividing $\mathfrak{n}$ and, hence, the corresponding Euler factor is also trivial. On the other hand, if we denote by K any congruence subgroup with the property that there is an $\mathfrak{n}$ such that $K_1(\mathfrak{n}) \subset K$ (as, for example, of the form $K_0(\mathfrak{n})$, the analo of the classical $\Gamma_0(N)$) and *we further assume* that our $\mathbf{f}$ is an eigenform for the Hecke algebra corresponding to such a K, then since $S_k(K) \subset S_k(K_1(\mathfrak{n}))$, we may consider the above L-function viewing $\mathbf{f} \in S_k(K_1(\mathfrak{n}))$. However, in such a situation, our L-function will be only the partial L-function for the Hecke algebra *with respect to* K since we will be simply setting $L_p(s, \mathbf{f}, \chi) = 1$ for $p|\mathfrak{n}$.

It is a delicate matter to define the missing Euler factor in such a situation. Indeed, it is a conjecture of Langlands [18] that one can associate with all places a local L-factor and local root number such that the global complete L-function satisfies a functional equation. In such a situation and using the doubling method, Yamana [38] gives a definition of local L-factors and proves the functional equation for cuspidal representations over classical groups. However, his local L-factors are not given explicitly, but rather an existential result is proved [38, Theorem 5.2].

We will return to this matter (complete vs. incomplete L-function) again after we prove our main theorem (see Remark 6.4).

4 Eisenstein series and integral representation of *L*-functions

4.1 Siegel Eisenstein series and its Fourier expansion

We fix an integer $0 \le t \le m$, and for $x \in G$, we write

$$x = \begin{bmatrix} a_1 & a_2 & b_1 & c_1 & c_2 \\ a_3 & a_4 & b_2 & c_3 & c_4 \\ g_1 & g_2 & e & f_1 & f_2 \\ h_1 & h_2 & l_1 & d_1 & d_2 \\ h_3 & h_4 & l_2 & d_3 & d_4 \end{bmatrix}$$

with block size $(t, m-t, r, t, m-t) \times (t, m-t, r, t, m-t)$. The t-Klingen parabolic subgroup of G_n is defined as

$$P_n^t = \{x \in G : a_2 = g_2 = h_2 = h_3 = h_4 = l_2 = d_3 = 0\}.$$

Clearly, we have $P_n^m = G_n$. We define a projection map $\pi_t : P_n^t \to G_{2t+r}$ by

$$\begin{bmatrix} a_1 & 0 & b_1 & c_1 & c_2 \\ a_3 & a_4 & b_2 & c_3 & c_4 \\ g_1 & 0 & e & f_1 & f_2 \\ h_1 & 0 & l_1 & d_1 & d_2 \\ 0 & 0 & 0 & 0 & d_4 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & b_1 & c_1 \\ g_1 & e & f_1 \\ h_1 & l_1 & d_1 \end{bmatrix}.$$

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In particular, if $r = 0$, then $n = 2m$. In this case, the parabolic subgroup for $t = 0$,

$$P_n := P_n^0 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_n : c = 0 \right\},$$

is called *Siegel parabolic subgroup*. We now fix weight $l \in \mathbb{N}$, and let χ be a Hecke character whose conductor divides $\mathfrak{n}$. The Siegel-type Eisenstein series is defined as

$$\mathbf{E}_l(x,s) = \mathbf{E}_l^m(x,s;\chi) = \sum_{\gamma \in P_n \backslash G_n} \varphi(\gamma x, s),$$

with

$$\varphi(x,s) = \chi_{\mathbf{h}}(\det(d_p))^{-1} j(x,z_0)^{-l} |\det(d_p)|_{\mathbf{h}}^{-s} |j(x,z_0)|^{l-s},$$

if $x = pkk_\infty \in P_n(\mathbb{A})K_1(\mathfrak{n})K_\infty$ and $\varphi(x,s) = 0$ if otherwise. Let $J_{\mathbf{h}} \in G_n(\mathbb{A})$ be an element defined by $J_v = J_m$ for $v \in \mathbf{h}$ and $J_\infty = 1$. We set $\mathbf{E}_l^*(x,s) = \mathbf{E}_l(xJ_{\mathbf{h}}^{-1}, s)$. Then we have the following proposition (see, for example, [3]).

Proposition 4.1 $\mathbf{E}_l^*(x,s)$ *has a Fourier expansion of form*

$$\mathbf{E}_l^*\left(\begin{bmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{bmatrix}, s \right) = \sum_{h \in S} c(h,q,s) e_{\mathbb{A}}(\lambda(h\sigma)),$$

where $q \in \mathrm{GL}_m(\mathbb{B}_{\mathbb{A}})$ *and* $\sigma \in S(\mathbb{A})$. *The Fourier coefficient* $c(h,q,s) \neq 0$ *only if* $(q^* hq)_v \in T(\mathbb{Q}_v) \cap M_m(\mathfrak{n}_v^{-1})$ *for all* $v \in \mathbf{h}$. *In this case, we have*

$$c(h,q,s) = A(n)\chi(\det(q_{\mathbf{h}}))^{-1} \det(q_\infty)^s |\det(q)|_{\mathbf{h}}^{2m-1-s} \alpha_{\mathfrak{n}}(q_{\mathbf{h}}^* h q_{\mathbf{h}}, s, \chi) \xi(q_\infty q_\infty^*, h, s+l, s-l).$$

Here:

(1) $A(n) \in \overline{\mathbb{Q}}^\times$ *is a constant depending on* n.

(2) *If* h *has rank* r, *then*

$$\alpha_{\mathfrak{n}}(q^* hq, s, \chi) = \frac{\prod_{i=1}^{m-r} L_{\mathfrak{n}}(2s - 4m + 2r + 2i + 1, \chi^2)}{\prod_{i=0}^{m-1} L_{\mathfrak{n}}(2s - 2i, \chi^2)} \prod_p P_{h,q,p}(\chi^*(p)p^{-s}),$$

where $P_{h,q,p}(X) \in \mathcal{O}[X]$ *and* $P_{h,q,p} = 1$ *if* $\det(h) \in 2^{m+1}\mathbb{Z}_p^\times$. *Here,* p *is the prime corresponding to* v.

(3) *Let* p *be the number of positive eigenvalues of* h, q *the number of negative eigenvalues of* h, *and* $t = m - p - q$, *then for* $y \in S_\infty^+$, $h \in S_\infty$,

$$\xi(y,h,s+l,s-l) = \frac{\Gamma_t(2s-2m+1)}{\Gamma_{m-q}(s+l)\Gamma_{m-p}(s-l)} \omega(y,h,s+l,s-l),$$

where

$$\Gamma_m(s) = \pi^{m(m-1)} \prod_{i=0}^{m-1} \Gamma(s-2i), m \in \mathbb{Z},$$

and ω *is holomorphic with respect to* $s+l, s-l$. *In particular, when* $p = m$,

$$\xi(y,h,2l,0) = 2^{2-2m}(2\pi i)^{2ml} \Gamma_m^{-1}(2l) \det(h)^{l - \frac{2m-1}{2}} e(i\lambda(hy)).$$

For $g \in G(\mathbb{A})$, write it as $g = \gamma t_j k p_\infty k_\infty$ with $\gamma \in G(\mathbb{Q})$, $k \in \eta K_0(\mathfrak{n})\eta^{-1}$, $k_\infty \in K_\infty$, and $t_j = \begin{bmatrix} q_j & \sigma_j \hat{q}_j \\ 0 & \hat{q}_j \end{bmatrix}$, $p_\infty = \begin{bmatrix} q_\infty & \sigma_\infty \hat{q}_\infty \\ 0 & \hat{q}_\infty \end{bmatrix}$. Set $q = q_j q_\infty$, $\sigma = \sigma_j \sigma_\infty$, then by modularity property, $\mathbf{E}_l^*(g,s)$ can be written as

$$\mathbf{E}_l^*(g,s) = j(k_\infty, z_0)^{-l} \sum_h c(h,q,s) e_{\mathbb{A}}(\lambda(h\sigma)),$$

with $c(h,q,s)$ in the above propositions. We are interested in the special values $\mathbf{E}_l^*(g,s)$ for $s = l$. From the Fourier expansion, we have the following proposition by counting poles and degree of π in those Gamma functions.

Proposition 4.2 *Assume that $l > n-1$. Then the Fourier coefficients $c(h,q,l) \neq 0$ unless $h > 0$ and in this case*

$$\det(q_\infty)^{-l} c(h,q,l) = C \cdot e_\infty(i\lambda(q^* h q)).$$

Here, up to a constant in $\overline{\mathbb{Q}}$, $C = \chi(\det(q_{\mathbf{h}}))^{-1} |\det(q)|_{\mathbf{h}}^{2m-1-l}$. In particular, $\mathbf{E}_l^*(g,l)$ is holomorphic in the sense that when viewed as a function on $G(\mathbb{A}_{\mathbf{h}}) \times \mathfrak{Z}$, $\mathbf{E}_l^*(g_{\mathbf{h}}, z, l)$ is holomorphic in z.

We note, in particular, that the proposition implies that also $\mathbf{E}_l(g,l)$ is holomorphic in the above sense since $\mathbf{E}_l^*(x,s) = \mathbf{E}_l(xJ_{\mathbf{h}}^{-1}, s)$.

4.2 Coset decompositions

Let

$$\rho : G_n \times G_n \to G_N, g_1 \times g_2 \mapsto R^{-1} \mathrm{diag}[g_1, g_2] R,$$

be the doubling embedding defined before. To ease the notation, we may omit the subscript n. Denote P_N for the Siegel parabolic subgroup of G_N and $P^t = P_n^t$ the t-parabolic subgroup of G.

Proposition 4.3 *For $0 \le t \le m$, let τ_t be the element of G_N given by*

$$\tau_t = \begin{bmatrix} 1_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_r & 0 & 0 & 0_r & 0 \\ 0 & 0 & 1_m & 0 & 0 & 0 \\ 0 & 0 & e_t & 1_m & 0 & 0 \\ 0 & 0_r & 0 & 0 & 1_r & 0 \\ e_t^* & 0 & 0 & 0 & 0 & 1_m \end{bmatrix}, e_t = \begin{bmatrix} 1_t & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{B}_m^m.$$

Then τ_t form a complete set of representatives of $P_N \backslash G_N / G \times G$.

Proof This can be proved similarly to the proofs of Lemmata 4.1 and 4.2 in [30]. Let $W = \{w \in \mathbb{B}_{2N}^N : wJ_n w^* = 0, \mathrm{rank}(w) = N\}$, then $P_N \backslash G_N \cong \mathrm{GL}_N \backslash W$. Therefore, it suffices to find representatives of $\mathrm{GL}_N \backslash W / G \times G$. Let $w = \begin{bmatrix} a_1 & b_1 & c_1 & a_2 & b_2 & c_2 \end{bmatrix} \in W$ of column size (m,r,m,m,r,m). The condition $wJ_n w^* = 0$ is equivalent to $wR^* \omega R w^* = 0$. Explicitly, $wR^* = \begin{bmatrix} x & y \end{bmatrix}$ with

$$x = \begin{bmatrix} a_1 & \frac{b_1}{2} + b_2\zeta^{-1} & a_2 \end{bmatrix}, y = \begin{bmatrix} -c_1 & -\frac{b_1}{2} + b_2\zeta^{-1} & c_2 \end{bmatrix}, x\phi_1 x^* = y\phi_2 y^*.$$

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Multiplying by some element in GL_N, we can assume that

$$x\phi_1 x^* = y\phi_2 y^* = \begin{bmatrix} 0 & 0 & e_t \\ 0 & \zeta I_r & 0 \\ e_t^* & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & e_t \\ 0 & 0_r & 0 \\ e_t^* & 0 & 0 \end{bmatrix}, 0 \le t \le m.$$

Let $V = \mathbb{B}^n$ with standard basis $\{\varepsilon_i\}$ and denote x_i be the ith row of x. Let $\tilde{U}$ be the subspace of V spanned by basis ε_i with $i \le t$ or $m + r \le i \le m + r + t$ and $\tilde{U}^\perp$ the subspace spanned by other basis, so $V = \tilde{U} \oplus \tilde{U}^\perp$. Let θ be the restriction of ϕ_1 on $\tilde{U}$ and η the restriction on $\tilde{U}^\perp$, then we can write $(V, \phi_1) = (\tilde{U}, \theta) \oplus (\tilde{U}^\perp, \eta)$. Assume $x\phi x^*$ is given by the first matrix as above, let U be the subspace of V spanned by vector x_i with $i \le m$ or $m + r \le i \le m + r + t$, and let U' be the subspace spanned by other x_i. We also denote $U^\perp = \{v \in V : u\phi v^* = 0 \text{ for any } u \in U\}$, then $V = U \oplus U^\perp$. Then there exists an automorphism γ of (V, ϕ) such that $U\gamma = \tilde{U}$, $U^\perp\gamma = \tilde{U}^\perp$. Now $U'\gamma \subset \tilde{U}^\perp$ is a totally η-isotropic subspace; thus; there exists an automorphism γ' of $(\tilde{U}^\perp, \eta)$ such that $U'\gamma\gamma' \subset \sum_{m+t+r+1 \le i \le n} \mathbb{B}\varepsilon_i$. Viewing γ' as an automorphism of (V, ϕ) and putting $g_1 = \gamma\gamma'$, we have (similarly for y)

$$xg_1 = \begin{bmatrix} 1_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1_t & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}, \; yg_2 = \begin{bmatrix} 1_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u' \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1_t & 0 \\ 0 & 0 & 0 & 0 & v' \end{bmatrix}.$$

We further modify

$$g_1 \mapsto g_1 \begin{bmatrix} 0 & 0 & 0 & 1_t & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ so } xg_1 = \begin{bmatrix} 0 & 0 & 0 & -1_t & 0 \\ 0 & 0 & 0 & 0 & u \\ 0 & 0 & 1 & 0 & 0 \\ 1_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}.$$

Therefore,

$$w(g_1 \times g_2) = \begin{bmatrix} 0 & 0 & 0 & -1_t & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & u' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\zeta & 0 & 0 \\ 1_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & v' \end{bmatrix}.$$

By our assumption $\mathrm{rank}(w) = N$, we must have $\begin{bmatrix} u & u' \\ v & v' \end{bmatrix}$ is of full rank. If $x\phi_1 x^*$ equals the second matrix as above, then by the same argument, we can obtain a similar result but without the term 1 in the middle of xg_1 and yg_2, which contradicts the assumption

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$\operatorname{rank}(w) = N$. Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} u & u' \\ v & v' \end{bmatrix}^{-1}$, and multiplying

$$\begin{bmatrix} -1_t & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b \\ 0 & 0 & -\zeta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1_t & 0 \\ 0 & c & 0 & 0 & d \end{bmatrix} \in \mathrm{GL}_N,$$

on the left, we then get the desired form in the proposition. ∎

Put $V_t = \tau_t(G_n \times G_n)\tau_t^{-1} \cap P_N$. Then, by straightforward computation,

$$V_t = \{\beta \times \gamma \in P_n^t \times P_n^t : \kappa_t \pi_t(\beta) = \pi_t(\gamma)\kappa_t\}, \kappa_t = \begin{bmatrix} 0 & 0 & -1_t \\ 0 & 1_r & 0 \\ 1_t & 0 & 0 \end{bmatrix}.$$

To simplify the computation, we may also use the modified representatives $\tilde{\tau}_t = \tau_t(1_n \times (\kappa_t \times 1_{2m-2t}))$ for $0 \le t \le m$. Put $\tilde{V}_t = \tilde{\tau}_t(G_n \times G_n)\tilde{\tau}_t^{-1} \cap P_N$. Then

$$\tilde{V}_t = \{\beta \times \gamma \in P_n^t \times P_n^t : \pi_t(\beta) = \pi_t(\gamma)\}.$$

One easily shows that (compare with Lemma 4.3 in [30])

$$P_n^t \times P_n^t = \bigcup_{\xi \in G_{2t+r}} \tilde{V}_t((\xi \times 1_{2m-2t}) \times 1_{2n}) = \bigcup_{\xi \in G_{2t+r}} \tilde{V}_t(1_{2n} \times (\xi \times 1_{2m-2t})),$$

$$P^N \tilde{\tau}_t(G_n \times G_n) = \bigcup_{\xi,\beta,\gamma} P^N \tilde{\tau}_t((\xi \times 1_{2m-2t})\beta \times \gamma) = \bigcup_{\xi,\beta,\gamma} P^N \tilde{\tau}_t(\beta \times (\xi \times 1_{2m-2t})\gamma),$$

where ξ runs over G_{2t+r} and β, γ run over $P_n^t \backslash G_n$. In particular,

$$P^N \tilde{\tau}_m(G_n \times G_n) = \bigcup_{\xi \in G_n} P^N \tilde{\tau}_m(\xi \times 1_n) = \bigcup_{\xi \in G_n} P^N \tilde{\tau}_m(1_n \times \xi).$$

We denote $K_1(\mathfrak{n}) = \prod_v K_v$ be the open compact subgroup of $G_n(\mathbb{A}_\mathbf{h})$ defined before. We also consider the open compact subgroup of G_N, $K_1^N(\mathfrak{n}) = \prod_{v \in \mathbf{h}} K_v^N$ with

$$K_v^N(\mathfrak{n}) = \left\{\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_N(\mathbb{A}_v) \cap M_N(\mathcal{O}_v) : \gamma \equiv 1_N \bmod \mathfrak{n}_v\right\}.$$

Proposition 4.4 *Assume $\mathfrak{n}$ is coprime to (2) and (ζ). Then, for $v|\mathfrak{n}$, $\tilde{\tau}_m(\xi \times 1)\tilde{\tau}_m^{-1} \in P_N(\mathbb{Q}_v)K_v^N$ if and only if $\xi \in K_v$.*

Proof Let $\xi = \begin{bmatrix} a & b & c \\ g & e & f \\ h & l & d \end{bmatrix} \in G_n$ where blocks has size $(m, r, m) \times (m, r, m)$. We calculate that

$$\tilde{\tau}_m(\xi \times 1)\tilde{\tau}_m^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ -h & -l & -d+1 & d & \frac{l\zeta}{2} & 0 \\ -\zeta^{-1}g & -\zeta^{-1}(e-1) & -\zeta^{-1}f & \zeta^{-1}f & \frac{e+1}{2} & 0 \\ a-1 & b & c & -c & \frac{-b\zeta}{2} & 1 \end{bmatrix}.$$

Suppose $\tilde{\tau}_m(\xi \times 1)\tilde{\tau}_m^{-1} \in P_N(\mathbb{Q}_v)K_v^N$, then there exist $p \in P_N(\mathbb{Q}_v)$, say

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & p_{11} & p_{12} & p_{13} \\ 0 & 0 & 0 & p_{21} & p_{22} & p_{23} \\ 0 & 0 & 0 & p_{31} & p_{31} & p_{33} \end{bmatrix},$$

such that $p\tilde{\tau}_m(\xi \times 1)\tilde{\tau}_m^{-1} \in K_v^N$. It is obvious that $p_{13} \equiv p_{23} \equiv 0 \bmod \mathfrak{c}$ and $p_{33} \equiv 1 \bmod \mathfrak{c}$. Comparing the third and fourth columns, we have

$$p_{11}(-d+1) + p_{12}(-\zeta^{-1}f) + p_{13}c \equiv 0 \bmod \mathfrak{n}, p_{11}d + p_{12}(\zeta^{-1}f) + p_{13}(-c) \equiv 1 \bmod \mathfrak{n};$$

$$p_{21}(-d+1) + p_{22}(-\zeta^{-1}f) + p_{23}c \equiv 0 \bmod \mathfrak{n}, p_{21}d + p_{22}(\zeta^{-1}f) + p_{23}(-c) \equiv 0 \bmod \mathfrak{n};$$

$$p_{31}(-d+1) + p_{32}(-\zeta^{-1}f) + p_{33}c \equiv 0 \bmod \mathfrak{n}, p_{31}d + p_{32}(\zeta^{-1}f) + p_{33}(-c) \equiv 0 \bmod \mathfrak{n}.$$

This forces $p_{11} \equiv 1 \bmod \mathfrak{n}$ and $p_{21} \equiv p_{31} \equiv 0 \bmod \mathfrak{n}$. Comparing the second and fifth columns and using our assumption on $\mathfrak{n}$, we have

$$p_{11}(-l) + p_{12}(-\zeta^{-1}(e-1)) + p_{13}b \equiv 0 \bmod \mathfrak{n}, p_{11}(l\zeta) + p_{12}(e+1) + p_{13}(-b\zeta) \equiv 0 \bmod \mathfrak{n};$$

$$p_{21}(-l) + p_{22}(-\zeta^{-1}(e-1)) + p_{23}b \equiv 0 \bmod \mathfrak{n}, p_{21}(l\zeta) + p_{22}(e+1) + p_{23}(-b\zeta) \equiv 2 \bmod \mathfrak{n};$$

$$p_{31}(-l) + p_{32}(-\zeta^{-1}(e-1)) + p_{33}b \equiv 0 \bmod \mathfrak{n}, p_{31}(l\zeta) + p_{32}(e+1) + p_{33}(-b\zeta) \equiv 0 \bmod \mathfrak{n}.$$

The second line shows that $p_{22} \equiv e \equiv 1 \bmod \mathfrak{n}$ and then $p_{12} \equiv p_{32} \equiv 0 \bmod \mathfrak{n}$ by other formulas. Therefore, $p \in P_N(\mathbb{Q}_v) \cap K_v^N$. From the above identities, we already have

$$d - 1 \equiv f \equiv c \equiv l \equiv e - 1 \equiv b \equiv 0 \bmod \mathfrak{n}.$$

The claim $\xi \in K_v$ then follows from the above identities together with

$$p_{i1}(-h) + p_{i2}(-\zeta^{-1}g) + p_{i3}(a-1) \equiv 0 \bmod \mathfrak{n}, i = 1, 2, 3. \qquad \blacksquare$$

4.3 Integral representation

Let χ be a Hecke character whose conductor divides the fixed ideal $\mathfrak{n}$. Recall that we write $N = 2n$. We define $\sigma \in G_N(\mathbb{A}_{\mathbf{h}})$ as $\sigma_v := 1$, the identity matrix, if $v \nmid \mathfrak{n}$, and $\sigma_v = \tilde{\tau}_m$ if $v|\mathfrak{n}$.

We then consider the weight k Siegel-type Eisenstein series (twisted by $\tilde{\tau}_m$) on G_N defined as

$$\mathbf{E}(\mathfrak{g}, s) := E_k^n(\mathfrak{g}\,\sigma^{-1}, s; \chi), \quad \mathfrak{g} \in G_N(\mathbb{A}).$$

By decomposition of G_N. we can write

$$\mathbf{E}(\mathfrak{g}, s) = \sum_{t=0}^{m} \sum_{\gamma \in P_N \backslash P_N \tilde{\tau}_t (G\times G)} \phi(\gamma \mathfrak{g}, s) =: \sum_{t=0}^{m} \mathbf{E}_t(\mathfrak{g}, s).$$

Assume $\mathfrak{g} = h \times g$ with $h, g \in G(\mathbb{A})$, and let $\mathbf{f} \in S_k(K_1(\mathfrak{n}))$ be a cusp form.

We are going to study the integral

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}(g \times h, s)\mathbf{f}(h)\mathbf{d}h.$$

Proposition 4.5 *Assume that $t < m$, then*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}_t(g \times h, s)\mathbf{f}(h)\mathbf{d}h = 0.$$

Proof Let U_t be the unipotent radical of P_n^t. Then the integral equals

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \sum_{\xi \in G_{2t+r}} \sum_{\beta, \gamma \in P_n^t \backslash G} \phi(\tilde{\tau}_t((\xi \times 1_{2n-2t})\beta g \times \gamma h), s)\mathbf{f}(h)\mathbf{d}h$$

$$= \int_{P_n^t \backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \sum_{\xi \in G_{2t+r}} \sum_{\gamma \in P_n^t \backslash G_n} \phi(\tilde{\tau}_t((\xi \times 1_{2n-2t}) g \times \gamma h), s)\mathbf{f}(h)\mathbf{d}h$$

$$= \int_{U_t(\mathbb{A})P_n^t(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \int_{U_t(\mathbb{Q})\backslash U_t(\mathbb{A})} \sum_{\xi,\gamma} \phi(\tilde{\tau}_t(\xi n g \times \gamma h), s)\mathbf{f}(nh)\mathbf{d}n\mathbf{d}h.$$

Since ξ normalizes U_t and $\tilde{\tau}_t(n \times 1) \in P_n$, this equals

$$= \int_{U_t(\mathbb{A})P_n^t(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \int_{U_t(\mathbb{Q})\backslash U_t(\mathbb{A})} \sum_{\xi,\gamma} \phi(\tilde{\tau}_t(\xi g \times \gamma h), s)\mathbf{f}(nh)\mathbf{d}n\mathbf{d}h,$$

which vanishes by the cuspidality of $\mathbf{f}$. ■

Therefore,

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}(g \times h, s)\mathbf{f}(h)\mathbf{d}h = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}_m(g \times h, s)\mathbf{f}(h)\mathbf{d}h$$

$$= \int_{G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \phi(\tilde{\tau}_m(g \times h), s)\mathbf{f}(h)\mathbf{d}h.$$

The infinite part is calculated in following lemma.

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Lemma 4.6 *For $k + Re(s) > 2n+1$, we have*

$$\int_{G_\infty/K_\infty} \phi_\infty(\tilde{\tau}_m(g_\infty \times h_\infty), s)\mathbf{f}(h_{\mathbf{h}} \cdot h_\infty)\mathbf{d}h_\infty = c_k(s)\mathbf{f}(h_{\mathbf{h}} \cdot g_\infty),$$

with

$$c_k(s) = \alpha(s)\pi^{\frac{n(n-1)}{2}} \frac{\Gamma(s+k-2n+3)\Gamma(s+k-2n+5)\dots\Gamma(s+k-1)}{\Gamma(s+k-n+2)\Gamma(s+k-n+3)\dots\Gamma(s+k)}$$

$$= \alpha(s)\pi^{\frac{n(n-1)}{2}} \frac{\prod_{j=0}^{q-1}\Gamma(s+k-n+1-t-2j)}{\prod_{j=0}^{q-1}\Gamma(s+k-2j)},$$

where $n = 2q + t$ with $q \in \mathbb{N}$ and $t \in \{0,1\}$ and $\alpha(s)$ is a holomorphic function on $s \in \mathbb{C}$ such that $\alpha(\lambda) \in \overline{\mathbb{Q}}$ for all $\lambda \in \mathbb{Q}$.

Proof Note that

$$j(\tilde{\tau}_m(g_\infty \times h_\infty), z_0 \times z_0) = j(\tilde{\tau}_m, g_\infty z_0 \times h_\infty z_0) j(g_\infty, z_0)\overline{j(h_\infty, z_0)}.$$

Put $w = h_\infty z_0, z = g_\infty z_0$, then since

$$f(w) = j(h_\infty, z_0)^k \mathbf{f}(h_\infty), j(\tilde{\tau}_m, z \times w) = \delta(w, z),$$

the integral becomes

$$j(g_\infty, z_0)^{-k}\delta(z)^{\frac{s-k}{2}} \int_{\mathfrak{Z}} \delta(w,z)^{-k}|\delta(w,z)|^{k-s}\delta(w)^{\frac{k+s}{2}} f(w)dw.$$

This kind of integral is calculated in [31, Appendix A2] and [14]. In particular, it is shown there that for $k + Re(s) > n + \frac{1}{2}$,

$$\int_{\mathfrak{Z}} \delta(w,z)^{-k}|\delta(w,z)|^{-2s}\delta(w)^{s+k} f(w)dw = \widetilde{c}_k(s) f(z)\delta(z)^{-s},$$

where $\widetilde{c}_k(s)$ is a function on s which does not depend on f. Indeed, as it is explained in [31], the quantity $\widetilde{c}_k(s)$ is independent of f and it is equal to

$$\widetilde{c}_k(s) = \alpha(s) \int_{\mathfrak{B}} \det(I + z\bar{z})^{s+k}\mathbf{d}z,$$

where $\mathbf{d}z$ is the invariant measure on the bounded domain and is given as $\mathbf{d}z = \det(I + z\bar{z})^{-n+1}dz$, and $\alpha(s)$ is a holomorphic function on s such that $\alpha(\lambda) \in \overline{\mathbb{Q}}$ for all $\lambda \in \mathbb{Q}$ (actually it can be made precise, but we do not need it here). But this last integral has been computed in [14, p. 46] from which we obtain that

$$\widetilde{c}_k(s) = \alpha(s)\pi^{\frac{n(n-1)}{2}} \frac{\Gamma(2\lambda+1)\Gamma(2\lambda+3)\dots\Gamma(2\lambda+2n-3)}{\Gamma(2\lambda+n)\Gamma(2\lambda+n+1)\dots\Gamma(2\lambda+2n-2)},$$

where $\lambda = s + k - n + 1$.

Setting now $s \mapsto \frac{s-k}{2}$, we obtain that

$$j(g_\infty, z_0)^{-k}\delta(z)^{\frac{s-k}{2}} \int_{\mathfrak{Z}} \delta(w,z)^{-k}|\delta(w,z)|^{k-s}\delta(w)^{\frac{k+s}{2}} f(w)dw =$$

$$\widetilde{c}_k((s-k)/2) j(g_\infty, z_0)^{-k} f(z) = c_k(s)\mathbf{f}(h_{\mathbf{h}} \cdot g_\infty),$$

where we have set $c_k(s) := \widetilde{c}_k((s-k)/2)$. ■

By changing variables, it remains to calculate

$$\int_{K_1(\mathfrak{n})\backslash G(\mathbb{A}_\mathbf{h})} \phi_\mathbf{h}(\tilde{\tau}_m(h\times 1), s)\mathbf{f}(gh^{-1})\mathbf{d}h.$$

Note that for $v|\mathfrak{n}$, ϕ_v is nonzero unless $h_v \in K_v$. Hence, it remains to consider the integral over $\prod_{v\nmid\mathfrak{n}} K_v\backslash G(\mathbb{Q}_v)$. These unramified integrals are well known by [19, 33, 37]. Indeed, by the Cartan decomposition, we can write

$$G(\mathbb{Q}_v) = \coprod_{\substack{e_1,\dots,e_{m_v}\in\mathbb{Z}\\ 0\le e_1\le\dots\le e_{m_v}}} K_{e_1,\dots,e_{m_v}},$$

$$K_{e_1,\dots,e_{m_v}} = K_v \mathrm{diag}[p_v^{e_1},\dots,p_v^{e_{m_v}}, 1_{r_v}, p_v^{-e_1},\dots,p_v^{-e_{m_v}}]K_v,$$

where m_v is the local Witt index of $G(\mathbb{Q}_v)$ and p_v the prime corresponds to v. Note that by definition of ϕ,

$$\phi_v(\tilde{\tau}_m(h_v\times 1)) = (\chi_v(p_v)p_v^{-s})^{e_1+\cdots+e_{m_v}}.$$

Assume that $\mathbf{f}$ is an eigenform such that $\mathbf{f}|[K_{e_1,\dots,e_{m_v}}] = \lambda_\mathbf{f}(K_{e_1,\dots,e_{m_v}})\mathbf{f}$. Then the integral over $K_v\backslash G(\mathbb{Q}_v)$ contributes a Dirichlet series

$$\sum_{\substack{e_1,\dots,e_{m_v}\in\mathbb{Z}\\ 0\le e_1\le\cdots\le e_{m_v}}} \lambda_\mathbf{f}(K_{e_1,\dots,e_{m_v}})(\chi_v(p_v)p_v^{-s})^{e_1+\cdots+e_{m_v}}.$$

In conclusion, we obtain

$$\prod_{v\nmid\mathfrak{n}} \sum_{\substack{e_1,\dots,e_{m_v}\in\mathbb{Z}\\ 0\le e_1\le\dots\le e_{m_v}}} \lambda_\mathbf{f}(K_{e_1,\dots,e_{m_v}})(\chi_v(p_v)p_v^{-s})^{e_1+\cdots+e_{m_v}}\cdot\mathbf{f}(g) = D(s,\mathbf{f},\chi)\mathbf{f}(g).$$

We summarize the discussion in the following theorem.

Theorem 4.7 *Let $\mathbf{f}\in\mathcal{S}_k^n(K_1(\mathfrak{n}))$ be an eigenform, and assume that $\mathfrak{n}$ is coprime to (2ζ). Then*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathbf{E}(g\times h, s)\mathbf{f}(h)\mathbf{d}h = c_k(s)D(s,\mathbf{f},\chi)\mathbf{f}(g).$$

5 Algebraic modular forms and differential operators

In order to move from the analytic considerations discussed so far to algebraic questions, we need to discuss the notion of an algebraic modular form in our setting. The notion of algebraic modular forms on Hermitian symmetric space is well understood. There are mainly four characterizations of algebraic modular forms: via Fourier–Jacobi expansion, CM-points, pullback to elliptic modular forms, and canonical model of automorphic vector bundle. For example, in [20, Section III.7], automorphic forms are interpreted as sections of certain automorphic vector bundles. The canonical model of automorphic vector bundles then defines a subspace of algebraic automorphic forms (see also [9, 10]). It is also proved there that this definition is equivalent to the definition in terms of values at CM-points. In [6], Garrett

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gives three characterizations of algebraicity for scalar-valued modular forms via CM-points, Fourier–Jacobi expansion, and pullback to elliptic modular forms. They are also proved to be equivalent.

However, in this work, instead of simply referring to the results of Harris as in [9, 10], we have decided to offer a definition of algebraic modular forms via CM-points using the rather more explicit language of Shimura as in [32], without need to refer to the more advanced and general theory as developed by Deligne, Milne, and others. Indeed, our approach of the definition of CM-points and the underlying periods follows an idea in the first works of Shimura on the subject [25], where one "tensors" a given embedding $h : K_1 \times \cdots \times K_n \hookrightarrow G$, of CM field K_i, with another CM field K, disjoint to the K_i's to obtain a point whose associated abelian variety is of CM type (see also [5, Proof of Theorem 6.4]). In this way, we will be able to define and study the CM-points in our case by considering an embedding of our group into a unitary group, after a choice of an imaginary quadratic field. However, we will show that our definition of CM-points and the attached periods is independent of the choice of the auxiliary imaginary quadratic field. This should be seen as our main contribution in this section, which we believe it is worth appearing in the literature and could be helpful to other researchers, thanks to its rather explicit nature and basic background, Finally, we will show that in certain case, when the underlying symmetric space is a tube domain, i.e., a Siegel Domain of Type I, our definition is equivalent to standard definition using the Fourier expansion.

5.1 CM-points

We introduce the following setting, with some small repetition of what we have discussed so far.

We let $\mathbb{B}$ be a definite quaternion algebra over $\mathbb{Q}$, $T^* = -T \in M_n(\mathbb{B})$ a skew-hermitian matrix, and define the algebraic group

$$G := G(T) := \{g \in \mathrm{SL}_n(\mathbb{B}) : g^* T g = T\}.$$

Let K_i, $i = 1, \ldots, n$, be imaginary quadratic fields and consider the CM algebra $Y = K_1 \times \cdots \times K_n$ and $Y^1 = \{y \in Y : yy^\rho = 1\}$ with ρ induced by the nontrivial involutions (i.e., complex conjugation) on each K_i. We are interested in embeddings $h : Y^1 \to G(T)$. Clearly, $h(Y^1) \subset G(T)(\mathbb{R})$ and $(Y^1 \otimes_{\mathbb{Q}} \mathbb{R})^\times$ is a compact subgroup of $G(T)(\mathbb{R})$. Let us show that there always exists such an embedding.

Without loss of generality, we may write $T = \mathrm{diag}[a_1, \ldots, a_n]$ in diagonal form. We then select as imaginary quadratic fields $K_i := \mathbb{Q}(a_i)$, for $i = 1, \ldots, n$, and define the embedding

$$h : Y^1 \to G(T), (y_1, \ldots, y_n) \mapsto \mathrm{diag}[y_1, \ldots, y_n].$$

Back to our general considerations, we select an imaginary quadratic field K which is different from K_i's above, and splits $\mathbb{B}$. It is easy to see that that there exists always such a field K. We now fix an embedding $M_n(\mathbb{B}) \to M_{2n}(K)$. Denote the image of T in $M_{2n}(K)$ by $\mathcal{T}$ and the unitary group

$$U(\mathcal{T}) := \{g \in \mathrm{GL}_{2n}(K) : g^* \mathcal{T} g = \mathcal{T}\}.$$

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We note that the group of $\mathbb{R}$-points of $U(\mathcal{T})$ is isomorphic to

$$U(n,n) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{C}) : g^* \begin{bmatrix} i\cdot 1_n & 0 \\ 0 & -i\cdot 1_n \end{bmatrix} g = \begin{bmatrix} i\cdot 1_n & 0 \\ 0 & -i\cdot 1_n \end{bmatrix} \right\}.$$

Its action on the bounded domain (see, for example, [32]),

$$\mathcal{B} = \{ z \in M_n(\mathbb{C}) : 1 - z^* z > 0 \},$$

is defined by $gz = (az+b)(cz+d)^{-1}$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with the obvious block matrices. The two factors of automorphy are given by $\lambda(g,z) = \bar{c}^t z + \bar{d}$, and $\mu(g,z) = cz + d$. The embedding $M_n(\mathbb{B}) \to M_{2n}(K)$ induces an embedding $\mathrm{i} : G(T) \to U(\mathcal{T})$ which is compatible with natural inclusion $\iota : \mathfrak{B} \to \mathcal{B}$. We will view $G(T)$ (resp. $\mathfrak{B}$) as a subgroup (resp. subspace) of $U(\mathcal{T})$ (resp. $\mathcal{B}$) under this embedding.

Lemma 5.1 (1) *Y is spanned by Y^1 over $\mathbb{Q}$. In particular, there exists an element $\beta \in Y^1$ such that $Y = \mathbb{Q}[\beta]$ and $\beta_1, \dots, \beta_n, \beta_1^\rho, \dots, \beta_n^\rho$ are pairwise distinct.*

(2) *There is a unique $w \in \mathfrak{B}$ which is a common fixed point for $h(Y^1)$.*

Proof The first part can be shown exactly as [32, Lemma 4.12], and for the second part, we adapt an idea of the proof of that lemma. Without loss of generality, we can assume that the origin 0 of $\mathfrak{B}$ is a fixed point for $h(Y^1)$ and our task is to show that it is the unique fixed point. We note that the maximal compact subgroup in $G(T)(\mathbb{R})$ fixing the origin is isomorphic to $U(n)$, and hence with respect to the embedding $G(T)(\mathbb{R}) \hookrightarrow U(n,n)$, we have that $U(n) \hookrightarrow U(n) \times U(n)$ diagonally, i.e., $a \mapsto (a, \bar{a})$. In particular, we have an embedding $h(Y^1) \hookrightarrow U(n) \hookrightarrow U(n,n)$. Assume now that there is another point $z \in \mathfrak{B}$ which is a fixed point of $h(Y^1)$. Then we must have that $z = az\bar{a}^{-1}$ for every element $\mathrm{diag}[a, \bar{a}] \in (U(n) \times U(n)) \cap h(Y^1)$. But for such a point we have that $a^* a = 1$ and hence $\bar{a}^{-1} = {}^t a$. That is, $z = az{}^t a$. Since $a \in U(n) \hookrightarrow U(n) \times U(n)$, we may diagonalize it, say with eigenvalues λ_i, $i = 1, \dots, 2n$, and hence we must have $z_{ij} = 0$ for every $\lambda_i \neq \lambda_j$. Taking a to be the element obtained from β above, we have that z has to be the origin. ∎

We call a point fixed by some $h(Y^1)$ as above a CM-point, and we note that this definition does not depend on the choice of the field K. For example, take $T = \mathrm{diag}[\zeta \cdot 1_m, \zeta \cdot 1_r, -\zeta \cdot 1_m]$. This is the group G' in ection 2, and we have described its embedding into unitary group and the action on $\mathfrak{B}$ explicitly there. Let h and Y be as above, and one easily checks that 0 is the fixed point of $h(Y^1)$ and thus a CM-point.

We now want to attach some CM periods to our CM-points. We will do this by relating our definition with the notion of CM-points of unitary groups. Indeed, our selection of the field K allows us to view our group as a subgroup of a unitary group, and hence an embedding $\mathfrak{B} \hookrightarrow \mathcal{B}$. Our next aim is to relate the just-defined CM-points in $\mathfrak{B}$ with the well-studied, as in [32], CM-points of $\mathcal{H}$. It is here that we employ the idea of Shimura, which was used in [25] (see also [26, Section 7]) to study CM-points in general type-C domains.

Let $w \in \mathfrak{B} \subset \mathcal{B}$ be a CM-point fixed by $h(Y^1) \subset G \subset U(n,n)$. Then, for such a point, we have that

$$\Lambda(\alpha, w) p(x, w) = p(x\alpha, w), \quad \alpha \in h(Y^1), \; x \in \mathbb{C}^{2n},$$

where $\Lambda(\alpha, w) \in GL_{2n}(\mathbb{C})$ and $p(x,z) : \mathbb{C}^{2n} \times \mathcal{B} \to \mathbb{C}^{2n}$ are the maps defined in [32, §4.7]. In this way, we can obtain an embedding $Y \to End_{\mathbb{C}}(\mathbb{C}^{2n})$ by sending $\alpha \mapsto \Lambda(\alpha, w)$ where we have used the fact that Y is spanned by Y^1 over $\mathbb{Q}$. We now extend this to an injection h of $K \otimes_{\mathbb{Q}} Y \cong \mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ into $End(\mathbb{C}^{2n})$, where $\mathcal{S}_i = KK_i$. Indeed, we set

$$h(\beta \otimes \alpha)p(x,w) = p(\beta x \alpha, w) = p(x\beta\alpha, w) = p(x\alpha\beta, w).$$

That is, the point w can be seen as a fixed point of $\mathcal{S}^1 \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathcal{S}^1 = \{s \in \mathcal{S} \mid ss^{\rho} = 1\}$ with ρ the involution on $\mathcal{S}$ induced by the complex conjugation on KK_i. Hence, w is a CM-point in $\mathcal{B}$ defined in [32, §4.11] for unitary groups. In particular, w has entries in $\overline{\mathbb{Q}}$ by [32, Lemma 4.13].

Remark 5.2 Following [32, Section 4], let $\Omega = \{K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^{s}\}$ be a PEL type and $\mathcal{F}(\Omega)$ family of polarized abelian varieties of PEL type. The abelian varieties in $\mathcal{F}(\Omega)$ are parameterized by $\mathcal{B}$. More precisely, there is a bijection

$$\Gamma\backslash\mathcal{B} \xrightarrow{\sim} \mathcal{F}(\Omega), \Gamma = \{\gamma \in U(\mathcal{T}) : L\gamma = L, u_i\gamma - u_i \in L\}.$$

As in [24], we can define $\Omega' = \{\mathbb{B}, \Psi', L, T, \{u_i\}_{i=1}^{s}\}$ for quaternions, and $\mathcal{F}(\Omega')$ are parameterized by $\mathfrak{B}$. The natural inclusion $\mathcal{F}(\Omega') \to \mathcal{F}(\Omega)$ is compatible with $\mathfrak{B} \to \mathcal{B}$. Moreover, similar to [6, 25], we actually have an embedding of canonical models between $\Gamma\backslash\mathfrak{B}$ and $\Gamma'\backslash\mathcal{B}$ for certain congruence subgroups Γ, Γ'.

As we have remarked, CM-points for unitary groups have been extensively studied in [32, Chapter II]. We recall some of their properties. For $\alpha \in \mathcal{S}^1$, we put $\psi(\alpha) := \lambda(h(\alpha), w) \in GL_n(\mathbb{C})$, $\phi(\alpha) := \mu(h(\alpha), w) \in GL_n(\mathbb{C})$, and $\Phi(\alpha) = \mathrm{diag}[\psi(\alpha), \phi(\alpha)] \in GL_{2n}(\mathbb{C})$. We can then find $B, C \in GL_n(\overline{\mathbb{Q}})$ (see [32, p. 78]) such that for all $\alpha \in \mathcal{S}$,

$$B\psi(\alpha)B^{-1} = \mathrm{diag}[\psi_1(\alpha), \ldots, \psi_n(\alpha)], \quad C\phi(\alpha)C^{-1} = \mathrm{diag}[\phi_1(\alpha), \ldots, \phi_n(\alpha)],$$

for some ring homomorphism $\phi_i, \psi_i : \mathcal{S} \to \mathbb{C}$, where we have $\mathbb{Q}$-linearly extended ψ and ϕ, from $\mathcal{S}^1$ to $\mathcal{S}$. We set

$$\mathfrak{p}_{\infty}(w) := C^{-1}\mathrm{diag}[p_{\mathcal{S}}(\phi_1, \Phi), \ldots, p_{\mathcal{S}}(\phi_n, \Phi)]C \in GL_n(\mathbb{C}),$$

$$\mathfrak{p}_{\infty\rho}(w) := B^{-1}\mathrm{diag}[p_{\mathcal{S}}(\psi_1, \Phi), \ldots, p_{\mathcal{S}}(\psi_n, \Phi)]B \in GL_n(\mathbb{C}),$$

where the CM periods $p_{\mathcal{S}}(\psi_i, \Phi) \in \mathbb{C}^{\times}$ and $p_{\mathcal{S}}(\phi_i, \Phi) \in \mathbb{C}^{\times}$ are defined as in [32, p. 78]. Actually, we should remark here that the periods $p_{\mathcal{S}}(\psi_i, \Phi)$, $p_{\mathcal{S}}(\phi_i, \Phi)$ are uniquely determined up to elements in $\overline{\mathbb{Q}}^{\times}$, but this is sufficient for our applications.

We now use the fact that $w \in \mathfrak{B} \subset \mathcal{B}$ is a CM-point for both (Y, h) and also for $(\mathcal{S}, h)$. Note that $\psi(\alpha) = \phi(\alpha)$ for $\alpha \in Y^1 \subset \mathcal{S}^1$. Indeed, for $\alpha \in G(\mathbb{R})$, we have that (see [25, equation (2.18.9)]) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$, and hence, in particular, we have that $\lambda(\alpha, z) = \mu(\alpha, z)$ since ${}^t z = -z$. In particular, the values $\psi(\alpha) = \phi(\alpha) = \lambda(\alpha, w) = \mu(\alpha, w)$ for $\alpha \in Y^1$, that is, the restrictions of ϕ and ψ to Y^1, are independent of the

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choice of the field K. Furthermore, we note that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in K$ with $\alpha\overline{\alpha} = 1$ seen as elements of $U(n,n)$, i.e., $\alpha 1_{2n} \in U(n,n)$.

In the following lemma, we use the notation $I_Y, J_Y, J_{\mathcal{S}_j}$ as defined in [32, p. 77].

Lemma 5.3 *With notation as above, for all $1 \le i \le n$, we have that*

$$p_{\mathcal{S}}(\psi_i, \Phi) = p_Y(Res_{\mathcal{S}/Y}(\psi_i), \Phi') = p_Y(Res_{\mathcal{S}/Y}(\phi_i), \Phi') = p_{\mathcal{S}}(\phi_i, \Phi),$$

where $\Phi' = Res_{\mathcal{S}/Y}\phi = Res_{\mathcal{S}/Y}\psi \in I_Y$.

Proof Let us write $\Phi = \sum_{j=1}^n \Phi_j$ with $\Phi_j \in I_{\mathcal{S}_j}$ and $\Phi' = \sum_{j=1}^n \Phi'_j$, with $\Phi'_j \in I_{K_j}$. Then we have that $\Phi_j = \mathrm{Inf}_{\mathcal{S}_j/K_j}(\Phi'_j)$. Indeed, first, we observe that $\Psi = \sum_{j=1}^n \mathrm{Res}_{\mathcal{S}_j/K}\Phi_j \in I_K$ (see [32, p. 85]), where Ψ as in Remark 5.2. Moreover, we know that $\Phi = \phi + \psi$ with $\phi, \psi \in I_{\mathcal{S}}$ as above, and we have seen that $\psi = \overline{\phi}$ when restricted to K via $K \hookrightarrow Y \otimes_{\mathbb{Q}} K = \mathcal{S}$. But, on the other hand, we have seen that $\psi = \phi$ when restricted to Y, from which we obtain that $\Phi_j = \Phi'_j \otimes \tau + \Phi'_j \otimes \overline{\tau}$, where τ is a fixed embedding of $K \hookrightarrow \mathbb{C}$ (i.e., a CM type for K). Since $\mathcal{S}_j = K_j \otimes_{\mathbb{Q}} K$, the claim that $\Phi_j = \mathrm{Inf}_{\mathcal{S}_j/K_j}(\Phi'_j)$ now follows.

The statement of the lemma is now obtained from the inflation-restriction properties of the periods (see [32, p. 84]):

$$p_{\mathcal{S}}(\psi_i, \Phi) = \prod_{j=1}^n p_{\mathcal{S}_j}(\psi_{ij}, \Phi_j) = \prod_{j=1}^n p_{K_j}(\mathrm{Res}_{\mathcal{S}_j/K_j}(\psi_{ij}), \Phi'_j) = p_Y(\mathrm{Res}_{\mathcal{S}/Y}(\psi_i), \Phi'),$$

where $\psi_{ij} \in J_{\mathcal{S}_j}$ induced by $\psi_i \in J_{\mathcal{S}} = \bigcup_{j=1}^n J_{\mathcal{S}_j}$. Similarly follows also the other equality. ∎

The above lemma shows that we have $\mathfrak{p}_\infty(w) = \mathfrak{p}_{\infty\rho}(w)$ for $w \in \mathfrak{B}$ and they are independent of the choice of the imaginary quadratic field K we chose above (and hence of the embedding to the unitary group). We then simply define $\mathfrak{p}(w) = \mathfrak{p}_\infty(w) = \mathfrak{p}_{\infty\rho}(w)$ for the period attached to CM-point $w \in \mathfrak{B}$. By [32, Proposition 11.5] and the definition of periods, we immediately have:

(1) The coset $\mathfrak{p}(w)GL_n(\overline{\mathbb{Q}})$ is determined by the point $w \in \mathfrak{B}$ independently of the embedding (Y, h) chosen above.
(2) $\mathfrak{p}(\gamma w)GL_n(\overline{\mathbb{Q}}) = \lambda(\gamma, w)\mathfrak{p}(w)GL_n(\overline{\mathbb{Q}})$ for all $\gamma \in G(\mathbb{Q})$.

Remark 5.4 Even though the definition of a CM-point in $\mathfrak{B}$ given above is enough for our applications, we mention here that there is a more general definition as follows. We may take Y above as $Y = M_{n_1}(K_1) \times \cdots \times M_{n_s}(K_s)$ with K_i CM fields and the condition that $n = \sum_{i=1}^s n_i[K_i : \mathbb{Q}]$ and assume that there exists an embedding $h : Y^1 \to G(T)$ where $Y^1 := \{y \in Y \mid yy^\rho = 1\}$ with the involution on Y induced by complex conjugation and transpose. Then one can show as above that $h(Y^1 \otimes_{\mathbb{Q}} \mathbb{R})$ has a unique fixed point $w \in \mathfrak{B}$. Picking as before an imaginary quadratic field K disjoint from all K_i, we can see that the point $w \in \mathfrak{B} \hookrightarrow \mathcal{B}$ corresponds to an abelian variety A_w with endomorphism ring equal to $Y \otimes_{\mathbb{Q}} K$. In particular, we have that A_w is isogenous to $A_1^{n_1} \times \cdots \times A_s^{n_s}$, where the abelian variety A_i has CM by the field $\mathcal{S}_i := KK_i$.

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5.2 Algebraic modular forms

We keep the notation from before. In particular, we write G for $G(T)$ and we have an embedding $\mathrm{i}: G \to U(\mathcal{T})$ as above. For the following considerations, we need to augment our definition of modular forms from scalar-valued to vector-valued.

We start with a $\overline{\mathbb{Q}}$-rational representation $\omega: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$. Given a function $f: \mathfrak{B} \to V$ and $g \in G$, define $(f|_\omega g)(z) = \omega(\lambda(g,z))^{-1} f(gz)$. For a congruence subgroup Γ, the space of modular forms $M_\omega(\Gamma)$ consists of holomorphic function with the property $f|_\omega \gamma = f$ for all $\gamma \in \Gamma$. Put $M_\omega = \bigcup M_\omega(\Gamma)$ where the union is over all congruence subgroups, and

$$\mathfrak{A}_\omega = \bigcup_e \{g^{-1} f : f \in M_{\tau_e}, 0 \neq g \in M_e\},$$

$$\mathfrak{A}_\omega(\Gamma) = \{h \in \mathfrak{A}_\omega : h|_\omega \gamma = h \text{ for } \gamma \in \Gamma\},$$

where e runs over $\mathbb{Z}$ and τ_e denotes the representation defined by $\tau_e(x) = \det(x)^e \omega(x)$.

Definition 5.1 Let $\mathcal{W}$ be a set of CM-points which is dense in $\mathfrak{B}$. Put $\mathfrak{P}_\omega(w) = \omega(\mathfrak{p}(w))$ for $w \in \mathcal{W}$.

(1) An element $f \in \mathfrak{A}_\omega$ is called *algebraic*, denoted by $f \in \mathfrak{A}_\omega(\overline{\mathbb{Q}})$, if $\mathfrak{P}_\omega(w)^{-1} f(w)$ is $\overline{\mathbb{Q}}$-rational for every $w \in \mathcal{W}$ where f is finite.

(2) We set $M_\omega(\overline{\mathbb{Q}}) := M_\omega \cap \mathfrak{A}_\omega(\overline{\mathbb{Q}})$, and $M_\omega(\Gamma, \overline{\mathbb{Q}}) := M_\omega(\Gamma) \cap M_\omega(\overline{\mathbb{Q}})$.

We can compare the definition for our group with the unitary group. Let $\omega: \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ be $\overline{\mathbb{Q}}$-rational representation. Denote $\mathcal{A}_\omega, \mathcal{A}_\omega(\Gamma)$ for modular function spaces for unitary group as in [32, §5.3]. The composition of ω with the diagonal embedding $\mathrm{GL}_n \to \mathrm{GL}_n \times \mathrm{GL}_n$ gives a representation $\omega: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$. Clearly, if $f \in \mathcal{A}_\omega$, then its pullback $f \circ \iota \in \mathfrak{A}_\omega$ is a quaternionic modular form. Moreover, $f \in \mathcal{A}_\omega(\overline{\mathbb{Q}})$ and $f \circ \iota$ is finite, then the pullback $f \circ \iota \in \mathfrak{A}_\omega(\overline{\mathbb{Q}})$.

Even though we have provided a definition of algebraicity for modular forms on the bounded domain $\mathfrak{B}$, we can transfer it also to the other realization of the symmetric spaces discussed in Section 2. Indeed, with the notation of Section 2.2, suppose we are given two of these realizations $(\mathrm{i}_1, \Phi_1, H_1, K_1, \mathcal{G}_1)$ and $(\mathrm{i}_2, \Phi_2, H_2, K_2, \mathcal{G}_2)$, with $\mathrm{i}_1 = \mathrm{i}_2$ both induced from an algebraic embedding $M_n(\mathbb{B}) \to M_{2n}(K)$, where K is an imaginary quadratic field which splits $\mathbb{B}$. In particular, the matrix R in equation (2.2) has algebraic entries, and hence we obtain that the bijective map $\rho: \mathcal{H}_1 \to \mathcal{H}_2$ as defined there is algebraic, in the sense that maps algebraic points of $\mathcal{H}_1$ to algebraic points of $\mathcal{H}_2$. We also conclude from this that $\mu(z)$, as defined in the same equation, is algebraic if z is. In particular, given any realization $\mathcal{H}$, there is a bijection $\rho: \mathcal{H} \to \mathfrak{B}$. We define the CM-points on $\mathcal{H}$ to be the inverse image with respect to ρ of the CM-points of $\mathfrak{B}$.

As we have discussed, every vector-valued modular form $g: \mathcal{H} \to V$ corresponds uniquely to a modular form $f: \mathfrak{B} \to V$ by the rule $g(z) = \omega(\mu(z))^{-1} f(\rho(z))$. So it is enough to now observe that if w is a CM-point of $\mathcal{H}$, which by definition means $\rho(w)$ is a CM-point of $\mathfrak{B}$, and we have established above $\mu(w) \in GL_n(\overline{\mathbb{Q}})$. Hence, we can use the same periods $\mathfrak{P}_\omega(w)$ for both f and g.

In particular, the algebraicity as defined for bounded domains can be transferred to the unbounded domain $\mathfrak{Z}$. Let now $f : \mathfrak{Z} \to \mathbb{C}$ be a weight k modular form defined in Section 3, we can take its Fourier–Jacobi expansion. We denote the Fourier–Jacobi coefficients by $c(\tau, f; v, w)$. When $r = 0, n = 2m$, we simply denote it by $c(\tau, f)$.

Proposition 5.5 (1) *For congruence subgroup* Γ*, we have* $M_k(\Gamma) = M_k(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

(2) *For every* $f \in M_k$ *and* $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$*, we have* $c(\tau, f^\sigma; 0, w) = c(\tau, f; 0, w)^\sigma$*, for all* τ *and* w.

(3) *Let* $r = 0, n = 2m$ *(the Tube Domain case), and* $f \in M_k$*, then* $f \in M_k(\overline{\mathbb{Q}})$ *if and only if* $c(\tau, f) \in \overline{\mathbb{Q}}$ *for all* τ.

(4) *For congruence subgroup* Γ *and* $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$*, we have* $S_k(\Gamma)^\sigma = S_k(\Gamma)$ *and* $S_k(\Gamma) = S_k(\Gamma, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

Proof This can be proved similarly as [32, Propositions 11.11, 11.15, and 26.8]. See also [20, Proposition 7.2] for (1) and [6] for (2) and (3).

We briefly explain the proof for (3). Let $r = 0, n = 2m$, and $f \in M_k(\Gamma)$ for a congruence subgroup Γ. Let V be the model of $\Gamma\backslash\mathfrak{Z}$ defined over $\overline{\mathbb{Q}}$, then $\mathfrak{A}_0(\Gamma)$ can be identified with the function field of V. By the same method in [32, Sections 6, 7], one can show that $g \in \mathfrak{A}_0(\Gamma)$ if and only if g has algebraic Fourier coefficients. We will reduce our problem for $f \in M_k$ to $\mathfrak{A}_0$ similarly to what is done in the proof of [32, Proposition 11.11].

Let $\mathcal{W}$ be a dense subset of CM-points in $\mathfrak{Z}$. We first assume that $\det(\mathfrak{p}(w))^{-k} f(w) \in \overline{\mathbb{Q}}$ for all $w \in \mathcal{W}$ where f is finite. Note that there exists a function $U \in \mathfrak{A}_k(\overline{\mathbb{Q}})$ on $\mathfrak{Z}$ holomorphic in w with $\det(U)(w) \neq 0$. Indeed, denote $\mathcal{H}$ for the unbounded realization of $\mathcal{B}$ via the Cayley transform, and we can simply put $U(z) = R(z)$ for $z \in \mathfrak{Z} \hookrightarrow \mathcal{H}$ with R the function in [32, Proposition 9.11]. We set $g := \det(U)^{-k} f$, and note that

$$g(w) = \det\left(U(w)^{-1}\mathfrak{p}(w)\right)^k \det(\mathfrak{p}(w))^{-k} f(w).$$

But now we have that $U(w)^{-1}\mathfrak{p}(w)$ is $\overline{\mathbb{Q}}$ -rational since this holds for the function R in unitary case. That is, $g(w)$ is $\overline{\mathbb{Q}}$-rational for every CM-point w where g is finite; thus, $g \in \mathfrak{A}_0(\overline{\mathbb{Q}})$. Since $f = \det(U)^{-k} g$, we obtain that f also has algebraic Fourier expansion.

For the other direction, we keep the same notation. If $f \in M_k$ has algebraic Fourier expansion, then $g \in \mathfrak{A}_0(\overline{\mathbb{Q}})$. For every CM-point w, we may choose the function U above such that U is finite at w and $U(w)$ is invertible. If f is finite at w, then so is g and $g(w)$ is $\overline{\mathbb{Q}}$ rational. The equality of $g(w)$ as above then shows that $\det(\mathfrak{p}(w))^{-k} f(w)$ is $\overline{\mathbb{Q}}$-rational, and hence $f \in M_k(\overline{\mathbb{Q}})$. ∎

We end this subsection by giving a definition for adelic modular forms. Let $\mathbf{f} \in \mathcal{M}_k$ be an (adelic) modular form of scalar weight k, i.e., we are taking $\omega = \det^k$. We say that $\mathbf{f}$ is algebraic, denoted by $\mathbf{f} \in \mathcal{M}_k(\overline{\mathbb{Q}})$ if for a dense subset $\mathcal{W}$ of CM-points in $\mathfrak{Z}$, $\mathfrak{P}_k(w)^{-1}\mathbf{f}(g_{\mathfrak{h}} g) \in \overline{\mathbb{Q}}$ for all $w \in \mathcal{W}$. Since $j(g, w) \in \overline{\mathbb{Q}}$ for CM-point w, this is the same as all component f_j under correspondence $\mathbf{f} \leftrightarrow (f_0, \ldots, f_h)$ are algebraic.

Proposition 5.6 (1) *Let* $K = K_1(\mathfrak{n})$ *or* $K_0(\mathfrak{n})$*, then* $\mathcal{M}_k(K) = \mathcal{M}_k(K, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, \mathcal{S}_k(K) = \mathcal{S}_k(K, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

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(2) *Let* $r = 0, n = 2m$, *and* $\mathbf{f} \in \mathcal{M}_k$, *then* $\mathbf{f} \in \mathcal{M}_k(\overline{\mathbb{Q}})$ *if and only if the Fourier coefficients* $c(\tau, q, \mathbf{f}) \in \overline{\mathbb{Q}}$.

Let $r = 0, n = 2m$, and keep the notation for Eisenstein series in previous sections. Let $\mathbf{E}_l(g, s)$ be a Siegel Eisenstein series for group G_n. By explicit computation of Fourier expansion, we have $\mathbf{E}_l^*(g, l) \in \mathcal{M}_l(\overline{\mathbb{Q}})$. Clearly, we also have $\mathbf{E}_l(g, l) \in \mathcal{M}_l(\overline{\mathbb{Q}})$.

5.3 Differential operators and nearly holomorphic functions

In this subsection, we summarize some of the result of [29] (see also [32, Chapter 3]) on differential operators on type-D domains and then apply these operators to Siegel-type Eisenstein series. We will be working with the bounded realization of our symmetric space, but thanks to the remark above, we can transfer the definitions from one realization to the other. We set

$$\mathfrak{B} = \{z \in \mathbb{C}_n^n : {}^t z = -z, z^* z < 1_n\},\ \mathfrak{T} := \{z \in \mathbb{C}_n^n : {}^t z = -z\},\ \eta(z) := 1 - z^* z.$$

Here, $\mathfrak{T}$ is the tangent space of $\mathfrak{B}$ at the origin 0.

Given a positive integer d and two finite-dimensional complex vector spaces W and V, we denote by $Ml_d(W, V)$ the vector space of all $\mathbb{C}$-multilinear maps of $W \times \cdots \times W$ (d copies) into V and $S_d(W, V)$ the vector space of all homogeneous polynomial maps of W into V of degree d. We omit the symbol V if $V = \mathbb{C}$. Given a representation $\omega : \mathrm{GL}_n(\mathbb{C}) \to V$, we define a representation $\{\omega \otimes \tau^d, Ml_d(\mathfrak{T}, V)\}$ by

$$[(\omega \otimes \tau^d)(a)h](u_1, \ldots, u_d) = \omega(a)h({}^t a u_1 a, \ldots, {}^t a u_d a),$$

for $a \in \mathrm{GL}_n(\mathbb{C}), h \in Ml_d(\mathfrak{T}, V), u_i \in \mathfrak{T}$. In particular, taking $d = 1$ and ω the trivial representation, we define the representation $\{\tau, S_1(\mathfrak{T})\}$ of $\mathrm{GL}_n(\mathbb{C})$ by $[\tau(a)h](u) = h({}^t a u a)$ for $h \in S_1(\mathfrak{T}), u \in \mathfrak{T}$.

Take an $\mathbb{R}$-rational basis $\{\varepsilon_\nu\}$ of $\mathfrak{T}$ over $\mathbb{C}$ and for $u \in \mathfrak{T} = \sum_\nu u_\nu \varepsilon_\nu$. For $z \in \mathfrak{B}$, write $z = \sum_\mu z_\nu \varepsilon_\nu$. For $f \in C^\infty(\mathfrak{B}, V)$, we define $\mathfrak{D}f, \overline{\mathfrak{D}}f, \mathfrak{C}f \in C^\infty(\mathfrak{B}, S_1(\mathfrak{T}, V))$ by

$$(\mathfrak{D}f)(u) = \sum_\nu u_\nu \frac{\partial f}{\partial z_\nu},\ (\overline{\mathfrak{D}}f)(u) = \sum_\nu u_\nu \frac{\partial f}{\partial \overline{z}_\nu},\ (\mathfrak{C}f)(u) = (\mathfrak{D}f)({}^t\eta(z) u \eta(z)).$$

We further define $\mathfrak{D}^d f, \overline{\mathfrak{D}}^d f, \mathfrak{C}^d f$ by

$$\mathfrak{D}^d f = \mathfrak{D}\mathfrak{D}^{d-1} f,\ \overline{\mathfrak{D}}^d f = \overline{\mathfrak{D}\mathfrak{D}}^{d-1} f,\ \mathfrak{C}^d f = \mathfrak{C}\mathfrak{C}^{d-1} f,\ \mathfrak{D}^0 f = \overline{\mathfrak{D}}^0 f = \mathfrak{C} f = f.$$

And, define $\mathfrak{D}_\omega^d f \in C^\infty(\mathfrak{B}, S_d(\mathfrak{T}, V))$ by

$$\mathfrak{D}_\omega^d f = (\omega \otimes \tau^d)(\eta(z))^{-1} \mathfrak{C}^d[\omega(\eta(z)) f].$$

We now recall the important fact, due to Hua, Schmid, Johnson, and Shimura (see, for example, [27]), that the representation $\{\tau^d, S_d(\mathfrak{T})\}$ is the direct sum of irreducible representations and each irreducible constituent has multiplicity one. In particular, for each $\mathrm{GL}_n(\mathbb{C})$-stable subspace $Z \subset S_d(\mathfrak{T})$, we can define the projection map ϕ_Z of $S_d(\mathfrak{T})$ onto Z. Define $\mathfrak{D}_\omega^Z f \in C^\infty(\mathfrak{B}, Z \otimes V)$ by $\mathfrak{D}_\omega^Z f = \phi_Z \mathfrak{D}_\omega^d f$.

Lemma 5.7 *With notation as above, we have:*

(1) $\pi^{-1}\mathfrak{D}f \in \mathfrak{A}_\tau(\overline{\mathbb{Q}})$ *for every* $f \in \mathfrak{A}_0(\overline{\mathbb{Q}})$.
(2) *Let Z be a* $\mathrm{GL}_n(\mathbb{C})$*-stable subspace of* $S_d(\mathfrak{T})$. *If* $f \in \mathfrak{A}_\omega(\overline{\mathbb{Q}})$, *then*

$$\pi^{-d}\mathfrak{P}_\omega(w)^{-1}\mathfrak{D}_\omega^Z f(w)$$

is $\overline{\mathbb{Q}}$*-rational for any CM-point* w.

Proof The proof is the same as the one in [32, Theorems 14.5 and 14.7] (see also [27, Sections 5 and 6]). Indeed, as we have a natural inclusion $\mathfrak{B} \to \mathcal{B}$, we can reduce our problem to unitary case. For example, for (1), denote Df for the differential operators in unitary case. The lemma is proved for this case in [32, Theorem 14.5]. Let p be the complex dimension of $\mathfrak{B}$, and we can take p elements $g_1, \dots, g_p \in \mathcal{A}_0(\overline{\mathbb{Q}})$ such that $g_1 \circ \varepsilon, \dots, g_p \circ \varepsilon$ are algebraically independent. Put $f_j = g_j \circ \varepsilon$. As shown in the last section, $g_j \circ \varepsilon \in \mathfrak{A}_0(\overline{\mathbb{Q}})$ so $\partial/\partial f_1, \dots, \partial/\partial f_p$ are well-defined derivations of $\mathfrak{A}_0(\overline{\mathbb{Q}})$. For every $f \in \mathfrak{A}_0(\overline{\mathbb{Q}})$, we have $\mathfrak{D}f = \sum_j(\partial f/\partial f_j)\mathfrak{D}f_j$. Now $\mathfrak{D}(f_j) = (Dg_j) \circ \varepsilon$ and $\pi^{-1}Dg_j$ is $\overline{\mathbb{Q}}$-rational. This proves our assertion. ∎

We now set $r(z) := -\eta(z)^{-1}\overline{z}$. Let d be a nonnegative integer and $\{\omega, V\}$ the representation as before. A function $f \in C^\infty(\mathfrak{B}, V)$ is called *nearly holomorphic of degree d* if it can be written as a polynomial in r, of degree less than d, with V-valued holomorphic functions on $\mathfrak{B}$ as coefficients. We denote the space of such functions by $\mathfrak{N}^d(\mathfrak{B}, V)$. Let $\mathfrak{N}_\omega^d$ be the space consisting of functions satisfying the modular properties as in M_ω but now replacing the holomorphic condition with nearly holomorphic. For a congruence subgroup Γ, we can similarly define the space $\mathfrak{N}_\omega^d(\Gamma)$. An exact same argument in the proof of [32, Lemma 14.3] shows that this space is finite-dimensional over $\mathbb{C}$.

Suppose V is $\overline{\mathbb{Q}}$-rational. A function $f \in \mathfrak{N}_\omega^d$ is called *algebraic*, denoted by $f \in \mathfrak{N}_\omega^d(\overline{\mathbb{Q}})$, if $\mathfrak{P}_\omega(w)^{-1}f(w)$ is $\overline{\mathbb{Q}}$-rational for $w \in \mathcal{W} = \{g \cdot 0 : g \in G(\overline{\mathbb{Q}})\}$. Put $\mathfrak{N}_\omega^d(\Gamma, \overline{\mathbb{Q}}) = \mathfrak{N}_\omega^d(\overline{\mathbb{Q}}) \cap \mathfrak{N}_\omega^d(\Gamma)$. The proof of the following lemma is the same as the one in [32, Theorem 14.9].

Lemma 5.8 *Let Z be an irreducible subspace of $S_p(T)$. Then* $\pi^{-p}\mathfrak{D}_\omega^Z f \in \mathfrak{N}_{\omega\otimes\tau_Z}^{d+p}(\overline{\mathbb{Q}})$ *for every* $f \in \mathfrak{N}_\omega^d(\overline{\mathbb{Q}})$. *Here, τ_Z is the restriction of τ^p to Z.*

We now extend the above definitions to adelic modular forms. Let $\mathbf{f} \in \mathcal{M}_k$ and viewing it as a function on $G(\mathbb{A}_\mathbf{h}) \times \mathfrak{B}$ by setting $\mathbf{f}(g_\mathbf{h}, z) = j(g_z, z_0)^k \mathbf{f}(g_\mathbf{h} g_z)$ with $z = g_z \cdot 0 \in \mathfrak{B}$. Then $\mathfrak{D}_k\mathbf{f}, \mathfrak{D}_k^Z\mathbf{f}$ is defined as applying differential operators on $z \in \mathfrak{B}$. A function $\mathbf{f} : G(\mathbb{A}_\mathbf{h}) \times \mathfrak{B} \to \mathbb{C}$ is called *nearly holomorphic* if it is nearly holomorphic in $z \in \mathfrak{B}$. We can then define the space $\mathcal{N}_k^d$ and of nearly holomorphic modular forms as before. Similarly, we can define subspace $\mathcal{N}_k^d(\overline{\mathbb{Q}})$. These definitions are equivalent to all components in the correspondence $\mathbf{f} \leftrightarrow (f_1, \dots, f_h)$ being nearly holomorphic or algebraic nearly holomorphic.

We now apply the differential operators to Siegel-type Eisenstein series and show that it is nearly holomorphic for certain values of s. We will keep the notation of Section 4, and so in particular $\mathbf{E}_l(g, s)$ is the Siegel-type Eisenstein series associated with group G_n, $n = 2m$ of weight l and character χ.

Proposition 5.9 *Assume $l > n-1$, and let $\mu \in \mathbb{Z}$ such that $n-1 < \mu \le l$. Then:*

(1) $\mathbf{E}_l(g,\mu) \in \pi^{\alpha}\mathcal{N}_l^{m(l-\mu)}(\overline{\mathbb{Q}})$ *with* $\alpha = m(l-\mu)$.

(2) *Denote* $\mathfrak{E}_l(g,s) = \Lambda_{\mathfrak{n}}(s,\chi)\mathbf{E}_l(g,s,\chi)$. *Then* $\mathfrak{E}_l(g,\mu) \in \pi^{\beta}\mathcal{N}_l^{m(l-\mu)}(\overline{\mathbb{Q}})$ *with* $\beta = m(l+\mu) - m(m-1)$.

Proof For this, we use [27, Theorem 2D], which classifies the irreducible representations of $(\tau^{mp}, S_{mp}(\mathfrak{T}))$. In particular, for $p \in \mathbb{Z}$ and a weight q, we can define the operator Δ_q^p by $\Delta_q^p\mathbf{f} = (\mathfrak{D}_\omega^Z\mathbf{f})(\psi)$ with $\omega = \det^q$, $Z = \mathbb{C}\psi \subset S_{mp}(\mathfrak{T})$, and $\psi = \det^{p/2}$. Here, the square root of the determinant denotes the Pfaffian of the skew-symmetric matrix. Then

$$\Delta_q^p\mathcal{N}_q^t(\overline{\mathbb{Q}}) \subset \pi^{mp}\mathcal{N}_{q+p}^{t+mp}(\overline{\mathbb{Q}}).$$

We have shown that $\mathbf{E}_l(g,l) \in \mathcal{M}_l(\overline{\mathbb{Q}})$, so $\Delta_l^p\mathbf{E}_l(g,l) \in \pi^{mp}\mathcal{N}_{l+p}^{mp}(\overline{\mathbb{Q}})$. Take $p = l-\mu$, then by the explicit formula in [27, Theorem 4.3], we have

$$\Delta_\mu^p\mathbf{E}_\mu(g,\mu) = C\cdot\mathbf{E}_l(g,\mu), \quad \text{with } C \in \overline{\mathbb{Q}}^\times.$$

This concludes the proof of the proposition. ∎

6 Main results

We now recall that we have established the integral representation of the L-function:

$$L(s,\mathbf{f},\chi)\mathbf{f}(g) = c_k(s)\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/K_1(\mathfrak{n})K_\infty} \mathfrak{E}(g\times h,s)\mathbf{f}(h)\mathbf{d}h.$$

Here, $\mathfrak{E}_k(\mathfrak{g},s) = \Lambda_{\mathfrak{n}}(s,\chi)\mathbf{E}(\mathfrak{g},s)$; $\mathbf{E}(\mathfrak{g},s) = E_k(\mathfrak{g}\sigma^{-1},s)$ for $\mathfrak{g} \in G_N(\mathbb{A})$ and $\sigma = 1$ if $v \nmid \mathfrak{n}$, $\sigma = \tilde{\tau}_m$ if $v|\mathfrak{n}$, where $E_k(\mathfrak{g},s)$ is the Siegel-type Eisenstein series defined on G_N of weight k and $N = 2n$.

We first prove a lemma which is the analog of [32, Lemma 26.12] in our setting.

Lemma 6.1 *Let $\mathbf{f} \in \mathcal{N}_k^d(\overline{\mathbb{Q}})$ be an algebraic nearly holomorphic form associated with group G_N. Then there exist $\mathbf{g}_j, \mathbf{h}_j \in \mathcal{N}_k^d(\overline{\mathbb{Q}})$ associated with group G_n such that*

$$\mathbf{f}(g\times h) = \sum_{j=1}^{e}\mathbf{g}_j(g)\overline{\mathbf{h}_j(h)}.$$

Proof Write

$$\mathbf{f}(g\times h) = j(g_\infty\times h_\infty, z_0\times z_0)^{-k}\mathbf{f}(g_{\mathbf{h}}\times h_{\mathbf{h}}, z\times w)$$

with $z = g_\infty z_0, w = h_\infty z_0 \in \mathfrak{Z}_{m,r}$. By definition, $f(z,w) := \mathbf{f}(g_{\mathbf{h}}\times h_{\mathbf{h}}, z\times w)$ is nearly holomorphic in $z\times w$. Similarly to the proof of [32, Lemma 26.12], one can show that it is also nearly holomorphic in z and $\overline{f(z,w)}$ is nearly holomorphic in w. Therefore, $\mathbf{f} \in \mathcal{N}_k^d(\overline{\mathbb{Q}})$ (resp. $\overline{\mathbf{f}} \in \mathcal{N}_k^d(\overline{\mathbb{Q}})$) as a function in g or h.

Let $\{\mathbf{g}_j\}_{j=1}^e$ be a $\overline{\mathbb{Q}}$-rational basis of $\mathcal{N}_k^d(\overline{\mathbb{Q}})$. For each fixed h, we have $\mathbf{f}(g\times h) = \sum_{j=1}^e\mathbf{g}_j(g)\overline{\mathbf{h}_j(h)}$ with $\mathbf{h}_j(h) \in \mathbb{C}$. Since $\mathbf{g}_j$ are linearly independent, we can find e points $g_1,\ldots,g_e$ such that $\det(\mathbf{g}_j(z_k))_{j,k=1}^e \neq 0$. Solving the linear equations $\overline{\mathbf{f}(g,h)} = \sum_{j=1}^e\overline{\mathbf{g}_j(z_k)}\mathbf{h}_j(h)$, we find functions $\mathbf{h}_j \in \mathcal{N}_k^d$.

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It suffices to prove that $\{\mathbf{h}_j\}$ are algebraic. Since $\mathcal{W} = \{g \cdot z_0 : g \in G(\overline{\mathbb{Q}})\}$ is a dense subset of $\mathfrak{Z}_{m,r}$, we can take g_j such that $g_j z_0 \in \mathcal{W}$. We easily calculate the period $\mathfrak{P}_k(z_0 \times z_0) = \mathfrak{P}_k(z_0)\mathfrak{P}_k(z_0)$; hence,

$$\mathfrak{P}_k(z_0 \times z_0)^{-1}\overline{\mathbf{f}(g_{\mathbf{h}} \times h_{\mathbf{h}})} = \sum_{j=1}^{e} \mathfrak{P}_k(z_0)^{-1}\overline{\mathbf{g}_j(g_{\mathbf{h}})}\mathfrak{P}_k(z_0)^{-1}h_j(\mathbf{h}).$$

By algebraicity of $\mathbf{f}, \mathbf{g}_j$, we have $\mathfrak{P}_k(z_0)^{-1}h_j(\mathbf{h}) \in \overline{\mathbb{Q}}$, and thus for all $w = h_\infty z_0 \in \mathcal{W}$, we have $\mathfrak{P}_k(w)^{-1}h_j(h) \in \overline{\mathbb{Q}}$. Hence, $\mathbf{h}_j \in \mathcal{N}_k^d(\overline{\mathbb{Q}})$, which completes the proof. ∎

Before stating the main theorem, we need to establish one more result.

Proposition 6.2 *Assume that $k > 2n - 1$, and let $\mu \in \mathbb{Z}$ such that $2n - 1 < \mu \leq k$. Then there exists a function $\mathbf{T}(g, h)$ with $\overline{\mathbf{T}(g, h)} \in \mathcal{N}_k^{m(k-\mu)}(\overline{\mathbb{Q}}) \times \mathcal{M}_k(\overline{\mathbb{Q}})$ such that*

$$\langle \mathbf{T}(g, h), \mathbf{f}(h)\rangle = \langle \mathfrak{E}(g \times h, \mu), \mathbf{f}(h)\rangle.$$

Proof This is the analog of Lemma 29.3 proved in [32] in the unitary case. Actually, it is even simpler in our case since we do not need to involve some more complicated differential operators needed in the unitary case. Here, we simply indicate some changes to the proof in [32] to cover our case. We follow the notation of Appendix A8 in [32] and write $\mathfrak{g}$ for the real Lie algebra of $\mathbf{G} := G(\mathbb{R})$. We then have the familiar decomposition of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{k}$ is the Lie algebra of the fixed maximal compact subgroup $\mathbf{K} \cong U(n)$. Finally, we write $\mathfrak{U}$ for the universal enveloping Lie algebra of $\mathfrak{g}_{\mathbb{C}}$, and $\mathbf{K}^c = GL_n(\mathbb{C})$ for the complexification of $\mathbf{K}$.

Given a representation (ρ, V) of $\mathbf{K}^c$, we write $C^\infty(\rho)$ for the functions $f \in C^\infty(\mathbf{G}, V)$ such that $f(xk^{-1}) = \rho(k)f(x)$ for all $k \in \mathbf{K} \subset \mathbf{K}^c$, and $x \in \mathbf{G}$. As in [32], there is a bijection between $C^\infty(\mathfrak{Z}, V)$ and $C^\infty(\rho)$ which we denote by $f \mapsto f^\rho$. We also write $H(\rho)$ for the functions in $C^\infty(\rho)$ such that $Yf = 0$ for all $Y \in \mathfrak{p}_-$. These functions correspond to holomorphic functions in $C^\infty(\mathfrak{Z}, V)$.

Recall (see [32]) that a $\mathfrak{U}$-module $\mathcal{Y}$ is called *unitarizable* if there exists a positive-definite hermitian form $\{\,,\,\} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{C}$ such that $\{Xg, h\} = -\{g, Xh\}$ for every $g, h \in \mathcal{Y}$ and $X \in \mathfrak{g}$.

Let us now take $\rho = \det^k$ for some $k > 2m - 1$. Then we have that for any nonzero $f \in H(\rho)$, the $\mathfrak{U}$-module structure of $\mathfrak{U}f$ depends only on the weight k, and that such a module is unitarizable. This follows exactly as in [32, Theorem A8.4] where the cases of unitary and symplectic groups are considered. What needs to be explained is the bound on the weight k. For this, there are two remarks that one needs to make: first that the bound follows from the fact that in our type-D setting, the function ψ_Z is given by (see [27])

$$\psi_Z(s) = \prod_{h=1}^{m}\prod_{i=1}^{r_{2h}}(s - i + 2h - 1).$$

For the notation, we refer to [27] since what is important in the proof is the fact that $\psi_Z(-k) \neq 0$ which is satisfied for the selected bound on k. The other remark is the existence of a nonzero $g \in H(\det)$ and a discrete subgroup Γ of $\mathbf{G}$ such that $\Gamma\backslash\mathbf{G}$ is compact and $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$. This is the analog of [32, Lemma A8.5], which covers the unitary case. But again, the existence of such a g and a discrete Γ can be derived from the existence of such elements in the unitary case, say $\widetilde{\Gamma}$ and $\widetilde{g}$ (the content of Lemma A8.5 and the natural closed embedding $\mathbf{G} \hookrightarrow U(n,n)$. In particular, we may take $\Gamma := \widetilde{\Gamma} \cap \mathbf{G}$ and g as the restriction of $\widetilde{g}$ to $\mathbf{G}$.

The importance of considering $\mathfrak{U}$-module structures which are unitarizable becomes clear from the following result on holomorphic projection. Namely, if we still write $\rho = \det^k$, and consider an $f \in N_k^d(\overline{\mathbb{Q}})$ for any $d \in \mathbb{N}$ such that $\mathfrak{U}f^\rho$ is unitarizable, then there exists an element $q \in M_k(\overline{\mathbb{Q}})$ such that $\langle f, h\rangle = \langle q, h\rangle$ for all $h \in S_k$. This is a rather general result and can be obtained exactly in the same way as it is done in [32, Lemma A8.7] in the unitary and symplectic case with little changes.

We can now complete the proof of the proposition. First, we note that unitarizable $\mathfrak{U}$-modules behave well with respect to the doubling mapping. Let us write $\mathbf{G}_i$ with $i = 1,2$ for groups of type similar to $\mathbf{G}$, and we insert the index i to all notations. Assume we have a doubling embedding $\mathbf{G}_1 \times \mathbf{G}_1 \hookrightarrow \mathbf{G}_2$ of the kind considered in this paper, and we write Δ for the corresponding embedding of symmetric spaces. Then, if $f \in H_2(\rho)$ such that the $\mathfrak{U}_2$-module $\mathfrak{U}_2 f$ is unitarizable, then the $\mathfrak{U}_1 \times \mathfrak{U}_1$ module $\Delta^*\mathfrak{U}f$ is unitarizable with respect to both variables. This follows similar to [32, Lemma A8.11]. Hence, in order to complete the proof, it is enough to show that the Eisenstein series $\mathfrak{E}_k(g,\mu)$ belongs to a unitarizable $\mathfrak{U}$-module, since then when we pull it back with respect to the diagonal embedding, we can keep the one variable constant (the variable g in the statement of the proposition) and take the holomorphic projection with respect to the other. This final claim follows from the fact shown in Proposition 5.9 that the Eisenstein series $\mathfrak{E}_k(g,\mu)$ on G_N with $N = 2n$ are obtained from holomorphic ones of weight $l \geq N-1 = 2n-1$ by applying the Shimura–Maass operators Δ_l^p. But these operators are well known to be obtained as operators of the universal enveloping algebra. Indeed, this is shown for the symplectic and unitary case in the first few lines of [32, §A8.8] and more generally in [7]. ∎

We can now prove the theorem on the algebraicity of the L-values (we remind the reader here of Remark 1.2 made in the Introduction).

Theorem 6.3 *Let $\mathfrak{n}$ be an ideal in $\mathbb{Z}$, and assume that all finite places v with $v \nmid \mathfrak{n}$ are split in $\mathbb{B}$. Let $\mathbf{f} \in \mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}})$ be an eigenform with $k > 2n-1$, and let χ be a Dirichlet character whose conductor divides the ideal $\mathfrak{n}$. Let $\mu \in \mathbb{Z}$ such that $2n-1 < \mu \leq k$, then*

$$\frac{L(\mu,\mathbf{f},\chi)}{\pi^{n(k+\mu)-\frac{3}{2}n(n-1)}\langle \mathbf{f},\mathbf{f}\rangle} \in \overline{\mathbb{Q}}.$$

Proof We prove the theorem following an idea used in the proof of [32, Theorem 29.5], which allows us to cover also the non-split (i.e., non-tube) case. By the above proposition, we can replace $\mathfrak{E}(g\times h,\mu)$ by $\overline{\mathbf{T}(g,h)}$ holomorphic in h such that the integral can be rewritten as

$$c_k(\mu)L(\mu,\mathbf{f},\chi)\mathbf{f}(g) = \langle \mathbf{T}(g,h), \mathbf{f}(h)\rangle.$$

By Lemma 6.1, we have

$$\pi^{-\beta}\mathbf{T}(g,h) = \sum_{j=1}^{e} \overline{\mathbf{g}_j(g)}\mathbf{h}_j(h),$$

with $\mathbf{g}_j \in \mathcal{N}_k^{n(k-\mu)}(\overline{\mathbb{Q}}), \mathbf{h}_j \in \mathcal{M}_k(\overline{\mathbb{Q}})$, and $\beta = n(k+\mu) - n(n-1)$. We note here that, indeed, $\mathbf{h}_j \in \mathcal{M}_k(\overline{\mathbb{Q}})$, as one can see in the proof of Lemma 6.1 that the analytic properties of the $\mathbf{h}_j$'s follow from that of the restricted function on the h variable since they are obtained as the solutions of a linear system where the "constant" vector consists of holomorphic functions.

Then the above equation can be written as

$$\frac{c_k(\mu)L(\mu,\mathbf{f},\chi)}{\pi^\beta}\mathbf{f}(g) = \sum_{j=1}^{e}\langle \mathbf{h}_j, \mathbf{f}\rangle \cdot \mathbf{g}_j(g).$$

Since we are assuming $k > 2n-1$, we may apply [8, Corollary 2.4.6] and write $\mathcal{M}_k(\overline{\mathbb{Q}}) = \mathcal{S}_k(\overline{\mathbb{Q}}) \oplus \mathcal{E}_k(\overline{\mathbb{Q}})$ as a direct sum of space of algebraic cusp form and the space of algebraic Eisenstein series. In particular, we can find $\mathbf{h}'_j \in \mathcal{S}_k(\overline{\mathbb{Q}})$ such that $\langle \mathbf{h}_j, \mathbf{f}\rangle = \langle \mathbf{h}'_j, \mathbf{f}\rangle$. Let $w = g_\infty z_0$ be a CM-point with period $\mathfrak{P}_k(w)$, then by definition, $\mathfrak{P}_k(w)^{-1}\mathbf{g}_j(g_\mathbf{h} g_\infty) \in \overline{\mathbb{Q}}$. Therefore, at $g = g_\mathbf{h} g_\infty$, we can further find some $\mathbf{h}" \in \mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}})$ such that

$$\frac{c_k(\mu)L(\mu,\mathbf{f},\chi)}{\pi^\beta}\mathbf{f}(g_\mathbf{h}\cdot g_\infty) = \mathfrak{P}_k(w)\langle \mathbf{h}", \mathbf{f}\rangle.$$

Denote

$$\mathcal{V} = \{\mathbf{f} \in \mathcal{S}_k(K_1(\mathfrak{n})) : \mathbf{f}|T_\xi = \lambda(\xi)\mathbf{f}\}, \mathcal{V}(\overline{\mathbb{Q}}) = \mathcal{V} \cap \mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}})$$

for the space consisting of eigenforms with the same eigenvalues as $\mathbf{f}$. Since $\mathcal{S}_k(K_1(\mathfrak{n})) = \mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ and $\mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}})$ is stable under the action of Hecke operators, we obtain that the eigenvalues $\lambda(\xi) \in \overline{\mathbb{Q}}$. Hence, we have $\mathcal{V} = \mathcal{V}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. We may now write $\mathcal{S}_k(K_1(\mathfrak{n}), \overline{\mathbb{Q}})) = \mathcal{V}(\overline{\mathbb{Q}}) \oplus \mathcal{U}$ for some $\overline{\mathbb{Q}}$-rational vector space $\mathcal{U}$ (compare with the first few lines of the proof of [32, Theorem 28.5]). With $w = g_\infty z_0$ as above, let $\mathbf{h}_w$ be the projection of $\mathbf{h}"$ to $\mathcal{V}(\overline{\mathbb{Q}})$, then for all $\mathbf{f} \in \mathcal{V}(\overline{\mathbb{Q}})$ and any CM-point $w = g_\infty z_0$, we have

$$\frac{c_k(\mu)L(\mu,\mathbf{f},\chi)}{\pi^\beta}\frac{\mathbf{f}(g_\mathbf{h}\cdot g_\infty)}{\mathfrak{P}_k(w)} = \langle \mathbf{h}_w, \mathbf{f}\rangle.$$

For a fixed μ, we can choose w such that $\langle \mathbf{h}_w, \mathbf{f}\rangle \neq 0$, since $L(\mu,\mathbf{f},\chi) \neq 0$, thanks to the Euler product expansion and absolute convergence for such an $\mu > 2n-1$. Such $\mathbf{h}_w$ span $\mathcal{V}(\overline{\mathbb{Q}})$, so for any $\mathbf{h}, \mathbf{h}' \in \mathcal{V}(\overline{\mathbb{Q}})$, we have $\langle \mathbf{h}, \mathbf{h}'\rangle \in \pi^\beta c_k(\mu)^{-1}L(\mu,\mathbf{f},\chi)\overline{\mathbb{Q}}$ and thus $\langle \mathbf{h}, \mathbf{h}'\rangle/\langle \mathbf{f}, \mathbf{f}\rangle \in \overline{\mathbb{Q}}$. Choose g_∞ such that $\mathbf{f}(g_\mathbf{h} g_\infty) \neq 0$. Then, by algebraicity of $\mathbf{f}$, we have

$$\frac{c_k(\mu)L(\mu,\mathbf{f},\chi)}{\pi^\beta\langle \mathbf{f}, \mathbf{f}\rangle} = \frac{\langle \mathbf{h}_w, \mathbf{f}\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle}\left(\frac{\mathbf{f}(g_\mathbf{h}\cdot g_\infty)}{\mathfrak{P}_k(w)}\right)^{-1} \in \overline{\mathbb{Q}},$$

and the result follows from the value of $c_k(\mu)$ in Lemma 4.6. ■

Remark 6.4 We finally give several remarks on our main theorem.

(1) As it was discussed in Remark 3.1, if $\mathbf{f}$ is assumed to be an eigenform in $S_k(K)$ for some $K \supset K_1(\mathfrak{n})$, then the L-function above is only the partial L-function corresponding to such an $\mathbf{f}$. However, in our case since our result are for values of

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L at the absolute convergent range (larger than $2n-1$), we can extend our results to include such missing Euler factors. Indeed, if we consider a $p|\mathfrak{n}$ and write $L_p(p^{-s}) = P_p(\chi(p)p^{-s})^{-1}$ for such an Euler factor, then for an integer $\mu > 2n-1$, we have $P_p(\chi(p)p^{-\mu}) \neq 0$ since both the complete and the partial L-functions are absolutely convergent there. Hence, if one further knows that $P_p(\chi(p)p^{-\mu}) \in \overline{\mathbb{Q}}$, then one can simply "add" the Euler factor to the algebraicity result above. As it was mentioned above, the existence of the correct Euler factors has been established by Yamana in [38]; however, to the best of our knowledge, there is no result on their algebraicity, that is, it has not yet been established that $P_p(X) \in \overline{\mathbb{Q}}[X]$ even though such a result is expected to hold.

(2) Of course, the motivation for a theorem as above stems from the celebrated Deligne's conjectures [4]. These conjectures are related to critical values of motivic L-functions and are of central importance in modern number theory. This critical values can be defined by the Γ-factors appearing in the functional equation of the motive. Of course, in our setting and to the best of our knowledge, we do not have a motive corresponding to our automorphic object $\mathbf{f}$. As it is often the case in such situations, one attempts to define the critical values by looking at the Γ-factors of the automorphic object. However, similarly to Remark 3.1, the explicit form of the Euler factors at infinity (i.e., Γ-factors) such that a functional equation is satisfied, is not known (see [38, Theorem 5.2]).

The next best thing one can do is to "declare" as the right Γ-factors the ones that are obtained by combining the expression $c_k(s)$ derived by the reproducing kernel in Lemma 4.6 and then include the Γ-factors which give good analytic properties to the Siegel-type Eisenstein series. Such an approach is taken, for example, by Shimura [32] for Hermitian and Siegel modular forms and also by Böcherer and Schmidt [1, Appendix] in the Siegel modular forms case. In our case, since we are taking $k \geq 2n$, the Γ-factors for the Siegel-type Eisenstein series are $\Gamma_n(s+k) := \pi^{n(n-1)} \prod_{i=0}^{n-1} \Gamma(s+k-2i)$ (see [3, Theorem 3.8]), and hence for the L-function is (using the notation of Lemma 4.6 and write $n = 2q + t$ with $t \in \{0,1\}$),

$$\Gamma(s) := c_k(s)\Gamma_n(s+k) = \alpha(s)\pi^{\frac{3n(n-1)}{2}} \frac{\prod_{j=0}^{q-1} \Gamma(s+k-n+1-t-2j)}{\prod_{j=0}^{q-1} \Gamma(s+k-2j)} \prod_{i=0}^{n-1} \Gamma(s+k-2i) =$$

$$\alpha(s)\pi^{\frac{3n(n-1)}{2}} \prod_{j=0}^{q-1} \Gamma(s+k-n+1-t-2j) \prod_{i=q}^{n-1} \Gamma(s+k-2i),$$

where, we recall, $\alpha(s)$ is a holomorphic function for all $s \in \mathbb{C}$. It is worth mentioning here that in the case of n even, the above Γ-factors agree with those computed in [3, Theorem 8.2] where a very different method was used involving theta series and an Eisenstein series of different group and weight.

Declaring now as critical values the integral values of s such that $\Gamma(s)$ and $\Gamma(2n-1-s)$ (note our normalization of the Satake parameters) have no poles, we find that the critical values are at the interval

$$\{\mu \in \mathbb{Z} : 2n-1-k \leq \mu \leq k\}.$$

In particular, the values in our theorem above are all within the critical range (in the above sense).

Let it now explain a bit more the assumption in our theorem, namely that we are assuming $k > 2n-1$ and why we obtain results only for the critical values $2n-1 < \mu \leq k$, and not for the whole range indicated above. The condition on the weight $k > 2n-1$ is required since in our proof we need to be able to separate *algebraically* the cuspidal part from the Eisenstein part, that is, $\mathcal{M}_k(\overline{\mathbb{Q}}) = \mathcal{S}_k(\overline{\mathbb{Q}}) \oplus \mathcal{E}_k(\overline{\mathbb{Q}})$. For this, we rely on a celebrated result of Harris in [8] which holds under the assumption that $k > 2n-1$. We remark that the case of $r = 0$ (split case) and $r \neq 0$ (non-split case) are very different, and a similar phenomenon shows up also in the unitary group for the case of $U(n,n)$ and the case $U(n,m), n \neq m$ as, for example, in Theorem [32, Theorem 27.12]. The restriction of the range $2n-1 < \mu \leq k$ is due to the use of nearly holomorphic Eisenstein series (see Proposition 6.2 and Theorem 5.9) and the nonvanishing $L(\mu, \mathbf{f}, \chi) \neq 0$ used in the proof above.

We mention here that in the split case ($r = 0$), we can obtain algebraicity result in [15] for $n+1 \leq \mu \leq k$. The techniques there are very different to the ones used here and are modeled to the seminal paper of Böcherer and Schmidt [1], and rely on the use of some holomorphic differential operators (no use of nearly holomorphic Eisenstein series) and some further assumptions on nonvanishing eigenvalues. We simply mention here that these techniques seem to be particular to the split case (especially the holomorphic operators) and it is not known how they can be applied to the non-split case, which is the main case of interest of this paper.

(3) One may expect a refined result of above theorem. That is, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\left(\frac{L(\mu, \mathbf{f}, \chi)}{\pi^{n(k+\mu)-\frac{3}{2}n(n-1)} G(\chi)^n \langle \mathbf{f}, \mathbf{f}\rangle}\right)^{\sigma} = \frac{L(\mu, \mathbf{f}^{\sigma}, \chi^{\sigma})}{\pi^{n(k+\mu)-\frac{3}{2}n(n-1)} G(\chi^{\sigma})^n \langle \mathbf{f}^{\sigma}, \mathbf{f}^{\sigma}\rangle},$$

where $G(\chi)$ is certain Gauss sum. It is known that the Fourier coefficients of Siegel Eisenstein series have these nice Galois properties, but it is not clear whether its pullback still preserves the Galois properties. The same problem also occurs in [32] for unitary groups. When our group G is a split (i.e., $r = 0$), $\mathbf{f}^{\sigma}$ can be simply defined as the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Fourier coefficients of $\mathbf{f}$. In this setting, our Lemma 6.1 has a refined version as in [32, Lemma 24.11] for Case UT there (the split unitary group) and we actually obtained the above refined algebraicity result in [15]. In the general case treated in this paper, which of course includes the case of non-split groups ($r \neq 0$), we have to characterize the algebraic modular forms using CM-points and thus, when pulling back the Eisenstein series, we can only prove our Lemma 6.1 over $\overline{\mathbb{Q}}$ as in [32, Lemma 26.12] for Case UB, which includes the non-split unitary groups. And, of course, as it is clear from the proof of Theorem 6.3, the more refined results require the understanding of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\left(\frac{\mathbf{f}(g_{\mathbf{h}} \cdot g_{\infty})}{\mathfrak{P}_k(w)}\right) \in \overline{\mathbb{Q}}$ in the notation used in the proof of the theorem.

(4) Finally, we mention that as in [30, 32], we can also applying the pullback of Siegel Eisenstein series for two different groups $G_n \times G_{n'}$. This will give the Klingen Eisenstein series, and thus the algebraicity result for the Klingen Eisenstein series can also be obtained.

References

[1] S. Böcherer and C.-G. Schmid, *p-adic measures attached to Siegel modular forms*. Ann. Inst. Fourier (Grenoble) **50**(2000), no. 5, 1375–1443.

[2] A. Borel, *Automorphic forms on reductive groups*. In: Automorphic forms and applications, IAS/Park City Mathematics Series, 12, American Mathematical Society, Providence, RI, 2007, pp. 7–39.

[3] T. Bouganis, *On the standard L-function attached to quaternionic modular forms*. J. Number Theory **222**(2021), 293–345.

[4] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*. Proc. Symp. Pure Math. **33**(1979), 313–346.

[5] P. Deligne, *Travaux de Shimura*. Seminaire Bourbaki, 1970/1971, no. 389.

[6] P. Garrett, *Imbedded modular curves and arithmetic of automorphic forms on bounded symmetric domains*. Duke Math. J. **51**(1984), no. 2, 431–458.

[7] M. Harris, *Maass operators and Eisenstein series*. Math. Ann. **258**(1981), 135–144.

[8] M. Harris, *Eisenstein series on Shimura varieties*. Ann. of Math. (2) **119**(1984), no. 1, 59–94.

[9] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. I*. Invent. Math. **82**(1985), no. 1, 151–189.

[10] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*. Compos. Math. **60**(1986), no. 3, 323–378.

[11] M. Harris, *L-functions and periods of polarized regular motives*, J. Reine Angew. Math. **483**(1997), 75–161.

[12] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, 80, Academic Press, Inc. [Harcourt Brace Jovanovich Publishers], New York–London, 1978, xv + 628 pp.

[13] H. Hijikata, *Hasse's principle on quaternionic anti-hermitian forms*. J. Math. Soc. Japan **15**(1963), no. 2, 165–175.

[14] L.-K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, American Mathematical Society, Providence, RI, 1963, iv + 164 pp., translated from the Russian by Leo Ebner and Adam Korányi.

[15] Y. Jin, *On p-adic measures for quaternionic modular forms*. Preprint, 2022. arXiv:2209.11822

[16] A. Krieg, *Modular forms on half-spaces of quaternions*, Lecture Notes in Mathematics, 1143, Springer, Berlin, 1985, xiii + 203 pp.

[17] K.-W. Lan, *An example-based introduction to Shimura varieties*. In: Proceedings of the ETHZ Summer School on Motives and Complex Multiplication, to appear.

[18] R. P. Langlands, *Problems in the theory of automorphic forms*. In: Lectures in modern analysis and applications. III, Lecture Notes in Mathematics, 170, Springer, Berlin, 1970, pp. 18–61.

[19] E. Lapid and S. Rallis, *On the local factors of representations of classical groups*. In: Automorphic representations, L-functions and applications: progress and prospects, Ohio State University Mathematical Research Institute Publications, 11, de Gruyter, Berlin, 2005, pp. 309–359.

[20] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*. In: Automorphic forms, Shimura varieties, and L-functions (Ann Arbor, MI, 1988). Vol. I, Perspectives in Mathematical, 10, Academic Press, Boston, MA, 1990, pp. 283–414.

[21] A. Pitale, A. Saha, and R. Schmidt, *On the standard L-function for* $GSp_{2n} \times GL_1$ *and algebraicity of symmetric fourth L-values for* GL_2. Ann. Math. Qué. **45**(2021), 113–159.

[22] V. Platonov and A. Rapinchuk, Algebraic groups and number theory, Pure and Applied Mathematics Series, 139, Academic Press, Cambridge, MA, 1994.

[23] I. I. Pyateskii-Shapiro, Automorphic functions and the geometry of classical domains. *Vol. 8*, Gordon and Breach Science Publishers, New York–London–Paris, 1969, viii + 264 pp.

[24] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*. Ann. of Math. (2) **78**(1963), no. 2, 149–192.

[25] G. Shimura, *Algebraic number fields and symplectic discontinuous groups*. Ann. of Math. (2) **86**(1967), 503–592.

[26] G. Shimura, *Automorphic forms and the periods of abelian varieties*. J. Math. Soc. Japan **31**(1979), no. 3, 561–592.

[27] G. Shimura, *On differential operators attached to certain representations of classical groups*. Invent. Math. 77(1984), no. 3, 463–488.

[28] G. Shimura, *Nearly holomorphic functions on Hermitian symmetric spaces*. Math. Ann. **278**(1987), nos. 1–4, 1–28.

[29] G. Shimura, *Differential operators, holomorphic projection, and singular forms*. Duke Math. J. **76**(1994), no. 1, 141–173.
[30] G. Shimura, *Eisenstein series and zeta functions on symplectic groups*. Invent. Math. **119**(1995), no. 3, 539–584.
[31] G. Shimura, *Euler products and Eisenstein series*. CBMS Regional Conference Series in Mathematics, 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997, xx + 259 pp.
[32] G. Shimura, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, 82, American Mathematical Society, Providence, RI, 2000, x + 302 pp.
[33] G. Shimura, Arithmetic and analytic theories of quadratic forms and Clifford groups, Mathematical Surveys and Monographs, 109, American Mathematical Society, Providence, RI, 2004, x + 275 pp.
[34] T. Tsukamoto, *On the local theory of quaternionic anti-Hermitian forms*. J. Math. Soc. Japan **13**(1961), no. 4, 387–400.
[35] Ç. Ürtiş, *Special values of L-functions by a Siegel–Weil–Kudla–Rallis formula*. J. Number Theory **125**(2007), 149–181.
[36] J. Voight, Quaternion algebras, Graduate Texts in Mathematics, 288, Springer, Cham, 2021, xxiii + 885 pp.
[37] S. Yamana, *On the lifting of elliptic cusp forms to cusp forms of quaternionic unitary groups*. J. Number Theory **130**(2010), 2480–2527.
[38] S. Yamana, *L-functions and theta correspondence for classical groups*. Invent. Math. **196**(2014), 651–732.

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