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LIMIT SETS OF UNFOLDING PATHS IN OUTER SPACE

MLADEN BESTVINA[1], RADHIKA GUPTA[2] AND JING TAO[3]

[1] *University of Utah, Department of Mathematics, Department of Mathematics, Salt Lake City, Utah, USA*
(bestvina@gmail.com)
[2] *Tata Institute of Fundamental Research, School of Mathematics, School of Mathematics, Mumbai, 400005*
(radhikagupta.maths@gmail.com)
[3] *University of Oklahoma, Department of Mathematics, Department of Mathematics, Norman, Oklahoma*
(jing@ou.edu)

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Abstract We construct an unfolding path in Outer space which does not converge in the boundary, and instead it accumulates on the entire 1-simplex of projectivized length measures on a nongeometric arational $\mathbb{R}$-tree T. We also show that T admits exactly two dual ergodic projective currents. This is the first nongeometric example of an arational tree that is neither uniquely ergodic nor uniquely ergometric.

1. Introduction

For the once-punctured torus, the Thurston compactification of the Teichmüller space by projective measured laminations coincides with the visual compactification of the hyperbolic plane. In this case, every geodesic ray has a unique limit point, and the dynamical behavior of the ray in moduli space is governed by the continued fraction of its limit point. For hyperbolic surfaces of higher complexity, Teichmüller space with the Teichmüller metric is no longer negatively curved [Mas75, MW95] (or even Riemannian), and the Thurston boundary is no longer its visual boundary [Ker80]. More surprisingly, geodesic rays do not always converge [Len08, LLR18].

For hyperbolic surfaces of higher complexity, another interesting phenomenon is the existence of nontrivial simplices in the Thurston boundary which correspond to measures on nonuniquely ergodic laminations. Particularly interesting is the case when

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the underlying lamination is minimal and filling, also called *arational*. Constructions of nonuniquely ergodic arational laminations have a long history and typically used flat structures on surfaces [Vee69, Sat75, KN76, Kea77]. A topological construction was introduced in [Gab09]. In [LLR18], Leininger, Lenzhen and Rafi combined this topological approach with some arithmetic parameters akin to continued fractions. This allowed them to show that it is possible for the full simplex of measures on a nonuniquely ergodic arational lamination to be realized as the limit set of a Teichmüller geodesic ray.

In this paper, we take the above construction into Culler–Vogtmann's Outer space [CV86]. A Thurston-type boundary for Outer space is given by the set of projective classes of minimal, very small $\mathbb{F}_n$-trees [CM87, BF94, CL95, Hor17] and the action of $\mathrm{Out}(\mathbb{F}_n)$ extends continuously to the compactified space. The analogue of arational laminations are *arational trees*; for example, trees dual to arational laminations on a once-punctured surface fall into this category. There are other examples, such as trees dual to minimal laminations on finite 2-complexes that are not surfaces, called *Levitt type*; and yet others, called *nongeometric*, that do not come from the latter two constructions. The nonuniquely ergodic phenomenon for laminations has two natural analogues for $\mathbb{F}_n$-trees: one in terms of length measures on trees, giving rise to *nonuniquely ergometric* trees [Gui00] and the other in terms of currents, giving *nonuniquely ergodic* trees; see [CHL07]. It is an open problem to determine whether these two notions coincide. An example of a nonuniquely ergometric arational tree of Levitt type, modeled on Keane's construction, was given in [Mar97]. In this paper, we construct the first nongeometric example of an arational tree that is neither uniquely ergodic nor uniquely ergometric.

In Outer space, the analogue of Teichmüller metric is the Lipschitz metric and that of Teichmüller geodesics are folding paths. However, unlike Teichmüller geodesics, a folding path in Outer space has a forward direction, reflecting the asymmetry of the Lipschitz metric. Even though the boundary of Outer space is not a visual boundary, a folding path always converges along its forward direction. Our main result is that this nice behavior does not persist in the backward direction; in fact, in the backward direction, folding paths can behave as badly as Teichmüller geodesics. Define an *unfolding path* in Outer space to be a folding path with the backward direction. Our main result, as follows, is a direct analogue of the results of [LLR18].

Theorem 1.1. *There exists an unfolding path in Outer space of free group of rank 7 which does not converge to a point in the boundary of Outer space. In fact, the limit set is a 1-simplex consisting of the full set of length measures on a nongeometric and arational tree T. Moreover, the set of projective currents dual to T is also a one-dimensional simplex. In particular, T is neither uniquely ergometric nor uniquely ergodic.*

We use the framework of folding and unfolding sequences. Every such sequence tracks the combinatorics of an appropriate folding path, resp. unfolding path, in Outer space. An infinite folding sequence has a naturally associated limiting tree in the boundary of Outer space and an unfolding sequence has a naturally associated algebraic lamination, called the *legal lamination*. The graphs in the folding sequence can be given compatible metrics, which are then used to parametrize the different length measures supported on the limiting tree. Compatible edge thicknesses on the graphs of the unfolding sequence

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parametrize the different currents with support contained in the legal lamination. The latter can then be used to study the currents dual to the trees in the limit set of the unfolding sequence. See [NPR14] or our Section 3 for definitions and more precise statements.

Modeling the construction of [LLR18] on a five-holed sphere, the folding and unfolding sequences we consider come from explicit sequences of automorphisms of the free group of rank 7. More explicitly, fix a nongeometric fully irreducible automorphism on three letters and extend it to an automorphism ϕ of $\mathbb{F}_7$ by identity on the other four basis elements. Also, let ρ be a finite-order automorphism of $\mathbb{F}_7$ that rotates the support of ϕ off itself. For an integer r, set $\phi_r = \rho\phi^r$. Given a sequence $(r_i)_{i\geq 1}$ of positive integers, define a sequence of automorphisms by

$$\Phi_i = \phi_{r_1} \circ \cdots \circ \phi_{r_i}.$$

From $(\Phi_i)_i$, we get an unfolding sequence using the train track map induced by ϕ_{r_i}, and from $(\Phi_i^{-1})_i$ we get a companion folding sequence. The parameters $(r_i)_i$ play the role of the continued fraction expansion for the limiting tree of the folding sequence, and adjusting them produces different types of trees and behaviors of the unfolding sequence. In particular, we show that if the sequence $(r_i)_i$ satisfies certain arithmetic conditions and grows sufficiently fast, then the limiting tree is arational, nongeometric, nonuniquely ergodic and nonuniquely ergometric. Moreover, the limit set of the unfolding sequence is the full simplex of length measures on the tree. We refer to Theorem 10 for the full technical statement.

To see how the parameters $(r_i)_i$ come into play, it is informative to look at the sequence of free factors $A_i = \Phi_i(A)$, where A is the support of ϕ. The A_i's are the projection of the folding sequence to the free factor complex $\mathcal{FF}_7$. By our construction, A_i and A_{i+1} are disjoint (meaning $\mathbb{F}_7 = A_i * A_{i+1} * B_i$ for some B_i), but A_i, A_{i+2} are not, and r_i measures the distance between the projections of A_{i-2} and A_{i+2} to the free factor complex of A_i. Morally, if r_i's are sufficiently large, then $(A_i)_i$ forms a quasi-geodesic in $\mathcal{FF}_7$. Hence, by [BR15, Ham16], the limiting tree of the folding sequence is arational. In addition, we show that the tree is nongeometric. To get two currents on the tree, we take loops in the A_i's, which correspond to currents on $\mathbb{F}_n$ and take projective limits of the odd and even subsequences. Nonunique ergometricity of the tree follows a similar principle.

Although our construction is general in spirit, the case of rank 7 is already fairly involved, and some computations used computer assistance. One issue is that there is no known algorithm to tell if a collection of free factors has a common complement. This issue appears in the proof of arationality of the limiting tree that led to the peculiar looking arithmetic conditions on the parameters; see Section 5.

Outline

- In Section 2, we review some background material, including train track maps, Outer space, currents, length measures and arational trees.
- In Section 3, we discuss folding and unfolding sequences. We relate length measures on a folding sequence with the length measures on the limiting tree when it is arational. We also define the legal lamination for an unfolding sequence and state a result from [NPR14] relating the currents supported on the legal lamination with those of the unfolding sequence.

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- In Section 4, we discuss our main construction to generate from a sequence $(r_i)_i$ of positive integers a sequence of automorphisms of $\mathbb{F}_7$. The associated transition matrices for these automorphisms have block shapes which we use to analyze their asymptotic behavior. From each sequence of automorphisms and their inverses, we get a folding and unfolding sequence of graphs of rank 7 induced by their train track maps.
- In Section 5, we show that under the right conditions on $(r_i)_i$, the folding sequence converges to a nongeometric and arational tree T in boundary of Outer space of rank 7. To show arationality, we project the folding sequence to the free factor complex and show it is a quasi-geodesic.
- In Section 6, we study the behavior of the unfolding sequence. The main result is that if the sequence $(r_i)_i$ grows sufficiently fast, then the legal lamination of the unfolding sequence supports a 1-simplex of projective currents.
- In Section 7, we show that if the sequence $(r_i)_i$ grows sufficiently fast, then the limiting tree of the folding sequence supports a 1-simplex of projective length measures. In particular, the limiting tree is not uniquely ergometric.
- In Section 8, we relate the legal lamination of the unfolding sequence to the dual lamination of the limiting tree of the folding sequence. This shows the limiting tree is not uniquely ergodic.
- In Section 9, we show that the unfolding sequence limits onto the full simplex of length measures on the limiting tree of the folding sequence, and thus does not have a unique limit in the boundary of Outer space.
- In Section 10, we collect the results to prove the main theorem.
- In Section A, we prove a technical lemma about convergence of products of matrices.

2. Background

Let $\mathbb{F}_n$ be the free group of rank n. We review some background on train track maps, Outer space, laminations, currents, arational trees and the free factor complex.

2.1. Train track maps

We recall some basic definitions from [BH92]. Identify $\mathbb{F}_n$ with $\pi_1(\mathrm{R}_n,*)$, where R_n is a rose with n petals. A *marked graph* G is a graph of rank n, all of whose vertices have valence at least three, equipped with a homotopy equivalence $m\colon \mathrm{R}_n \to G$ called a *marking*.

A *length vector* on G is a vector $\lambda \in \mathbb{R}^{|EG|}$ that assigns a positive number, that is, a length, to every edge of G. The volume of G with respect to λ is the total length of all the edges of G. This induces a path metric on G where the length of an edge e is $\lambda(e)$.

A direction d based at a vertex $v \in G$ is an oriented edge of G with initial vertex v. A *turn* is an unordered pair of distinct directions based at the same vertex. A *train track structure* on G is an equivalence relation on the set of directions at each vertex $v \in G$. The classes of this relation are called *gates*. A turn (d,d') is *legal* if d and d' do not belong to the same gate, it is called *illegal* otherwise. A path is legal if it only crosses legal turns.

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A map $f\colon G \to G'$ between two graphs is called a *morphism* if it is locally injective on open edges and sends vertices to vertices. If G and G' are metric graphs, then we can homotope f relative to vertices such that it is linear on edges. Similarly, for an $\mathbb{R}$-tree T, a map $\tilde{G} \to T$ from the universal cover of G is a morphism if it is injective on open edges. To a morphism $f\colon G \to G'$ we associate the *transition matrix* as follows: Enumerate the (unoriented) edges $e_1, e_2, \cdots, e_m$ of G and $e'_1, e'_2, \cdots, e'_n$ of G'. Then the transition matrix M has size $n \times m$ and the ij-entry is the number of times $f(e_j)$ crosses e'_i, that is, it is the cardinality of the set $f^{-1}(x) \cap e_j$ for a point x in the interior of e'_i. If f is in addition a homotopy equivalence, then f is a *change-of-marking*.

A homotopy equivalence $f\colon G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element ϕ of $\mathrm{Out}(\mathbb{F}_n)$. If f is a morphism, then we say that f is a *topological representative* of ϕ. A topological representative $f\colon G \to G$ induces a *train track structure* on G as follows: The map f determines a map Df on the directions in G by defining $Df(e)$ to be the first (oriented) edge in the edge path $f(e)$. We then declare $e_1 \sim e_2$ if $(Df)^k(e_1) = (Df)^k(e_2)$ for some $k \geq 1$.

A topological representative $f\colon G \to G$ is called a *train track map* if every vertex has at least two gates, and f maps legal turns to legal turns and legal paths (equivalently, edges) to legal paths. Equivalently, every positive power f^k is a topological representative. If f is a train track map with transition matrix M, then the transition matrix of f^k is M^k for every $k \geq 1$. If M is *primitive*, that is, M^k has positive entries for some $k \geq 1$, then Perron–Frobenius theory implies that there is an assignment of positive lengths to all the edges of G so that f uniformly expands lengths of legal paths by some factor $\lambda > 1$, called the *stretch factor* of f.

If σ is a path (or a circuit) in G, we denote by $[\sigma]$ the reduced path homotopic to σ (rel endpoints if σ is a path). A path or circuit σ in G is called a *periodic Nielsen path* if $[f^k(\sigma)] = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path*. A Nielsen path that cannot be written as a concatenation of nontrivial Nielsen paths is called an *indivisible* Nielsen path, denoted INP.

The following lemma is an important property of train track maps. For a very rudimentary form, see [BH92, Lemma 3.4] showing that INPs have exactly one illegal turn, and for a more involved version see [BFH97] (some details can also be found in [KL14, Proposition 3.27, 3.28]). We will need it for the proof of Lemma 4.8 and include a proof here.

Lemma 2.1. *Let $h\colon G \to G$ be a train track map with a primitive transition matrix. There exists a constant $R > 0$ such that for any edge path γ, either*

1. *the number of illegal turns in $[h^R(\gamma)]$ is less than that of γ, or*
2. *$\gamma = u_1 v_1 u_2 v_2 \dots u_n$, where each u_i is a legal subpath, possibly degenerate, and each $[h^R(v_i)]$ is a periodic INP.*

Proof. Let $\lambda > 1$ be the stretch factor of h, and equip G with the metric so that h uniformly expands the length of every legal path by λ. It goes back to the work of Thurston (see [Coo87]) that there is a constant $BCC(h)$, called the bounded cancellation constant for h, such that if $\alpha\beta$ is a reduced edge path, then $[h(\alpha)][h(\beta)]$ have cancellation

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bounded by $BCC(h)$. The existence of this constant is really a consequence of the Morse lemma and the fact that h is a quasi-isometry. Define $C = BCC(h)/(\lambda - 1)$.

Here is the significance of C. To fix ideas, let us assume that γ has only one illegal turn, so $\gamma = \alpha\beta$ with both α,β legal. Say α has length $|\alpha| = C + \epsilon > C$. Then $h(\alpha)$ has length $\lambda|\alpha|$ and after cancellation with $h(\beta)$ the length is $\geq \lambda|\alpha| - BCC(h) = |\alpha| + \lambda\epsilon$. Thus, assuming $[h^i(\gamma)]$ still has an illegal turn, the length of the initial subpath to the illegal turn has length growing exponentially in i, assuming it is long enough.

We now prove the lemma for paths $\gamma = \alpha\beta$ with one illegal turn and with α,β legal. Consider the finite collection of paths consisting of those with length at most C with both endpoints at vertices or with length exactly C with only one endpoint at a vertex. Let R be a number larger than the square of the size of this collection. If $[h^i(\gamma)] = \alpha_i\beta_i$ has one illegal turn (with α_i,β_i legal) for $i = 1,2,\cdots,R$, then by the pigeon-hole principle there will be $i < j$ in this range so that the C-neighborhoods of the illegal turns of $[h^i(\gamma)]$ and $[h^j(\gamma)]$ are the same (if α_i or β_i has length $< C$ this means $\alpha_i = \alpha_j$ or $\beta_i = \beta_j$). We can lift h^{j-i} and γ to the universal cover of the graph and arrange that (the lift of) γ and $[h^{j-i}(\gamma)]$ have the same illegal turn. Thus, h^{j-i} maps the terminal C-segment of α_i (or α_i itself) over itself (by the above calculation) and therefore fixes a point in α_i and similarly for β_i. The subpath of $[h^i(\gamma)]$ between these fixed points is a periodic INP, proving the lemma in the case γ has one illegal turn.

The general case is similar. Write $\gamma = \gamma_1\gamma_2\cdots\gamma_s$ with all γ_k legal and with the turn between γ_k and γ_{k+1} illegal. Also, assume that $[h^i(\gamma)]$ has the same number of illegal turns for $i = 1,\cdots,R$. We can write $[h^i(\gamma)] = \gamma_1^i\gamma_2^i\cdots\gamma_s^i$ with all γ_k^i legal and the turns between them illegal. For each illegal turn corresponding to the pair $(k,k+1)$, there will be $i < j$ in this range so that the C-neighborhoods of the illegal turn in $[h^i(\gamma)]$ and in $[h^j(\gamma)]$ are the same. This gives fixed points of h^{j-i} in γ_k^i and γ_{k+1}^i, and these fixed points split γ into periodic INPs and legal segments, as claimed. □

We will use the lemma in the situation that h has no periodic INPs, in which case the conclusion is that whenever γ is not legal, then $[h^R(\gamma)]$ has fewer illegal turns than γ.

2.2. Outer space and its boundary

An $\mathbb{F}_n$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}_n$. An $\mathbb{F}_n$-tree T has dense orbits if some (every) orbit is dense in T. An $\mathbb{F}_n$-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic and tripod stabilizers are trivial. We review the definition of Outer space first introduced in [CV86].

Unprojectivized Outer space, denoted by cv_n, is the set of free, minimal and simplicial $\mathbb{F}_n$-trees. By considering the quotient graphs, cv_n is also equivalently the set of marked metric graphs, that is, the set of triples (G,m,λ), where G is a graph of rank n with all valences at least 3, $m\colon \mathrm{R}_n \to G$ is a marking and λ is a positive length vector on G. By [CM87], the map of $\mathrm{cv}_n \to \mathbb{R}^{\mathbb{F}_n}$ given by $T \mapsto (\|g\|_T)_{g\in\mathbb{F}_n}$, where $\|g\|_T$ is the translation length of g in T, is an inclusion. This endows cv_n with a topology. The closure $\overline{\mathrm{cv}}_n$ in $\mathbb{R}^{\mathbb{F}_n}$ is the space of very small $\mathbb{F}_n$-trees [BF94, CL95]. The boundary $\partial\mathrm{cv}_n = \overline{\mathrm{cv}}_n - \mathrm{cv}_n$ consists of very small trees that are either not free or not simplicial.

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Culler Vogtmann's *Outer space*, CV_n, is the image of cv_n in the projective space $\mathbb{PR}^{\mathbb{F}_n}$. Elements in CV_n can also be described as free, minimal, simplicial $\mathbb{F}_n$-trees with unit covolume. Topologically, CV_n is a complex made up of simplices with missing faces, where there is an open simplex for each marked graph (G,m) spanned by positive length vectors on G of unit volume. The closure $\overline{\mathrm{CV}}_n$ of CV_n in $\mathbb{PR}^{\mathbb{F}_n}$ is compact and the boundary $\partial \mathrm{CV}_n = \overline{\mathrm{CV}}_n - \mathrm{CV}_n$ is the projectivization of $\partial \overline{\mathrm{cv}}_n$.

The spaces cv_n and CV_n and their closures are equipped with a natural (right) action by $\mathrm{Out}(\mathbb{F}_n)$. That is, for $\Phi \in \mathrm{Out}(\mathbb{F}_n)$ and $T \in \overline{\mathrm{cv}}_n$ the translation length function of $T\Phi$ on $\mathbb{F}_n$ is $\|g\|_{T\Phi} = \|\phi(g)\|_T$, where ϕ is any lift of Φ to $\mathrm{Aut}(\mathbb{F}_n)$.

2.3. Laminations, currents and nonuniquely ergodic trees

In [BFH00], Bestvina, Feighn and Handel defined a dynamical invariant called the attracting lamination associated to a train track map. In this article, we will consider the more modern definition of a lamination as given in [CHL08a].

Let $\partial \mathbb{F}_n$ denote the Gromov boundary of $\mathbb{F}_n$, and let Δ be the diagonal in $\partial \mathbb{F}_n \times \partial \mathbb{F}_n$. The *double boundary* of $\mathbb{F}_n$ is $\partial^2 \mathbb{F}_n = (\partial \mathbb{F}_n \times \partial \mathbb{F}_n - \Delta)/\mathbb{Z}_2$, which parametrizes the space of unoriented bi-infinite geodesics in a Cayley graph of $\mathbb{F}_n$. By an *(algebraic) lamination*, we mean a nonempty, closed and $\mathbb{F}_n$-invariant subset of $\partial^2 \mathbb{F}_n$.

Associated to $T \in \overline{\mathrm{cv}}_n$ is a *dual lamination* $L(T)$, defined as follows in [CHL08b]. For $\epsilon > 0$, let

$$L_\epsilon(T) = \overline{\{(g^{-\infty}, g^{\infty}) \quad | \quad \|g\|_T < \epsilon, g \in \mathbb{F}_n\}},$$

so $L_\epsilon(T)$ is a lamination and set $L(T) = \bigcap_{\epsilon>0} L_\epsilon(T)$. Elements of $L(T)$ are called *leaves*. For trees in cv_n, $L(T)$ is empty.

A *current* is an additive, nonnegative, $\mathbb{F}_n$-invariant function on the set of compact open sets in $\partial^2 \mathbb{F}_n$. Equivalently, it is an $\mathbb{F}_n$-invariant Radon measure on the σ-algebra of Borel sets of $\partial^2 \mathbb{F}_n$. Let Curr_n denote the space of currents, equipped with the weak* topology. The quotient space of $\mathbb{P}\mathrm{Curr}_n$ of projectivized currents (i.e., homothety classes of nonzero currents) is compact.

For $\mu \in \mathrm{Curr}_n$, let $\mathrm{supp}(\mu) \subset \partial^2 \mathbb{F}_n$ denote the support of μ, which is in fact a lamination. For $T \in \overline{\mathrm{cv}}_n$ and $\mu \in \mathrm{Curr}_n$, if $\mathrm{supp}(\mu) \subseteq L(T)$, then we say μ is *dual* to T. Denote by $\mathrm{Curr}(T)$ the convex cone of currents dual to T and by $\mathbb{P}\mathrm{Curr}(T)$ the set of projective currents dual to T. $\mathbb{P}\mathrm{Curr}(T)$ is a compact, convex space and its extremal points are called the *ergodic currents* dual to T. We say T is *uniquely ergodic* if there is only one projective class of currents dual to T, and *nonuniquely ergodic* otherwise. In [CH16], the authors show that if $T \in \partial \mathrm{cv}_n$ has dense orbits, then $\mathbb{P}\mathrm{Curr}(T)$ is the convex hull of at most $3n-5$ projective classes of ergodic currents dual to T.

In [KL09], Kapovich and Lustig established a length pairing, $\langle \cdot, \cdot \rangle$, between $\overline{\mathrm{cv}}_n$ and the space of measured currents Curr_n. They also showed in [KL10, Theorem 1.1] that for $T \in \overline{\mathrm{cv}}_n$ and $\mu \in \mathrm{Curr}_n$, $\langle T, \mu \rangle = 0$ if and only if μ is dual to T.

Given two trees T and T', we say a map $h\colon T \to T'$ is *alignment-preserving* if whenever $b \in T$ is contained in an arc $[a,c] \subset T$, then $h(b)$ is contained in the arc $[h(a), h(c)]$.

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Theorem 2.2 [CHL07]. *Let $T,T' \in \partial CV_n$ be two trees with dense orbits. The following are equivalent:*

- $L(T) = L(T')$.
- *There exists an $\mathbb{F}_n$-equivariant alignment-preserving bijection between T and T'.*

2.4. Length measures and nonuniquely ergometric trees

Since $\mathbb{R}$-trees need not be locally compact, classical measure theory is not well suited for them. In [Pau95], a *length measure* was introduced for $\mathbb{R}$-trees. See [Gui00] for details.

A *length measure* on an $\mathbb{F}_n$-tree T is a collection of finite Borel measures λ_I for every compact interval I in T such that if $J \subset I$, then $\lambda_J = (\lambda_I)|_J$. We require the length measure to be invariant under the $\mathbb{F}_n$ action. The collection of the Lebesgue measures of the intervals of T is $\mathbb{F}_n$-invariant, and this will be called the Lebesgue measure of T. A length measure λ is *nonatomic* or *positive* if every λ_I is nonatomic or positive. If every orbit is dense in some segment of T, then T cannot have an invariant measure with atoms. Further, if T is *indecomposable*, that is, if for any pair of nondegenerate arcs I and J in T, there exist $g_1,\dots,g_m \in \mathbb{F}_n$ such that $I \subset \bigcup g_i J$ and $g_i J \cap g_{i+1} J$ is nondegenerate, then every nonzero length measure is positive (in fact, the condition of mixing [Gui00] suffices).

Let $\mathcal{D}(T)$ be the cone of $\mathbb{F}_n$-invariant length measures on T, with projectivization $\mathbb{P}\mathcal{D}(T)$, that is, the homothety classes of $\mathbb{F}_n$-invariant length measures on T. $\mathbb{P}\mathcal{D}(T)$ is a compact convex set and we will call its extremal points the *ergodic length measures* on T. When T has dense orbits there are at most $3n-4$ such measures for any T (see [Gui00, Corollary 5.2, Lemma 5.3]) and $\mathcal{D}(T)$ is naturally a subset of ∂cv_n. In fact,

Lemma 2.3. *[Gui00] If $T \in cv_n$ is indecomposable, then $\mathcal{D}(T)$ is in one-to-one correspondence with the set of isometry classes of $\mathbb{F}_n$-invariant metrics on T, denoted $X_T \subset cv_n$.*

Proof. Let $\lambda \in \mathcal{D}(T)$ be a length measure on T. Consider the pseudo-metric d_λ on T, where $d_\lambda(x,y) = \lambda([x,y])$ for $x,y \in T$. In fact, since T is indecomposable, d_λ is a metric on T. For the converse, let $T' \in X_T$. Then the pull back of Lebesgue measure on T' under identity map $\mathrm{id}\colon T \to T'$ gives a positive length measure on T. □

We say T is *uniquely ergometric* if there is only one projective class of length measures on T, which necessarily is the homothety class of the Lebesgue measure on T. It is called *nonuniquely ergometric* otherwise.

2.5. Arational trees and the free factor complex

For a tree $T \in \overline{\mathrm{cv}}_n$ and a free factor H of $\mathbb{F}_n$, let T_H denote the minimal H-invariant subtree of T (this tree is unique unless H fixes an arc). A tree $T \in \partial \mathrm{cv}_n$ is *arational* if every proper free factor H of $\mathbb{F}_n$ has a free and simplicial action on T_H. By [Rey12], every arational tree is free and indecomposable or it is the dual tree to an arational measured lamination on a surface with one puncture. The arational trees of the first kind are either Levitt type or nongeometric.

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Let $\mathcal{AT} \subset \partial \mathrm{CV}_n$ denote the set of arational trees with the subspace topology. Using Lemma 2.3, define an equivalence relation $\sim$ on $\mathcal{AT}$ by 'forgetting the metric', that is, $T \sim T'$ if $T' \in \mathbb{PD}(T)$, and endow $\mathcal{AT}/\sim$ with the quotient topology. The following lemma is implicit in [Gui00] and we include a proof for completeness.

Lemma 2.4. *Let T, T' be arational trees. Then $T \sim T'$ if and only if $L(T) = L(T')$.*

Proof. If $T \sim T'$, then the identity map id: $T \to T'$ is an alignment-preserving bijection. Therefore, by Theorem 2.2, $L(T) = L(T')$.

If $L(T) = L(T')$, then by Theorem 2.2 there is an alignment preserving bijection $f\colon T \to T'$. Pulling back the Lebesgue measure on T' induces a length measure on T, and the corresponding metric d_μ on T is isometric to T', so $T' \sim T$. □

The *free factor complex* $\mathcal{FF}_n$ is a simplicial complex whose vertices are given by conjugacy classes of proper free factors of $\mathbb{F}_n$ and a k-simplex is given by a nested chain $[A_0] \subset [A_1] \subset \cdots \subset [A_k]$. When the rank $n = 2$ the definition is modified and an edge connects two conjugacy classes of rank 1 factors if they have complementary representatives. The free factor complex can be given a metric as follows: Identify each simplex with a standard simplex and endow the resulting space with path metric. By result of [BF14a], the metric space $\mathcal{FF}_n$ is Gromov hyperbolic. The Gromov boundary of $\mathcal{FF}_n$ was identified with $\mathcal{AT}/\sim$ in [BR15] and [Ham16].

There is a projection map $\pi\colon \mathrm{CV}_n \to \mathcal{FF}_n$ defined as follows [BF14a, Section 3]: for $G \in \mathrm{CV}_n$, $\pi(G)$ is the collection of free factors given by the fundamental group of proper subgraphs of G which are not forests. This map is coarsely well defined, that is, $\mathrm{diam}_{\mathcal{FF}_n}(\pi(G)) \leq K$ for some universal K. Note that if G, G' belong to the same open simplex of CV_n, then $\pi(G) = \pi(G')$, so the projection of a simplex of CV_n has uniformly bounded diameter.

3. Folding and unfolding sequences

In this section we introduce (un)folding sequences and review some work of Namazi-Pettet–Reynolds [NPR14].

A *folding/unfolding sequence* is a sequence

$$G_a \longrightarrow \cdots \longrightarrow G_{-1} \longrightarrow G_0 \longrightarrow G_1 \longrightarrow \cdots \longrightarrow G_b$$

of graphs, together with maps $f_i\colon G_i \to G_{i+1}$ such that for any $j \leq i$, $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_j\colon G_j \to G_i$ is a change-of-marking morphism. Equivalently, a sequence as above is called a folding/unfolding sequence, if there exists a train track structure on each G_i and $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_j$ maps legal paths to legal paths. We allow the sequence to be infinite in one or both directions. We assume that a marking on G_0 has been specified, so a folding/unfolding sequence determines a sequence of open simplices in Outer space.

Let Q_i be the transition matrix of f_i. A *length measure* for a folding/unfolding sequence $(G_i)_{a \leq i \leq b}$ is a sequence $(\lambda_i)_{a \leq i \leq b}$, where $\lambda_i \in \mathbb{R}^{|EG_i|}$ is a length vector on G_i, and for $a \leq i < b$,

$$\lambda_i = Q_i^T \lambda_{i+1}.$$

In this way, f_i restricts to a local isometry on every edge of G_i. When $b < \infty$, a length vector on G_b determines a length measure on the sequence. When the sequence is infinite in the forward direction we denote by $\mathcal{D}((G_i)_i)$ the space of length measures on $(G_i)_i$, and $\mathbb{P}\mathcal{D}((G_i)_i)$ its projectivization. Observe that the dimension of $\mathcal{D}((G_i)_i)$ is bounded by $\liminf_{i\to\infty} |EG_i|$.

A *current* for a folding/unfolding sequence $(G_i)_{a\le i\le b}$ is a sequence $(\mu_i)_{a\le i\le b}$, where $\mu_i \in \mathbb{R}^{|EG_i|}$ is a length vector on G_i (but thought of as a vector of thicknesses of edges), and for $a \le i < b$, we require

$$\mu_{i+1} = Q_i \mu_i.$$

Likewise, when the sequence is infinite in the backward direction, we denote by $\mathrm{Curr}((G_i)_i)$ the space of currents on $(G_i)_i$, and $\mathbb{P}\mathrm{Curr}((G_i)_i)$ its projectivization. The dimension of $\mathrm{Curr}((G_i)_i)$ is bounded by $\liminf_{i\to-\infty} |EG_i|$.

3.1. Isomorphism between length measures

In this section, we identify the space of length measures on a folding sequence with that of the limiting tree when it is an arational tree.

Consider a folding sequence of marked graphs of rank n

$$G_0 \xrightarrow{f_1} G_1 \longrightarrow \cdots\cdots \xrightarrow{f_i} G_i \xrightarrow{f_{i+1}} \cdots\cdots.$$

Let $\tilde{G}_i$ be the universal cover of G_i, and let $\tilde{f}_i$ be a lift of f_i. For any positive length measure $(\lambda_i)_i \in \mathcal{D}((G_i)_i)$, we can realize $(\tilde{G}_i, \tilde{\lambda}_i)_i$ as a sequence in cv_n, which can be 'filled in' by a folding path in cv_n (see [BF14a] for details on folding paths). In particular, $(\tilde{G}_i, \tilde{\lambda}_i)_i$ always converges to a point $T \in \partial\mathrm{cv}_n$. Furthermore, we have morphisms $h_i\colon \tilde{G}_i \to T$ such that $h_i = h_{i+1}\tilde{f}_{i+1}$. With respect to the length measure $\tilde{\lambda}_i$, $\tilde{f}_i$ and h_i restrict to isometries on edges [BR15, Lemma 7.6].

Let $(U_i)_i$ be the sequence of open simplices CV_n associated to the sequence $(G_i)_i$. Recall the projection map $\pi\colon \mathrm{CV}_n \to \mathcal{FF}_n$ is coarsely well defined on simplices of CV_n. We will say the folding sequence $(G_i)_i$ *converges to an arational tree* T if $\pi(U_i)$ converges to $[T] \in \partial\mathcal{FF}_n$.

Proposition 3.1. *Suppose a folding sequence $(G_i)_i$ converges to an arational tree T. Then there is a linear isomorphism between $\mathcal{D}((G_i)_i)$ and $\mathcal{D}(T)$.*

Proof. Fix a positive length measure $(\lambda_i)_i \in \mathcal{D}((G_i)_i)$ and let $T \in \partial\mathrm{cv}_n$ be the limiting tree of $(\tilde{G}_i, \tilde{\lambda}_i)$ with corresponding morphism $h_i\colon \tilde{G}_i \to T$. Recall from Section 2.5 that if T is arational, then we can identify $\mathcal{D}(T)$ with the subspace of $\mathbb{F}_n$-metrics on T in $\partial\mathrm{cv}_n$. We will let $\lambda \in \mathcal{D}(T)$ be a length measure, and T_λ its image in $\partial\mathrm{cv}_n$.

By [BR15, Proposition 8.5], if $\pi(U_i)$ converges to $[T''] \in \partial\mathcal{FF}_n$, then for any positive $(\lambda'_i)_i \in \mathcal{D}((G_i)_i)$, $(\tilde{G}_i, \tilde{\lambda}'_i)$ also converges to an arational tree $T' \in \partial\mathrm{cv}_n$, such that $[T''] = [T'] = [T]$; in other words, $T' = T_{\lambda'}$ for some $\lambda' \in D(T)$. This gives a linear map $\mathcal{D}((G_i)_i) \to \mathcal{D}(T)$.

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Conversely, for any positive length measure $\lambda' \in \mathcal{D}(T)$, we can use the morphism h_i to pull back λ' from T to a length measure λ'_i on $\tilde{G}_i$. The fact that $h_i = h_{i+1}\tilde{f}_{i+1}$ implies $(\lambda'_i)_i \in \mathcal{D}((G_i)_i)$. Moreover, the sequence $(\tilde{G}_i, \tilde{\lambda}'_i)_i$ converges to $T_{\lambda'} \in \mathrm{cv}_n$. This gives a linear map $\mathcal{D}(T) \to \mathcal{D}((G_i)_i)$ which is the inverse of $\mathcal{D}((G_i)_i) \to \mathcal{D}(T)$ defined above. This shows $\mathcal{D}((G_i)_i) \to \mathcal{D}(T)$ is an isomorphism. □

Remark 3.2. A more general statement of Proposition 3.1 which doesn't involve the assumption that T is arational can be found in [NPR14, Proposition 5.4], but we will not need such a general statement here.

3.2. Isomorphism between currents

In this section, we state an analogous result identifying the space of currents on an unfolding sequence with the space of currents of a legal lamination associated to a unfolding sequence. We record some definitions from [NPR14] first.

Consider an unfolding sequence of marked graphs of rank n

$$\cdots\cdots \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} \cdots\cdots \xrightarrow{f_2} G_1 \xrightarrow{f_1} G_0.$$

Denote the composition $F_i = f_1 \circ \cdots \circ f_i$. Let $\Omega^L_\infty(G_i)$ denote the set of bi-infinite legal paths in G_i. Define the *legal lamination* of the unfolding sequence $(G_i)_i$ to be

$$\Lambda = \bigcap_i F_i(\Omega^L_\infty(G_i)) \subseteq \Omega^L_\infty(G_0).$$

Use the marking on G_0 to identify $\partial^2\pi_1(G_0)$ with $\partial^2\mathbb{F}_n$. The preimage, in $\partial^2\mathbb{F}_n$, of the lift of Λ to $\partial^2\pi_1(G_0)$ is a lamination $\tilde{\Lambda}$. We denote by $\mathrm{Curr}(\Lambda)$ the convex cone of currents supported on $\tilde{\Lambda}$, with projectivization $\mathbb{P}\mathrm{Curr}(\Lambda)$.

An *invariant sequence of subgraphs* is a sequence of nondegenerate (i.e., not forests) proper subgraphs $H_i \subset G_i$ such that f_i restricts to a morphism $H_i \to H_{i-1}$. We will need the following theorem from [NPR14], which we will include a sketch of the proof for completeness.

Theorem 3.3 (Theorem 4.4 [NPR14]). *Given an unfolding sequence $(G_i)_{i\geq 0}$ without an invariant sequence of subgraphs and with legal lamination Λ, then there is a natural linear isomorphism between Curr$((G_i)_i)$ and Curr(Λ).*

Sketch of proof. The lamination Λ consists of biinfinite lines in G_0 that lift to every G_i. All such lines are legal, and we view Λ as a subset of $(\partial\mathbb{F})^2$ invariant under the involution that flips the factors. An element in $\mathrm{Curr}((G_i)_i)$ is a compatible sequence $(\mu_i)_i$, where μ_i assigns a nonnegative weight to each edge of G_i. The compatibility condition is that the transition matrix of $G_{i+1} \to G_i$ takes the vector μ_{i+1} to the vector μ_i. An alternative way to describe compatibility is this. Let $\tilde{G}_i$ be the universal cover of G_i, and let $F_{i+1} : \tilde{G}_{i+1} \to \tilde{G}_i$ be a lift of the folding map. The weights μ_{i+1}, μ_i lift to the edges of

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$\tilde{G}_{i+1}, \tilde{G}_i$. If e is an edge of $\tilde{G}_i$, then $F_{i+1}^{-1}(e)$ is a finite collection of partial edges in $\tilde{G}_{i+1}$, and we complete them to edges, say $e_1, e_2, \cdots, e_k$. The compatibility condition is

$$\mu_i(e) = \mu_{i+1}(e_1) + \mu_{i+1}(e_2) + \cdots + \mu_{i+1}(e_k)$$

Since F_{i+1} is injective on the leaves of Λ, no such leaf passes through more than one of the e_j's. Let $\mathrm{Cyl}_\Lambda(e)$ be the set of leaves of Λ that pass through e and similarly for $\mathrm{Cyl}_\Lambda(e_j)$. Thus, we have

$$\mathrm{Cyl}_\Lambda(e) = \bigsqcup_j \mathrm{Cyl}_\Lambda(e_j)$$

Define measure μ on the cylinder sets corresponding to edges:

$$\mu(\mathrm{Cyl}_\Lambda(e)) = \mu_i(e).$$

The compatibility condition states that this measure is additive. The assumption that the sequence has no invariant subgraphs implies that cylinder sets corresponding to edges form a basis for the topology on Λ. This allows us to extend μ to general cylinder sets $\mathrm{Cyl}_\Lambda(\gamma)$, where γ is a finite segment of Λ in $\tilde{G}_i$. The key is that folding cannot identify vertices in the same orbit. Thus, there is a uniform upper bound on the number of vertices that map to the same vertex for any $\tilde{G}_j \to \tilde{G}_i$. When γ is a segment, the preimages of γ in $\tilde{G}_j$, for j sufficiently large, will be contained in either single edges or concatenations of two edges (see Lemma 8.2). By the above remark, the number of the length-2 paths is bounded by the combinatorial length of γ times the number of vertices in G_j. While we don't have enough information from μ_j alone to assign measure to these cylinder sets, we know their contribution goes to 0 as $j \to \infty$. So for each j, we take the sum of the measures of cylinder sets of the single edges in the preimage of γ. This is an increasing and bounded sequence as $j \to \infty$, so we define $\mu(\mathrm{Cyl}_\Lambda(\gamma))$ to be the limit, and this is the only possible definition. It is now an exercise to check that μ induces a premeasure on the semiring of cylinder sets $\mathrm{Cyl}_\Lambda(\gamma)$. Carathéodory's theorem then implies that μ extends to a unique (Radon) measure on Λ, which finishes the proof.

4. Main setup

In this section, we will construct an unfolding sequence $(\tau_i)_i$ and a folding sequence $(\tau_i')_i$ in CV_7 that intersect the same infinite set of simplices, which we will eventually use to show the existence of a nonuniquely ergodic and ergometric tree. The construction is done via a family of outer automorphisms. We will describe these automorphisms and then analyze the asymptotic behavior of their train track maps.

4.1. The automorphisms

Let $\mathbb{F}_7 = \langle a,b,c,d,e,f,g \rangle$. Denote by $\bar{x}$ the inverse of $x \in \mathbb{F}_7$. First, consider the map induced on the three-petaled rose by the automorphism

$$\theta\colon\ a \mapsto b, b \mapsto c, c \mapsto ca \in \mathrm{Aut}(\mathbb{F}_3)$$

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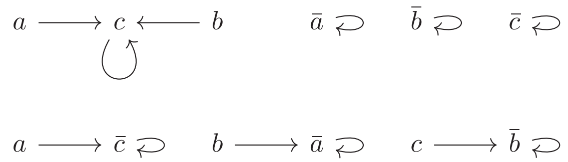
and the map induced by the inverse automorphism

$$\vartheta\colon\ a \mapsto \bar{b}c, b \mapsto a, c \mapsto b.$$

Using θ and ϑ to also denote the corresponding graph maps and using the convention that a also denotes the initial direction of the oriented edge a, while $\bar{a}$ denotes the terminal direction, we have the maps $D\theta^3$ and $D\vartheta^3$ given, respectively, as:



Observation 4.1. From the structure of the above maps, for $n \equiv 0 \bmod 3$, $D\theta^n = D\theta^3$ and $D\vartheta^n = D\vartheta^3$.

Lemma 4.2. *The map on the three-petaled rose labeled a,b,c induced by ϑ is a train track map with respect to the train track structure with gates $\{a,\bar{c}\},\{b,\bar{a}\},\{c,\bar{b}\}$. Moreover, this train track map does not have any periodic INPs.*

The map on the three-petaled rose labeled a,b,c induced by θ is also a train track map with respect to the train track structure with gates $\{a,b,c\},\{\bar{a}\},\{\bar{b}\},\{\bar{c}\}$ and it has one periodic Nielsen path (see [BF94, Example 3.4]).

Proof. The train track structure on the rose induces a metric on the graph coming from Perron–Frobenius theory. Every INP has length at most twice the volume of the graph, one illegal turn and the endpoints are fixed. Since there are only finitely many fixed points in G, it is easy to enumerate all such paths and check if they are Nielsen. For periodic INPs one knows that the period is bounded by a function of the rank of $\mathbb{F}_n$ [FH18], so one can take a suitable power and check for INPs (though there are more efficient ways, see [Kap19]). Coulbois' train track package [Cou] for the mathematics software system Sage [Sag] computes periodic INPs of train track maps. □

Now, let $\phi \in \mathrm{Aut}(\mathbb{F}_7)$ be the automorphism:

$$a \mapsto b, b \mapsto c, c \mapsto ca, d \mapsto d, e \mapsto e, f \mapsto f, g \mapsto g$$

and $\rho \in \mathrm{Aut}(\mathbb{F}_7)$ be the rotation by four clicks:

$$a \mapsto e, b \mapsto f, c \mapsto g, d \mapsto a, e \mapsto b, f \mapsto c, g \mapsto d.$$

Thus, ϕ is the extension of θ by identity, and ρ rotates the support of ϕ off itself.

Lemma 4.3. *For any $r \geq 3$, the map on the seven-petaled rose induced by $\phi_r = \rho\phi^r$ is a train track map with respect to the train track structure with gates*

$$\{a,b,c\},\{d,e,f\}$$

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and eight more gates consisting of single half edges. The transition matrix M_r has block form

$$\begin{pmatrix} 0 & I \\ B^r & 0 \end{pmatrix},$$

where I is the 4×4 identity matrix, and B is the transition matrix of θ:

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proof. By Observation 4.1, we only have to check the lemma for ϕ_3, ϕ_4, ϕ_5, which can be done by hand or using the train track package for Sage. □

Lemma 4.4. *For any $r \geq 3$ and $r \equiv 0$ mod 3, the map on the seven-petaled rose induced by $\psi_r = (\rho\phi^r)^{-1}$ is a train track map with respect to the train track structure with gates*

$$\{a,e,\bar{g}\},\{b,\bar{d}\},\{c,\bar{b}\},\{d,\bar{c}\},\{f,\bar{e}\},\{g,\bar{f}\},\{\bar{a}\}$$

The transition matrix N_r has block form

$$\begin{pmatrix} 0 & C^r \\ I & 0 \end{pmatrix},$$

where I is the 4×4 identity matrix, and C is the transition matrix of ϑ:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. By Observation 4.1, we only have to check the lemma for ψ_3, which can be done by hand or using the train track package for Sage. □

4.2. Asymptotics of transition matrices

Let θ, ϑ, ϕ_r, ψ_r be the maps defined in the last section. We now analyze the behavior of the transition matrices M_r and N_r for ϕ_r and ψ_r, respectively.

Lemma 4.5. *Let B be the transition matrix for θ, with Perron–Frobenius eigenvalue λ_B. There exists a constant $\kappa_B > 0$ such that if $r, s-r \to \infty$, then*

$$\frac{1}{\kappa_B \lambda_B^s} M_r M_s \to Y,$$

where Y is an idempotent matrix of the form

$$Y = \begin{pmatrix} u & pu & qu & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } u = (0,u_1,u_2,u_3,0,0,0)^T \text{ and } p,q > 0,$$

and $(u_1,u_2,u_3)^T$ is a Perron–Frobenius eigenvector of B.

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Proof. There exists a Perron–Frobenius eigenvector $x = (x_1, x_2, x_3)^T$ for B and constants $p, q > 0$ such that

$$P = \lim_{s\to\infty} \frac{B^s}{\lambda_B^s} = \begin{pmatrix} x & px & qx \end{pmatrix}.$$

We have

$$M_r M_s = \left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline & & & & & & 0 \\ & B^s & & & 0 & & 0 \\ & & & & & & 0 \\ \hline & & & & & & 0 \\ & 0 & & & B^r & & 0 \\ & & & & & & 0 \end{array}\right) \implies \frac{1}{\lambda_B^s} M_r M_s \to \left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & 0 \\ & P & & & 0 & & 0 \\ & & & & & & 0 \\ \hline & & & & & & 0 \\ & 0 & & & 0 & & 0 \\ & & & & & & 0 \end{array}\right).$$

The square of the limiting matrix above has a nonzero block where P is of the form

$$(px_1 + qx_2)P,$$

and zero elsewhere, so we set

$$\kappa_B = px_1 + qx_2 \qquad \text{and} \qquad (u_1, u_2, u_3)^T = \frac{1}{\kappa_B}(x_1, x_2, x_3)^T. \qquad \square$$

We have a similar statement for the matrices N_r.

Lemma 4.6. *Let C be the transition matrix for $\vartheta = \theta^{-1}$, with Perron–Frobenius eigenvalue λ_C. There exists a constant $\kappa_C > 0$ such that if $r, s - r \to \infty$, then*

$$\frac{1}{\kappa_C \lambda_C^s} N_s N_r \to Z,$$

where Z is an idempotent matrix of the form

$$Z = \begin{pmatrix} 0 & v & pv & qv & 0 & 0 & 0 \end{pmatrix} \text{ with } v = \begin{pmatrix} v_1, v_2, v_3, 0, 0, 0, 0 \end{pmatrix}^T \text{ and } p, q > 0,$$

and $(v_1, v_2, v_3)^T$ is a Perron–Frobenius eigenvector of C.

Proof. We observe that the matrix $N_s N_r$ has shape that is the transpose of the matrix in Lemma 4.5, with powers of the PF matrix C forming the nonzero blocks:

$$N_s N_r = \left(\begin{array}{c|ccc|ccc} 0 & & & & & & \\ 0 & & C^s & & & 0 & \\ 0 & & & & & & \\ \hline 0 & & & & & & \\ 0 & & 0 & & & C^r & \\ 0 & & & & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \qquad \square$$

For future reference, we also record the following. Let $P = \lim_{r\to\infty} B^r/\lambda_B^r$ and $Q = \lim_{r\to\infty} C^r/\lambda_C^r$. Set

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$$M_\infty = \lim_{r\to\infty} \frac{M_r}{\lambda_B^r} = \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix} \qquad \text{and} \qquad N_\infty = \lim_{r\to\infty} \frac{N_r}{\lambda_C^r} = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.7. *There are* $p,q,r,s > 0$ *such that*

$$M_\infty Y = \begin{pmatrix} y & py & qy & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } y = \left(0,0,0,0,y_1,y_2,y_3\right)^T,$$

$(y_1,y_2,y_3)^T$ *is a Perron–Frobenius eigenvector of B, and*

$$ZN_\infty = \begin{pmatrix} 0 & 0 & 0 & 0 & z & rz & sz \end{pmatrix} \text{ with } z = \left(z_1,z_2,z_3,0,0,0,0\right)^T,$$

and $(z_1,z_2,z_3)^T$ *is a Perron–Frobenius eigenvector of C.*

4.3. Folding and unfolding sequence

Consider a sequence of positive integers $(r_i)_{i\geq 1}$ and the sequence of automorphisms ϕ_{r_i}, with transition matrix M_{r_i} and $\phi_{r_i}^{-1} = \psi_{r_i}$ with transition matrix N_{r_i}. Let $\tau_i \to \tau_{i-1}$ (resp. $\tau'_{i-1} \to \tau'_i$) be the train track map induced on the rose by ϕ_{r_i} (resp. ψ_{r_i}) as given by Lemma 4.3 (resp. Lemma 4.4). Thus, we have an unfolding sequence

$$\cdots \longrightarrow \tau_{i+1} \xrightarrow{\phi_{r_{i+1}}} \tau_i \xrightarrow{\phi_{r_i}} \tau_{r_{i-1}} \longrightarrow \cdots \xrightarrow{\phi_{r_3}} \tau_2 \xrightarrow{\phi_{r_2}} \tau_1 \xrightarrow{\phi_{r_1}} \tau_0,$$

and a folding sequence

$$\cdots \longleftarrow \tau'_{i+1} \xleftarrow{\psi_{r_{i+1}}} \tau'_i \xleftarrow{\psi_{r_i}} \tau'_{i-1} \longleftarrow \cdots \xleftarrow{\psi_{r_3}} \tau'_2 \xleftarrow{\psi_{r_2}} \tau'_1 \xleftarrow{\psi_{r_1}} \tau'_0,$$

Let $\Phi_i = \phi_{r_1} \circ \ldots \circ \phi_{r_i}$ and $\Phi_i^{-1} = \Psi_i = \psi_{r_i} \circ \ldots \circ \psi_{r_1}$. Here, τ_0 is a rose with petals labeled by elements in $\{a,b,c,d,e,f,g\}$ and hence for $i \geq 1$, τ_i is a rose labeled by $\{\Phi_i(a),\ldots,\Phi_i(g)\}$. Also, τ'_0 is a rose labeled by $\{a,b,c,d,e,f,g\}$, so τ'_i is also a rose labeled by $\{\Phi_i(a),\ldots,\Phi_i(g)\}$. Thus, for every $i \geq 0$, τ_i and τ'_i have the same marking but different train track structures. In other words, they belong to the same simplex in CV_7.

The next lemma studies the behavior of illegal turns in a path along the folding sequence. This will be used in the proof of Proposition 5.10 to show that the limit tree of the folding sequence is nongeometric.

Lemma 4.8. *Let* $(r_i)_{i\geq 1}$ *be strictly increasing such that* $r_i \equiv 0$ *mod 3 and* $r_1 > R$*, where* R *is the constant from Lemma 2.1. Let* $(\tau'_i)_i$ *be the corresponding folding sequence. Then for any edge path* β *in* τ'_j *with at least one illegal turn, the number of illegal turns in* $[\psi_{r_{j+3}}\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)]$ *is less than the number of illegal turns in* β*.*

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Proof. By Lemma 4.4, the illegal turns in τ'_j are

$$\{a,e\},\{a,\bar{g}\},\{e,\bar{g}\},\{b,\bar{d}\},\{c,\bar{b}\},\{d,\bar{c}\},\{f,\bar{e}\},\{g,\bar{f}\}$$

and we have

$$\{a,e\} \xrightarrow{\psi_{r_{j+1}}} \{d,\bar{c}\} \xrightarrow{\psi_{r_{j+2}}} \{g,\bar{f}\}$$

$$\{a,\bar{g}\} \xrightarrow{\psi_{r_{j+1}}} \{d,\bar{c}\} \xrightarrow{\psi_{r_{j+2}}} \{g,\bar{f}\}$$

$$\{b,\bar{d}\} \xrightarrow{\psi_{r_{j+1}}} \{e,\bar{g}\}$$

$$\{c,\bar{b}\} \xrightarrow{\psi_{r_{j+1}}} \{f,\bar{e}\}$$

$$\{d,\bar{c}\} \xrightarrow{\psi_{r_{j+1}}} \{g,\bar{f}\}.$$

Thus, for any illegal edge path $\beta \subset \tau'_j$, one of $\beta,\psi_{r_{j+1}}(\beta),\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)$ has an illegal turn $\{x,y\}$, where $x,y \in \{e,f,g,\bar{e},\bar{f},\bar{g}\}$.

Consider the automorphism ϑ and corresponding train track map $h\colon \mathrm{R}_3 \to \mathrm{R}_3$ as in Lemma 4.2. Then h does not have any periodic INPs. Since R is the constant from Lemma 2.1, we get that one of $[\psi_{r_{j+1}}(\beta)],[\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)],[\psi_{r_{j+3}}\psi_{r_{j+2}}\psi_{r_{j+1}}(\beta)]$ has fewer illegal turns than β. □

5. Limiting tree of folding sequence

In this section, we will show that for appropriate choices of $(r_i)_i$, the projection of the folding sequences $(\tau'_i)_i$ to the free factor complex $\mathcal{FF}_7$ is a quasi-geodesic and hence converges to the equivalence class of an arational tree. We will also show that this tree is nongeometric.

5.1. Sequence of free factors

Given a sequence $(r_i)_{i\geq 1}$, recall that $\Phi_i = \phi_{r_1}\phi_{r_1}\cdots\phi_{r_i}$, where $\phi_r = \rho\phi^r$. For convenience, also set $\Phi_0 = \mathrm{id}$. We have the folding sequence

$$\cdots \longleftarrow \tau'_{i+1} \xleftarrow{\psi_{r_{i+1}}} \tau'_i \xleftarrow{\psi_{r_i}} \tau'_{i-1} \longleftarrow \cdots \xleftarrow{\psi_{r_3}} \tau'_2 \xleftarrow{\psi_{r_2}} \tau'_1 \xleftarrow{\psi_{r_1}} \tau'_0,$$

where τ'_i is a rose labeled by $\{\Phi_i(a),\cdots,\Phi_i(g)\}$, and $\psi_r = \phi_r^{-1}$. From the markings, we can associate τ'_i to an open simplex U_i in CV_7. Consider a sequence of free factors $A_i \in \pi(U_i)$, where $\pi\colon \mathrm{CV}_7 \to \mathcal{FF}_7$. For an appropriate sequence of $(r_i)_i$, we will see that $(A_i)_i$ is a quasi-geodesic (with infinite diameter). The key will be Lemma 5.3 which is the main goal of this section.

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We now consider the following explicit sequence of free factors. Let $A_0 = \langle d,e,f\rangle$ be the free factor in $\mathbb{F}_7$, and define

$$A_i := \Phi_i(A_0) = \langle \Phi_i(d), \Phi_i(e), \Phi_i(f)\rangle.$$

Note that for any $r,s,t > 0$, the following holds:

$$\begin{aligned} A_0 &= \langle d,e,f\rangle \\ A_1 &= \phi_r(A_0) = \langle a,b,c\rangle \\ A_2 &= \phi_s\phi_r(A_0) = \langle e,f,g\rangle \\ A_3 &= \phi_t\phi_s\phi_r(A_0) = \langle b,c,d\rangle. \end{aligned} \tag{1}$$

Thus, for any sequence $(r_i)_i$,

$$A_i = \Phi_i(A_0) = \Phi_{i-1}(A_1) = \Phi_{i-2}(A_2) = \Phi_{i-3}(A_3). \tag{2}$$

We say two free factors A and A' are *disjoint* if (possibly after conjugating) $\mathbb{F}_n = A * A' * B$ for a (possibly trivial) free factor B, and A' is *compatible* with A if it either contains A (up to conjugation) or is disjoint from A.

Lemma 5.1. *For any sequence $(r_i)_{i\geq 1}$, if $|i-j| = 1$, then A_i, A_j are disjoint, and if $|i-j| = 2$ or 3, then they are distinct and not disjoint.*

Proof. We see from Equation 1 that the statement of the lemma holds for A_0, A_1, A_2 and A_3. Now, for $i \geq 1$ and $k \in \{1,2,3\}$, by Equation 2, the pair A_i, A_{i+k} differs from A_0, A_k by the automorphism Φ_i, whence the lemma. □

Recall the transition matrix M_r for ϕ_r, and the 3×3 matrix B whose power B^r forms a block of M_r. For each $i \geq 1$, let $\overline{M}_i = M_i \bmod 2$. By a simple computation, we see that $B^7 = I \bmod 2$. Thus, when $i = j \bmod 7$, $\overline{M}_i = \overline{M}_j$. We have the following lemma.

Lemma 5.2. *Let V_0 be the three-dimensional vector space of $(\mathbb{Z}/2\mathbb{Z})^7$ spanned by the vectors $(0,0,0,1,0,0,0)^T, (0,0,0,0,1,0,0)^T, (0,0,0,0,0,1,0)^T$. Then for all $i \geq 0$,*

$$V_0 \bigcup \left(\bigcup_{j=0}^{107} \overline{M}_i \overline{M}_{i+1} \ldots \overline{M}_{i+j} V_0 \right) = (\mathbb{Z}/2\mathbb{Z})^7.$$

Proof. Since $\overline{M}_i = \overline{M}_j$ whenever $i = j \bmod 7$, it is enough to verify the statement for $i \in \{0,\ldots,6\}$. In these cases, we can check the validity of the statement using Sage with the following code:

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```
B = matrix(GF(2), [
        [0,0,1],
        [1,0,0],
        [0,1,1]
    ])

def M(i):
    return block_matrix([
        [ matrix(4,3,0) , identity_matrix(4) ],
        [ B^i           , matrix(3,4,0)      ]
    ])

V0 = set(
    (0,0,0,i,j,k,0)
    for i in (0,1)
    for j in (0,1)
    for k in (0,1)
)

for i in range(0,7):
    W = set(V0)
    P = identity_matrix(7)
    for j in range(i,200):
        P = P*M(j)
        for v in V0:
            w = tuple(P*vector(v))
            W.add(w)
        if len(W) >= 2^7:
            break
    print(i,j)

# Output:
# 0 107
# 1 107
# 2 107
# 3 107
# 4 107
# 5 107
# 6 107
```

□

Lemma 5.3. *For any sequence $(r_i)_i$, if $r_i \equiv i \mod 7$, then 109 consecutive A_i's cannot be contained in the same free factor or be disjoint from a common factor.*

Proof. For any $i \geq 1$ and $k \geq 0$, let

$$B_{i+k} = \phi_i \phi_{i+1} \cdots \phi_{i+k} A_0.$$

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Abelianizing and reducing mod 2, we have $A_0 \equiv V_0$, and $B_{i+k} \equiv \overline{M}_i \cdots \overline{M}_{i+k} V_0$. Thus, by Lemma 5.2, the sequence $\{A_0, B_i, \ldots, B_{i+107}\}$ cannot be contained in the same free factor or be disjoint from a common factor.

Now, consider any sequence $(r_i)_i$ with $r_i \equiv i \bmod 7$ so that $\overline{M}_{r_i} = \overline{M}_i$ for all i. Let $A_i = \Phi_i A_0 = \phi_{r_1} \cdots \phi_{r_i} A_0$. Set $\Phi_0 = \mathrm{id}$. For any $i \geq 1$, by applying the automorphism Φ_{i-1}^{-1}, the sequence of free factors $\{A_{i-1}, \ldots, A_{i+107}\}$ is isomorphic to the sequence

$$\{A_0, \phi_{r_i} A_0, \ldots, \phi_{r_i} \phi_{r_{i+1}} \cdots \phi_{r_{i+107}} A_0\}.$$

The latter sequence after abelianization and reducing mod 2 is equivalent to the sequence $\{A_0, B_i, \ldots, B_{i+107}\}$. Thus, $\{A_{i-1}, \ldots, A_{i+107}\}$ cannot be contained in the same factor or be disjoint from a common factor.here □

5.2. Subfactor projection

We will now use subfactor projection theory originally introduced in [BF14b] and further developed in [Tay14] to show that $(A_i)_i$ is a quasi-geodesic for appropriate choices of sequence $(r_i)_i$.

We first define subfactor projection and recall the main results about them. For $G \in \mathrm{CV}_n$ and a rank ≥ 2 free factor A, let $A|G$ denote the core subgraph of the cover of G corresponding to the conjugacy class of A. Pulling back the metric on G, we obtain $A|G \in \mathrm{CV}(A)$. Denote by $\pi_A(G) := \pi(A|G) \subset \mathcal{F}(A)$ the projection of $A|G$ to $\mathcal{F}(A)$. Here, $\mathrm{CV}(A)$ is the Outer space of the free group A and $\mathcal{F}(A)$ is the corresponding free factor complex.

Recall two free factors A and B are *disjoint* if they are distinct vertex stabilizers of a free splitting of $\mathbb{F}_n$. If B is not compatible with A, then we say B *meets* A, that is, B and A are not disjoint and A is not contained in B, up to conjugation. In this case, define the projection of B to $\mathcal{F}(A)$ as follows:

$$\pi_A(B) := \{\pi_A(G) | G \in \mathrm{CV}_n \text{ and } B|G \subset G \text{ is embedded }\}$$

If B is compatible with A, then define $\pi_A(B)$ to be empty. If A meets B and B meets A, then we say A and B *overlap*.

Theorem 5.4 [Tay14]. *Let A,B,C be free factors of $\mathbb{F}_n$. There is a constant D depending only on n such that the following statements hold.*

1. *If* $\mathrm{rank}(A) \geq 2$, *then either $A \subseteq B$ (up to conjugation), A and B are disjoint, or $\pi_A(B) \subset \mathcal{F}(A)$ is defined and has diameter $\leq D$.*
2. *If* $\mathrm{rank}(A) \geq 2$, *B and C meet A and B is compatible with C, then*
$$d_A(B,C) = \mathrm{diam}_{\mathcal{F}(A)}(\pi_A(B) \cup \pi_A(C)) \leq D.$$
3. *If A and B overlap, have rank at least 2 and C meets both, then*
$$\min\{d_A(B,C), d_B(A,C)\} \leq D.$$

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Theorem 5.5 (Bounded geodesic image theorem [Tay14]). *For $n \geq 3$, there exists $D' \geq 0$ such that if A is a free factor with rank$(A) \geq 2$ and γ is a geodesic of $\mathcal{FF}_n$ with each vertex of γ having a well-defined projection to $\mathcal{F}(A)$, then diam$(\pi_A(\gamma)) \leq D'$.*

We now prove the following lemma.

Lemma 5.6. *For any $K > 0$, there exists a constant $r = r(K)$ such that for any sequence $(r_i)_{i\geq 1}$, if $r_i \geq r$ for all i, then the following statements hold:*

1. *For any $j \geq 2$, the projections of A_{j-2} and A_{j+2} to the free factor complex $\mathcal{F}(A_j)$ are defined and the distance between them is at least K.*
2. *Let D be the constant of Theorem 5.4. If $K > 3D$, then for any $i < j < k$, if $j - i \geq 2$ and $k - j \geq 2$, the projections of A_i and A_k to $\mathcal{F}(A_j)$ are defined and have distance at least $K - 2D$.*

Proof. Recall for any r, $\phi_r = \rho\phi^r$, where ϕ restricts to a fully irreducible outer automorphism of $\langle a,b,c\rangle$. In particular, ϕ acts as a loxodromic isometry of the free factor complex $\mathcal{F}(\langle a,b,c\rangle)$, Thus, for any K, there exists $r = r(K)$ such that for all $s \geq r$, the distance between $\phi^s(\langle b,c\rangle)$ is at least $K + 2D$ away from $\langle a,b\rangle$ in $\mathcal{F}(\langle a,b,c\rangle)$.

Now consider any sequence $(r_i)_i$ with $r_i \geq r$ for all i. By Lemma 5.1 and Theorem 5.4, the projections of A_{j-2} and A_{j+2} to $\mathcal{F}(A_j)$ are defined. Moreover, by Equation 2, we see that, by applying an automorphism, the distance between projections of A_{j-2} and A_{j+2} in $\mathcal{F}(A_j)$ is the same as the distance between the projections of $A_0 = \langle d,e,f\rangle$ and $\phi_{r_{j-1}}(A_3) = \langle \phi_{r_{j-1}}(b), \phi_{r_{j-1}}(c), a\rangle$ to $\mathcal{F}(A_2) = \mathcal{F}(\langle e,f,g\rangle)$. Note that the rotation ρ sends the free factor $\langle a,b,c\rangle$ to A_2, thus inducing an isometry from $\mathcal{F}(\langle a,b,c\rangle)$ to $\mathcal{F}(A_2)$. The projection of A_0 to $\mathcal{F}(A_2)$ is D-close to the factor $\langle e,f\rangle = \rho(\langle a,b\rangle)$, and the projection of $\phi_{r_{j-1}}(A_3)$ to $\mathcal{F}(A_2)$ is D-close to the factor $\rho\phi^{r_{j-1}}(\langle b,c\rangle)$. Thus, the distance in $\mathcal{F}(A_2)$ of the two projections is at least K. This shows the first statement of the lemma.

Now, fix $K > 3D$ and let $(r_i)_i$ be any sequence with $r_i \geq r(K)$ for all i. We will prove the second statement by inducting on $l = k - i$ with the previous statement giving the base case $l = 4$. Suppose we are given A_i, A_j, A_k with $l = k - i > 4$, $j - i, k - j \geq 2$. We first claim that projections of $A_{j+2}, A_{j+3}, \cdots, A_k$ to $\mathcal{F}(A_j)$ are defined, that is, none of them are equal to or disjoint from A_j. For suppose A_s is the first on the list that is equal to or disjoint from A_j. By Lemma 5.1, we have $4 \leq s - j < k - i$. By induction, the projections of both A_j and A_s to $\mathcal{F}(A_{j+2})$ are defined and the distance between their projections is $\geq K - 2D > D$. Using statement 2 of Theorem 5.4, this implies that A_s and A_j cannot coincide or be disjoint, proving the claim. By the same argument, we also have that the projections of $A_i, A_{i+1}, \cdots, A_{j-2}$ to $\mathcal{F}(A_j)$ are all defined.

By the first statement of the lemma, we have $d_{A_j}(A_{j-2}, A_{j+2}) \geq K$. We now claim that $d_{A_j}(A_{j+2}, A_k) \leq D$. If $k = j + 3$, then A_{j+2} and A_k are disjoint, and the claim holds by statement 2 of Theorem 5.4. If $k \geq j + 4$, then applying induction again to j, $j+2$ and k, we see that A_j and A_k have well-defined projections to $\mathcal{F}(A_{j+2})$ and $d_{A_{j+2}}(A_j, A_k) \geq K - 2D > D$. Now, the claim follows by the third statement of Theorem 5.4. By the same

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argument, we also see that $d_{A_j}(A_i, A_{j-2}) \leq D$. We now conclude $d_{A_j}(A_i, A_k) \geq K - 2D$ by the triangle inequality. □

We are now ready to prove the main results of this section.

Proposition 5.7. *There exists $R > 0$ such for any sequence $(r_i)_{i\geq 1}$, if $r_i \geq R$, and $r_i \equiv i$ mod 7, then the sequence $(A_i)_{i\geq 0}$ is a quasi-geodesic in $\mathcal{FF}_7$.*

Proof. Let D be the constant of Theorem 5.4, and let D' be the constant of Theorem 5.5. Fix $K = 4D + D'$. Let $R = r(K)$ be the constant of Lemma 5.6. Let $(r_i)_{i\geq 1}$ be any sequence with $r_i \geq R$ and $r_i \equiv i \mod 7$ for all i. We will show that the sequence $(A_i)_i$ goes to infinity with linear speed. More precisely, we will show that for any $d > 0$, if $k - i \geq 110d + 4$, then $d_{\mathcal{FF}_7}(A_i, A_k) \geq d$. Suppose not. Let γ be a geodesic between A_i and A_k of length $< d$.

For every $j \in \{i+2, \ldots, k-2\}$, there exists a free factor in γ that is compatible with A_j. Indeed, if every free factor in γ meets A_j, then by Theorem 5.5, projection of γ to A_j will be well defined and has diameter bounded by D'. However, by Lemma 5.6, the projections of A_i and A_k to $\mathcal{F}(A_j)$ has distance at least $K - 2D > D'$.

By the pigeonhole principle, there exists a vertex B of γ compatible with at least 110 free factors among $\{A_{i+2}, \ldots, A_{k-2}\}$. By Lemma 5.3, it is not possible for B to be compatible with 109 consecutive A_j's. Therefore, it must be possible to find $i + 2 \leq i' < j' < k' \leq k - 2$ with $j' - i' \geq 2$ and $k' - j' \geq 2$ such that B is compatible with $A_{i'}$ and $A_{k'}$, but B meets $A_{j'}$. In particular, $\pi_{A_{j'}}(B)$ is defined. By Lemma 5.6, $A_{i'}$, $A_{k'}$ also have well-defined projections to $\mathcal{F}(A_{j'})$ with $d_{A_{j'}}(A_{i'}, A_{k'}) \geq K - 2D > 2D$. On the other hand, since B is compatible with both $A_{i'}$ and $A_{k'}$, we have $d_{A_{j'}}(A_{i'}, B) \leq D$ and $d_{A_{j'}}(A_{k'}, B) \leq D$ by Theorem 5.4. This is a contradiction, finishing the proof that $d_{\mathcal{FF}_7}(A_i, A_k) \geq d$ for all $k - i \geq 110d + 4$. □

Recall that $\mathcal{FF}_n$ is Gromov hyperbolic and that its Gromov boundary is the space of equivalence class of arational trees. Also, recall we say a folding sequence $(G_i)_i$ *converges to an arational tree* T, if $\pi(U_i)$ converges to $[T] \in \partial\mathcal{FF}_n$, where U_i is the open simplex in in CV_n associated to G_i. We have the following corollary.

Corollary 5.8. *Given any strictly increasing sequence $(r_i)_{i\geq 1}$ satisfying $r_i \equiv i \mod 7$, the folding sequence $(\tau'_i)_i$ converges to an arational tree T.*

5.3. Nongeometric tree

We will now show that the arational tree obtained in the previous section as the limit of the free factors $(A_i)_i$ is nongeometric. This section will use the terminology of band complexes and resolutions; for details see [BF95].

Definition 5.9 (Geometric tree). [BF94, LP97] Let X be a band complex and T a $G = \pi_1(X)$-tree. A resolution $f : \widetilde{X} \to T$ is *exact* if for every G-tree T' and equivariant factorization

$$\widetilde{X} \xrightarrow{f'} T' \xrightarrow{h} T$$

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of f with f' a surjective resolution it follows that h is an isometry onto its image. We say T is *geometric* if every resolution is exact.

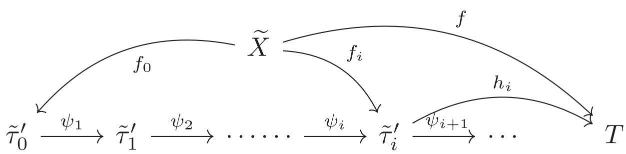
The proof of the following proposition is based on [BF94, Proposition 3.6].

Proposition 5.10. *For any strictly increasing sequence* $(r_i)_{i\geq 1}$*, if the corresponding folding sequence* $(\tau'_i)_i$ *converges to an arational tree* T*, then* T *is not geometric.*

Proof. Let $\tilde{\psi}_i\colon \tilde{\tau}'_{i-1} \to \tilde{\tau}'_i$ be a lift of the train track map to the universal covers fixing a base vertex. Pick a length measure on $(\tau'_i)_i$, so we get a folding sequence $\tilde{\tau}'_0 \xrightarrow{\tilde{\psi}_1} \tilde{\tau}'_1 \xrightarrow{\tilde{\psi}_2} \cdots$ in cv_7 that converges to T. Recall that there are morphisms $h_i\colon \tilde{\tau}'_i \to T$ such that $h_i = h_{i+1}\tilde{\psi}_{i+1}$. Since T is arational, h_i's are not isometries though they restrict to isometries on edges. Let X be a finite band complex with resolution $f\colon \tilde{X} \to T$. We will show that the resolution factors through $\tilde{\tau}'_i$ for sufficiently large i. This will imply T is not geometric.

Let Γ be the underlying real graph of X (disjoint union of metric arcs) with preimage $\tilde{\Gamma}$ in $\tilde{X}$. We may assume f embeds the components of $\tilde{\Gamma}$. A vertex v of $\tilde{X}$ is either a vertex of $\tilde{\Gamma}$ or a corner of a band or a 0-cell of $\tilde{X}$. For every such vertex v, choose a point $f_0(v) \in \tilde{\tau}_0$ so that f_0 is equivariant and $f = h_0 f_0$ on the vertices of $\tilde{X}$.

An edge in $\tilde{X}$ is either a subarc of $\tilde{\Gamma}$ or a vertical boundary component of a band or a one-cell in $\tilde{X}$. Up to the action of $\mathbb{F}_7$, there are only finitely many edges. Using Lemma 4.8, we can find $i > 0$ such that for every edge e in $\tilde{X}$, the edge path in $\tilde{\tau}'_i$ joining the two vertices of $\tilde{\psi}_i \cdots \tilde{\psi}_1 f_0(\partial e)$ is legal. Now, extend $\tilde{\psi}_i \cdots \tilde{\psi}_1 f_0$ to an equivariant map $f_i\colon \tilde{X} \to \tilde{\tau}'_i$ that sends edges to legal paths (or points) and is constant on the leaves. Thus, f_i is a resolution of $\tilde{\tau}'_i$.



This yields a factorization

$$\tilde{X} \xrightarrow{f_i} \tilde{\tau}'_i \xrightarrow{h_i} T$$

but h_i is not an isometry. This shows T is nongeometric. $\square$

6. Nonuniquely ergodic unfolding sequence

The goal of this section is to show that if a sequence $(r_i)_{i\geq 1}$ grows sufficiently fast, then the set of currents supported on the legal lamination Λ of the unfolding sequence $(\tau_i)_{i\geq 0}$ is a 1-simplex in $\mathbb{P}\mathrm{Curr}_7$.

Recall that M_r is a 7×7 matrix of the block form

$$\begin{pmatrix} 0 & I \\ B^r & 0 \end{pmatrix},$$

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where I is the 4×4 identity matrix, and B is the transition matrix of θ; all that matters is that some positive power of B has all entries positive. Let λ_B be the Perron–Frobenius eigenvalue of B. Recall the constant $\kappa_B > 0$ from Lemma 4.5. Given a sequence $(r_i)_i$, define for each $i \geq 1$

$$P_i = \frac{1}{\kappa_B \lambda_B^{r_{i+1}}} M_{r_i} M_{r_{i+1}}.$$

Let $\{e_k : k = 1,\dots,7\}$ be the standard basis for $\mathbb{R}^7$. Denote by $\mathbb{PR}^7_{\geq 0}$ the projectivization of $\mathbb{R}^7_{\geq 0}$, and the projective class of a vector v by $[v]$. Fix a metric d on $\mathbb{PR}^7_{\geq 0}$. We view M_r as a projective transformation $\mathbb{PR}^7_{\geq 0} \to \mathbb{PR}^7_{\geq 0}$. For a sequence $(r_i)_{i\geq 1}$ and for $i < j$ denote by $S_{i,j} \subset \mathbb{PR}^7_{\geq 0}$ the image of the composition

$$M_{ij} := M_{r_i} M_{r_{i+1}} \cdots M_{r_j}$$

and by $S_i = \bigcap_{j>i} S_{i,j}$. We denote by v_B a positive Perron–Frobenius eigenvector of B, and by v_B^{234} (resp. v_B^{567}) the vector in $\mathbb{R}^7$ which is v_B in coordinates 2,3,4 (resp. 5,6,7) and 0 in all other coordinates. The main result of this section is the following.

Proposition 6.1. *Let $(r_i)_{i\geq 1}$ be a sequence of positive integers with $r_{i+1} - r_i \geq i$. Then for all i the set S_i is a 1-simplex, that is, it is the convex hull of two distinct points $p_i, q_i \in \mathbb{PR}^7_{>0}$. Moreover, as $i \to \infty$, $\{p_i, q_i\}$ converges (as a set) to $\{[v_B^{234}], [v_B^{567}]\}$.*

Before we give a technical proof of Proposition 6.1, we will give a simpler, more intuitive proof where the sequence $r_1 < r_2 < \cdots$ is chosen inductively so that r_1 is sufficiently large and each r_i is sufficiently large depending on $r_1, r_2, \cdots, r_{i-1}$. Later, we do a more careful analysis where we can control the growth of the sequence.

Proof idea of Proposition 6.1. For $\epsilon > 0$, we will write $x \overset{\epsilon}{=} y$ if $d(x,y) < \epsilon$ in $\mathbb{PR}^7_{\geq 0}$. Each S_{ij} is the convex hull of the M_{ij}-images of the vectors e_i, $i = 1, \cdots, 7$. The proof consists of computing these images using the Perron–Frobenius dynamics. We first observe that there is a sequence $\epsilon_r \to 0$ such that

- $M_r(e_7) = e_4$, $M_r(e_6) = e_3$, $M_r(e_5) = e_2$, $M_r(e_4) = e_1$,
- $M_r(e_i) \overset{\epsilon_r}{=} v_B^{567}$, $i = 1,2,3$,
- $M_r(v_B^{567}) = v_B^{234}$, $M_r(v_B^{234}) \overset{\epsilon_r}{=} v_B^{567}$.

Next, we consider the composition $M_s M_r$ for $r >> s$. The third bullet uses uniform continuity of M_s and the assumption that r is sufficiently large compared to s.

- $M_s M_r(e_7) = e_1$,
- $M_s M_r(e_i) \overset{\epsilon_s}{=} v_B^{567}$, $i = 4,5,6$,
- $M_s M_r(e_i) \overset{\epsilon_s}{=} v_B^{234}$, $i = 1,2,3$.

Finally, for $r >> s >> t$ we see similarly:

- $M_t M_s M_r(e_7) \overset{\epsilon_t}{=} v_B^{567}$,
- $M_t M_s M_r(e_i) \overset{\epsilon_t}{=} v_B^{234}$, $i = 4,5,6$,
- $M_t M_s M_r(e_i) \overset{\epsilon_t}{=} v_B^{567}$, $i = 1,2,3$.

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It follows that if we make suitably large choices for the r_i's, the set $S_{i,i+3}$ will be contained in the ϵ_{r_i}-neighborhood of the 1-simplex $[v_B^{567}, v_B^{234}]$. Moreover, given any $\epsilon > 0$ and $j > i+3$ we can choose r_j large (depending on uniform continuity constants of M_{ij}) to ensure that $S_{i,j+3} = M_{ij}(S_{j,j+3})$ is contained in the ϵ-neighborhood of the 1-simplex with endpoints $M_{ij}(v_B^{567})$ and $M_{ij}(v_B^{234})$. Thus, each S_i is the nested intersection of simplices of dimension ≤ 6 such that for all $\epsilon > 0$ they are eventually all contained in the ϵ-neighborhood of a 1-simplex with definite distance between the endpoints. This proves the proposition. □

We now present a more detailed proof of Proposition 6.1. For a sequence of integers $(r_i)_{i\geq 1}$ such that $r_i, r_{i+1} - r_i \to \infty$, by Lemma 4.5 $(P_i)_i$ converges to an idempotent matrix Y. Let $\Delta_i = Y - P_i$ and let $||Y||$ be the operator norm.

Lemma 6.2. *Let $(r_i)_{i\geq 1}$ be a sequence of positive integers such that $r_{i+1} - r_i \geq i$. Then there exists an $I \geq 1$ such that for all $i \geq I$, $||\Delta_i|| \leq 1/(2\cdot 2^i)$.*

Proof. Let λ_B, μ_B, μ'_B be the modulus of the three eigenvalues of B; we have $\lambda_B \sim 1.46$ and $\mu_B = \mu'_B \sim 0.826$. Then

$$||\Delta_i|| = ||P_i - Y|| \leq \max\left(\frac{\mu^{r_{i+1}}}{\lambda^{r_{i+1}}}, \frac{\lambda^{r_i}}{\lambda^{r_{i+1}}}\right) \leq \frac{\lambda^{r_i}}{\lambda^{r_{i+1}}},$$

where the two terms comes from the two blocks in P_i. For the last inequality, note that $\mu < 1 < \lambda$ and r_i are positive integers. Therefore, $\mu^{r_{i+1}} < 1 < \lambda^{r_i}$.

Now, we claim that there exists an $I \geq 1$ such that for all $i \geq I$,

$$\frac{\lambda^{r_i}}{\lambda^{r_{i+1}}} \leq \frac{1}{2^{i+1}} \qquad \text{equivalently,} \qquad 2 \leq \lambda^{\frac{r_{i+1}-r_i}{i+1}}.$$

We only need to show that the sequence $\frac{r_{i+1}-r_i}{i+1}$ is eventually increasing. Indeed, by assumption, $r_{i+1} - r_i \geq i$, so

$$i \leq r_{i+1} - r_i$$
$$\frac{i}{i+1} \leq \frac{r_{i+1}-r_i}{i+1}.$$

Since $i/(i+1)$ is an increasing sequence, it follows that our sequence is also increasing. □

The following lemma is a consequence of Lemma 4.5 and Lemma A.1.

Lemma 6.3. *Let $(r_i)_{i\geq 1}$ be a sequence of positive integers such that $r_{i+1} - r_i \geq i$, Y be the idempotent matrix of Lemma 4.5 and $M_\infty = \lim_{r\to\infty} M_r/\lambda_B^r$. Then the following statements hold.*

(1) *For all $i \geq 1$, the sequence of matrices $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=1}^\infty$ converges to a matrix Y_i. Furthermore, for all sufficiently large i,*

$$\|Y_i - Y\| \leq \frac{2}{2^i}\left(\|Y\| + \|Y\|^2\right).$$

(2) *The kernel of Y is a subspace of the kernel of Y_i for all $i \geq 1$.*

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(3) *For all* $i \geq 1$, $Y_i(e_1) \neq 0$ *with nonnegative entries and* $Y_i(e_2)$ *and* $Y_i(e_3)$ *are positive multiples of* $Y_i(e_1)$.

(4) *For all* $i \geq 1$, $M_{r_i} Y_{i+1}(e_1) \neq 0$ *with nonnegative entries, and* $M_{r_i} Y_{i+1}(e_1)$ *and* $Y_i(e_1)$ *are not scalar multiples of each other.*

(5) *Projectively,* $[Y_i(e_1)] \to [Y(e_1)]$ *and* $[M_{r_i} Y_{i+1}(e_1)] \to [M_\infty Y(e_1)]$ *as* $i \to \infty$.

Proof. For (1), it suffices to show convergence for all i greater than some I. Indeed, if such I exists and $i < I$, then let $i_0 \geq I$ be such that $i = i_0 \pmod 2$ and observe that

$$\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=\frac{i_0-i}{2}}^{\infty} = P_i P_{i+2} \cdots P_{i_0-2} \{P_{i_0} P_{i_0+2} \cdots P_{i_0+2k}\}_{k=0}^{\infty}.$$

By assumption, $\{P_{i_0} P_{i_0+2} \cdots P_{i_0+2k}\}_{k=0}^{\infty}$ converges. Since matrix multiplication is continuous, the sequence $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=0}^{\infty}$ also converges.

For each i, let

$$\Delta_i = P_i - Y.$$

By Lemma 6.2, there exists $I \geq 1$ such that for all $i \geq I$, $\|\Delta_i\| \leq \frac{1}{2 \cdot 2^i}$. Also, choose I sufficiently large so that $\frac{1}{2^I}\|Y\| \leq 1/2$. Then, by Lemma A.1, for all $i \geq I$, the sequence $\{P_i P_{i+2} \cdots P_{i+2k}\}_{k=0}^{\infty}$ converges to some matrix Y_i, with

$$\|Y_i - Y\| \leq \frac{2}{2^i}\left(\|Y\| + \|Y\|^2\right). \tag{3}$$

For (2), it again suffices to show the statement is true for all sufficiently large i, and the statement holds for all $i \geq I$ by Lemma A.1.

For (3), first note that since all the matrices involved are nonnegative, the resulting vectors are all also nonnegative. So we only need to show that they are not the zero vector. It suffices to check that $Y_i(e_1) \neq 0$ for all sufficiently large i since each P_i is nonnegative and has full rank. For large i, the statement follows because $Y(e_1)$ is not equal to 0 and $\|Y_i(e_1) - Y(e_1)\| \leq \|Y_i - Y\|$ can be made arbitrarily small. For the second statement, we know that $Y(e_2)$ and $Y(e_3)$ are positive multiples of $Y(e_1)$, so there are $s, t > 0$ such that $se_2 - e_1$ and $te_3 - e_1$ are in the kernel of Y. Then $Y_i(se_2 - e_1) = Y_i(te_3 - e_1) = 0$ for all i by (2).

For (4), $M_{r_i} Y_{i+1}(e_1) \neq 0$ with nonnegative entries since $Y_{i+1}(e_1)$ is so by (3). To see that $M_{r_i} Y_{i+1}(e_1)$ and $Y_i(e_1)$ are projectively distinct, it is enough to do this for all sufficiently large i. Let $M_\infty = \lim_{r \to \infty} M_r / \lambda_B^r$. By Lemma 4.5 and Lemma 4.7, $Y(e_1)$ and $M_\infty Y(e_1)$ are orthogonal. Since $r_i \to \infty$, we can make $\frac{M_{r_i}}{\lambda_B^{r_i}} Y_{i+1}(e_1)$ arbitrarily close to $M_\infty Y(e_1)$, and $Y_i(e_1)$ close to $Y(e_1)$. This means $M_{r_i} Y_{i+1}(e_1)$ and $Y_i(e_1)$ are near orthogonal, so they can't be scalar multiples of each other.

Statement (5) is clear. □

Proof of Proposition 6.1. By Lemma 4.5 and Lemma 4.7, $[v_B^{234}] = [Y(e_1)]$ and $[v_B^{567}] = [M_\infty Y(e_1)]$. Using notation from Lemma 6.3, set

$$p_i = [Y_i(e_1)] \qquad \text{and} \qquad q_i = [M_{r_i} Y_{i+1}(e_1)].$$

By Lemma 6.3 (3)–(5),

- p_i and q_i are well defined and distinct.
- $p_i = [Y_i(e_k)]$, and $q_i = [M_{r_i} Y_{i+1}(e_k)]$, for $k = 1,2,3$.
- $p_i \to [v_B^{234}]$ and $q_i \to [v_B^{567}]$.
- $[M_{r_i}(p_{i+1})] = q_i$ and $[M_{r_i}(q_{i+1})] = p_i$.

Our goal is to show S_i is the 1-simplex spanned by p_i and q_i. To do this, we consider S_{ij}, which is the convex hull of the M_{ij}-images of the vectors e_k, $k = 1,\cdots,7$. That is, we have to show that $[M_{ij}(e_k)]$ is close to either p_i or q_i for each k. We first observe that for all $r,s > 0$:

- $M_r(e_4) = e_1$, $M_r(e_5) = e_2$, $M_r(e_6) = e_3$, $M_r(e_7) = e_4$,
- $M_r M_s(e_7) = e_1$.

We may assume that $j - 1 = i + 2m$, so M_{ij} breaks up into pairs, that is, for all k,

$$[M_{ij}(e_k)] = [P_i \cdots P_{j-1}(e_k)].$$

Let $\epsilon > 0$ be arbitrary. Choose $\delta > 0$ such that for any vector $u \in \mathbb{R}^7_+$ and any $v \in \left\{ Y_i(e_k), \frac{M_{r_i}}{\lambda_B^{r_i}} Y_{i+1}(e_k) : k = 1,2,3 \right\}$, if $\|u - v\| \leq \delta$, then $d([u],[v]) \leq \epsilon$. Now, by Lemma 6.3, we can choose J sufficiently large so that whenever $i + 2m \geq J$, then

- $\|P_i \cdots P_{i+2m} - Y_i\| \leq \delta$
- $\|P_{i+1} \cdots P_{i+2m+1} - Y_{i+1}\| \leq \frac{\delta}{\|M_{r_i}/\lambda_B^{r_i}\|}$.

Now, we may assume that $j - 3 \geq J$. Then,

- For $k = 1,2,3$, we have

$$\|P_i \cdots P_{j-1}(e_k) - Y_i(e_k)\| \leq \delta \quad \Longrightarrow \quad d\Big([M_{ij}(e_k)], p_i\Big) \leq \epsilon.$$

- For $k = 7$, we have $M_{ij}(e_7) = M_{i,j-2}(e_1)$, so $[M_{ij}(e_7)] = [P_i \cdots P_{j-3}(e_1)]$ is ϵ-close to $[p_i]$ by the same reasoning as the previous bullet point.
- For $k = 4,5,6$, $M_{ij}(e_k) = M_{i,j-1}(e_{k-3})$. In this case, we consider $\frac{M_{r_i}}{\lambda^{r_i}} P_{i+1} \cdots P_{j-2}(e_{k-3})$ and approximate it by $\frac{M_{r_i}}{\lambda^{r_i}} Y_{i+1}(e_{k-3})$, as follows:

$$\begin{aligned} &\left\| \frac{M_{r_i}}{\lambda^{r_i}} P_{i+1} \cdots P_{j-2}(e_{k-3}) - \frac{M_{r_i}}{\lambda^{r_i}} Y_{i+1}(e_{k-3}) \right\| \\ &\leq \left\| \frac{M_{r_i}}{\lambda^{r_i}} \right\| \|P_{i+1} \cdots P_{j-2} - Y_{i+1}\| \\ &\leq \delta. \end{aligned}$$

  Thus, for $k = 4,5,6$, $d\Big([M_{ij}(e_k)], q_i\Big) \leq \epsilon$.

We have shown that for any ϵ, the vertices of the simplex $S_{i,j}$ come ϵ-close to p_i and q_i for all sufficiently large j. Since $S_{i,j+1} \subset S_{i,j}$ and $S_i = \bigcap_{j>i} S_{i,j}$, it follows that S_i must be the 1-simplex spanned by p_i and q_i. This proves the Proposition. □

Recall the unfolding sequence $(\tau_i)_{i \geq 0}$, where M_{r_i} is the transition matrix of the train track map $\phi_{r_i} : \tau_i \to \tau_{i-1}$. Let Λ be the legal lamination of $(\tau_i)_{i \geq 0}$.

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Corollary 6.4. *If $(r_i)_{i\geq 1}$ is a positive sequence with $r_{i+1}-r_i \geq i$, then $\mathbb{P}Curr(\Lambda)$ is a 1-simplex.*

Proof. In light of Theorem 3.3, it is enough to show $\mathbb{P}\mathrm{Curr}((\tau_i)_i)$ is a 1-simplex. For each $i \geq 0$, we have a well-defined projection

$$p_i \colon \mathbb{P}\mathrm{Curr}((\tau_i)_i) \to \mathbb{PR}^7_+ \quad \text{given by} \quad p_i([(\mu_i)_i]) = [\mu_i].$$

The image of the projection is S_{i+1}, which is always a 1-simplex by Proposition 6.1. Therefore, $\mathbb{P}\mathrm{Curr}((\tau_i)_i)$ is a 1-simplex. □

7. Nonuniquely ergometric tree

The goal of this section is to show that if a sequence $(r_i)_{i\geq 1}$ grows sufficiently fast, then the set of projectivized length measures $\mathbb{P}\mathcal{D}((\tau'_i)_i)$ on the folding sequence $(\tau'_i)_i)$ is a 1-simplex. By Proposition 3.1, if $(\tau'_i)_i$ converges to an arational tree T, then $\mathbb{P}\mathcal{D}(T)$ is also a 1-simplex in $\partial\mathrm{CV}_7$.

Recall that N_r is a 7×7 matrix of the block form

$$\begin{pmatrix} 0 & C^r \\ I & 0 \end{pmatrix},$$

where I is the 4×4 identity matrix, and C is the transition matrix of ϑ. The transpose of N_r has the same shape as M_r. Therefore, the same theory from Section 6 holds true. For brevity, we record only the essential statements that will be used later and omit all proofs from this section.

Let λ_C be the Perron–Frobenius eigenvalue of C. Let κ_C be the constants of Lemma 4.6. Given a sequence $(r_i)_i$, define for each $i \geq 1$

$$Q_i = \frac{1}{\kappa_C \lambda_C^{r_{i+1}}} N_{r_{i+1}} N_{r_i}.$$

Lemma 7.1. *Given a sequence $(r_i)_{i\geq 1}$ of positive integers such that $r_{i+1}-r_i \geq i$. Then for all $i \geq 1$, the sequence of matrices $\{Q_{i+2k} \cdots Q_{i+2} Q_i\}_{k=0}^{\infty}$ converges to a matrix Z_i. Furthermore, $\lim_{i\to\infty} Z_i = Z$, where Z is the idempotent matrix of Lemma 4.6.*

Corollary 7.2. *If $(r_i)_{i\geq 1}$ is a positive sequence with $r_{i+1}-r_i \geq i$, then $\mathbb{P}\mathcal{D}((\tau'_i)_i)$, and hence $\mathbb{P}\mathcal{D}(T)$, is a 1-simplex.*

8. Nonuniquely ergodic tree

In this section, we relate the legal lamination Λ associated to the unfolding sequence $(\tau_i)_i$ defined in Section 6 and the limiting tree T of the folding sequence $(\tau'_i)_i$ defined in Section 5, to show that T is not uniquely ergodic.

Recall the automorphism $\Phi_i = \phi_{r_1} \circ \cdots \phi_{r_i}$, with $\Phi_0 = \mathrm{id}$. We also use Φ_i to denote the induced graph map from τ_i to τ_0. If each τ_i and τ'_i as a marked graph is the rose labeled

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by $\{a_i,b_i,c_i,d_i,e_i,f_i,g_i\}$, then x_i is represented by $\Phi_i(x)$ for $x \in \{a,b,c,d,e,f,g\}$ as a word in $\mathbb{F}_7 = \langle a,b,c,d,e,f,g\rangle = \pi_1(\tau_0) = \pi_1(\tau'_0)$. We denote x_0 as above simply by x.

Lemma 8.1. *If $(r_i)_{i\geq 1}$ is positive, then for any length measure $(\lambda_i)_i \in \mathcal{D}((\tau'_i)_i)$, the λ_i-volume of τ'_i goes to 0 as $i \to \infty$.*

Proof. The composition $\psi_{r_i}\psi_{r_{i-1}}\psi_{r_{i-2}} : \tau'_{i-3} \to \tau'_i$ has the property that the preimage of every point of τ'_i consists of at least two (in fact, many more) points of τ'_{i-3}, and so the λ_i-volume of τ'_i is at most half of the λ_{i-3}-volume of τ'_{i-3}. □

Lemma 8.2. *Suppose $(r_i)_{i\geq 1}$ is positive. Let Λ be the legal lamination of the unfolding sequence $(\tau_i)_i$. Then every leaf in Λ is obtained as a limit of a sequence $\{\Phi_i(w)\}_i$, where w is a legal word in τ_0 of length at most two in $\{a,b,c,d,e,f,g\}$ and their inverses. Moreover, w can be closed up to a legal loop which is a cyclic word of length ≤ 3.*

Proof. Let l be a leaf of Λ realized as a bi-infinite line in τ_0, and let s be any subsegment of l, with combinatorial edge length $\ell_s > 0$ in τ_0. By definition, for every i there is a bi-infinite legal path l_i in τ_i such that $l = \Phi_i(l_i)$. Let $i = i(s) \geq 0$ such that the edge length of x_i in τ_0 under the graph map Φ_i is $\geq \ell_s$ for all $x \in \{a,b,c,d,e,f,g\}$. Thus, there is a segment s_i of l_i of combinatorial length at most two in $\{a_i,b_i,c_i,d_i,e_i,f_i,g_i\}$ such that $s \subset \Phi_i(s_i)$ (here, Φ_i is a graph map). Now, if $s_i = x_i y_i$ for $x,y \in \{a,b,c,d,e,f,g\}$, take $w = xy$. Thus, we see that $\Phi_i(w)$ (here, Φ_i is an automorphism) covers s in τ_0. Since this is true for any segment of l, we conclude the lemma by taking a nested sequence of subsegments of l with edge length in τ_0 going to infinity. The fact that legal paths of length ≤ 2 can be closed up to legal loops of length ≤ 3 follows from the description of the train track in Lemma 4.3. □

Recall that if $(\tau'_i)_i$ converges to an arational tree T, then we can identify $\mathcal{D}((\tau'_i)_i)$ with $\mathcal{D}(T)$ by Proposition 3.1.

Lemma 8.3. *Suppose $(r_i)_{i\geq 1}$ is positive and that the folding sequence $(\tau'_i)_i$ converges to an arational tree T. Let w be any conjugacy class in $\mathbb{F}_7$ represented by a cyclic word in $\{a,b,c,d,e,f,g\}$ and their inverses, and let $\lambda \in \mathcal{D}(T)$ correspond to a length measure $(\lambda_i)_i \in \mathcal{D}((\tau'_i)_i)$. Then*

$$\lim_{i\to\infty} \|\Phi_i(w)\|_{(T,\lambda)} = 0.$$

Proof. Under the isomorphism from $\mathcal{D}((\tau'_i)_i) \to \mathcal{D}(T)$ that maps $(\lambda_i)_i \mapsto \lambda$, the sequence $(\tau'_i, \lambda_i) \subset \mathrm{cv}_7$ also converges to $(T,\lambda) \in \partial\mathrm{cv}_7$. Thus, for any $x \in \mathbb{F}_7$,

$$\|x\|_{(T,\lambda)} = \lim_{i\to\infty} \|x\|_{(\tau'_i,\lambda_i)}.$$

In fact, the sequence $\|x\|_{(\tau'_i,\lambda_i)}$ is monotonically nonincreasing. Recall that τ'_0 as a marked graph is the rose labeled by $\{a,b,c,d,e,f,g\}$. Represent w by a loop c_w in τ'_0. The graph τ'_i

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is the rose labeled by $\{\Phi_i(a),\dots,\Phi_i(g)\}$. Thus, the loop c_w in τ_i' represents the conjugacy class $\Phi_i(w)$. This shows

$$\|\Phi_i(w)\|_{(T,\lambda)} \leq \|\Phi_i(w)\|_{(\tau_i',\lambda_i)} \leq \|w\|_{\text{word}}\,\text{vol}(\tau_i',\lambda_i)$$

where $\|w\|_{\text{word}}$ is the word length of w. By Lemma 8.1, the last term goes to 0. □

We now come to the main statement of this section.

Proposition 8.4. *Suppose $(r_i)_{i\geq 1}$ is positive and that the folding sequence $(\tau_i')_i$ converges to an arational tree T. Let $\tilde{\Lambda}$ be the lamination corresponding to the legal lamination Λ of the unfolding sequence$(\tau_i)_i$, and let $L(T)$ be the lamination dual to T. Then $\tilde{\Lambda}\subseteq L(T)$. In particular, if T is nongeometric, then Curr$(\Lambda) =$ Curr(T).*

Proof. Recall by Lemma 2.4, the lamination dual to an arational tree is independent of the length measure on the tree. So fix an arbitrary length measure $\lambda \in \mathcal{D}(T)$ on T.

Let W_3 be the set of legal loops of length at most three in $\{a,b,c,d,e,f,g\}$ and their inverses. By Lemma 8.3, for every $\epsilon > 0$, there exists $I_\epsilon > 0$ such that for all $i \geq I_\epsilon$, $\left\|\Phi_i(w)_{(T,\lambda)}\right\| < \epsilon$, for every $w \in W_3$. Then the bi-infinite line $(\Phi_i(w)^{-\infty},\Phi_i(w)^{\infty})$ is in $L_\epsilon(T)$ for all $i \geq I_\epsilon$. Therefore,

$$\bigcap_{\epsilon>0}\overline{\bigcup_{\substack{w\in W_3\\ i\geq I_\epsilon}}(\Phi_i(w)^{-\infty},\Phi_i(w)^{\infty})} \subseteq \bigcap_{\epsilon>0} L_\epsilon(T).$$

By Lemma 8.2, we conclude that $\tilde{\Lambda}\subseteq L(T)$.

If T is nongeometric and arational, then it is freely indecomposable by [Rey12]. By [CHR15, Corollary 1.4], Curr$(\Lambda) =$ Curr(T). □

The following is the consequence of Proposition 8.4 and Corollary 6.4.

Corollary 8.5. *For a positive sequence $(r_i)_{i\geq 1}$ of integers with $r_{i+1}-r_i \geq i$, if the folding sequence $(\tau_i')_i$ converges to a nongeometric arational tree T, then $\mathbb{P}$Curr(T) is a 1-simplex. In particular, T is not uniquely ergodic.*

9. Nonconvergence of unfolding sequence

In this section, fix a sequence $(r_i)_{i\geq 1}$ such that $r_{i+1}-r_i \geq i$. We will show that the corresponding unfolding sequence $(\tau_i)_i$ does not converge to a unique point in ∂CV_7. In fact, we will show in Corollary 9.3 that it converges to a 1-simplex in ∂CV_7.

Recall the folding and unfolding sequences $(\tau_i')_i$ and $(\tau_i)_i$, respectively, from Section 4.

$$\cdots \longrightarrow \tau_{i+1} \xrightarrow{\phi_{r_{i+1}}} \tau_i \xrightarrow{\phi_{r_i}} \tau_{r_{i-1}} \longrightarrow \cdots \xrightarrow{\phi_{r_3}} \tau_2 \xrightarrow{\phi_{r_2}} \tau_1 \xrightarrow{\phi_{r_1}} \tau_0,$$

$$\cdots \longleftarrow \tau_{i+1}' \xleftarrow{\psi_{r_{i+1}}} \tau_i' \xleftarrow{\psi_{r_i}} \tau_{i-1}' \longleftarrow \cdots \xleftarrow{\psi_{r_3}} \tau_2' \xleftarrow{\psi_{r_2}} \tau_1' \xleftarrow{\psi_{r_1}} \tau_0',$$

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Here, τ_i and τ_i' as marked graphs belong to the same simplex in CV_7. Also, recall the matrices defined for all $i \geq 0$

$$P_i = \frac{1}{\kappa_B \lambda_B^{r_{i+1}}} M_{r_i} M_{r_{i+1}} \qquad \text{and} \qquad Q_i = \frac{1}{\kappa_C \lambda_C^{r_{i+1}}} N_{r_{i+1}} N_{r_i},$$

and the existence of the limiting matrices from Lemma 6.3 and Lemma 7.1

$$Y_i = \lim_{k \to \infty} P_i P_{i+2} \cdots P_{i+2k} \qquad \text{and} \qquad Z_i = \lim_{k \to \infty} Q_{i+2k} \cdots Q_{i+2} Q_i.$$

For all even $2m \geq 0$,

$$c_{2m} = \left(\kappa_B^m \lambda_B^{r_2} \lambda_B^{r_4} \cdots \lambda_B^{r_{2m}}\right)\left(\kappa_C^m \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{2m}}\right).$$

Similarly, for all odd $2m+1 \geq 1$, set

$$c_{2m+1} = \left(\kappa_B^m \lambda_B^{r_1} \lambda_B^{r_3} \cdots \lambda_B^{r_{2m+1}}\right)\left(\kappa_C^m \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{2m+1}}\right).$$

Let $\ell = \ell_0 \in \mathbb{R}^{|E\tau_0|}$ be a positive length vector on τ_0. Then ℓ determines a length vector ℓ_i on each τ_i given by $\ell_i = M_{r_i}^T \ldots M_{r_1}^T \ell \in \mathbb{R}^{|E\tau_i|}$. We set $\ell_e^T = \ell^T Y_1$ and $\ell_o^T = \ell^T \frac{M_{r_1}}{\lambda_B^{r_1}} Y_2$. Note that both ℓ_e and ℓ_o are positive vectors. For ℓ_e, this follows since ℓ is a positive vector and Y_1 is a nonnegative matrix. Similarly, $\ell^T M_{r_1}$ is positive and Y_2 is nonnegative, so ℓ_o is also positive.

We will show the sequence $(\tau_i, \ell_i)_i \subset \mathrm{CV}_7$, up to rescaling, does not have a unique limit in $\partial\mathrm{CV}_7$. We start by showing the even sequence and the odd sequence do converge, up to scaling. More precisely:

Lemma 9.1. *For any positive length vector $\ell = \ell_0$ on τ_0, the corresponding even sequence $\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)$ and odd sequence $\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)$ of metric graphs converge to two points T_e and T_o, respectively, in ∂CV_7. In fact, for any conjugacy class $x \in \mathbb{F}_7$, there exists an index $i_x \geq 0$, a vector $v_x \in \mathbb{R}^{|E\tau_{i_x}'|}$ and matrices Y_x^e and Y_x^o such that*

$$\|x\|_{T_e} = \ell_e^T Y_x^e v_x \qquad \text{and} \qquad \|x\|_{T_o} = \ell_o^T Y_x^o v_x.$$

Proof. Let $x \in \mathbb{F}_7$ be a cyclically reduced representative of its conjugacy class. By Lemma 4.8, there exists $i \geq 0$ such that x is legal in τ_i'. Let i_x be the smallest index among such i. Then we can represent x by a vector v_x in $\mathbb{R}^{|E\tau_{i_x}'|}$ and by the vector $N_{r_i} \ldots N_{i_x+1} v_x$ in $\mathbb{R}^{|E\tau_i|}$ for $i \geq i_x$. Thus, for all $i \geq i_x$, we have

$$\|x\|_{(\tau_i, \ell_i)} = \left(\ell^T M_{r_1} \cdots M_{r_i}\right)\left(N_{r_i} \cdots N_{i_x+1} v_x\right).$$

If i_x is even, then write $i_x = 2m_x$, and set

$$c_x^e = \kappa_C^{m_x} \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{i_x}} \qquad \text{and} \qquad c_x^o = \kappa_C^{m_x} \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{i_x - 1}}.$$

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If i_x is odd, then write $i_x = 2m_x + 1$, and set

$$c_x^e = \kappa_C^{m_x+1} \lambda_C^{r_2} \lambda_C^{r_4} \cdots \lambda_C^{r_{i_x-1}} \qquad \text{and} \qquad c_x^o = \kappa_C^{m_x} \lambda_C^{r_1} \lambda_C^{r_3} \cdots \lambda_C^{r_{i_x}}.$$

First, suppose i_x is even. Then for all even $2m \geq i_x$, we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} &= \frac{\|x\|_{(\tau_{2m}, \ell_{2m})}}{c_{2m}} \\ &= \frac{\ell^T \Big(P_1 P_3 \cdots P_{2m-1}\Big) \Big(Q_{2m-1} \cdots Q_{i_x+3} Q_{i_x+1}\Big) v_x}{c_x^e} \\ &\xrightarrow{m\to\infty} \frac{\ell^T Y_1 Z_{i_x+1} v_x}{c_x^e} = \ell_e^T \left(\frac{Z_{i_x+1}}{c_x^e}\right) v_x \end{aligned}$$

and for odd $2m+1 \geq i_x$, we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)} &= \frac{\|x\|_{(\tau_{2m+1}, \ell_{2m+1})}}{c_{2m+1}} \\ &= \frac{\ell^T \frac{M_{r_1}}{\lambda_B^{r_1}} \Big(P_2 P_4 \cdots P_{2m}\Big) \Big(Q_{2m} \cdots Q_{i_x+4} Q_{i_x+2}\Big) N_{i_x+1} v_x}{c_x^o \lambda_C^{r_{i_x+1}}} \\ &\xrightarrow{m\to\infty} \frac{\ell^T \frac{M_{r_1}}{\lambda_B^{r_1}} Y_2 Z_{i_x+2} N_{i_x+1} v_x}{c_x^o} = \ell_o^T \left(\frac{Z_{i_x+2}}{c_x^o} \frac{N_{i_x+1}}{\lambda_C^{i_x+1}}\right) v_x. \end{aligned}$$

Now, suppose i_x is odd. Then for all even $2m \geq i_x$, we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} &= \frac{\ell^T \Big(P_1 P_3 \cdots P_{2m-1}\Big) \Big(Q_{2m-1} \cdots Q_{i_x+3}\Big) N_{r_{i_x+1}} v_x}{c_x^e \lambda_C^{r_{i_x+1}}} \\ &\xrightarrow{m\to\infty} \ell_e^T \left(\frac{Z_{i_x+2}}{c_x^e} \frac{N_{r_{i_x+1}}}{\lambda_C^{r_{i_x+1}}}\right) v_x, \end{aligned}$$

and for odd $2m+1 \geq i_x$, we have

$$\begin{aligned} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)} &= \frac{\ell^T \frac{M_{r_1}}{\lambda_B^{r_1}} \Big(P_2 P_4 \cdots P_{2m}\Big) \Big(Q_{2m} \cdots Q_{i_x+3} Q_{i_x+1}\Big) v_x}{c_x^o} \\ &\xrightarrow{m\to\infty} \ell_o^T \left(\frac{Z_{i_x+1}}{c_x^o}\right) v_x. \end{aligned}$$

Either way, for any conjugacy class x in $\mathbb{F}_7$, both

$$\|x\|_{T_e} = \lim_{m\to\infty} \|x\|_{\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)} \qquad \text{and} \qquad \|x\|_{T_o} = \lim_{m\to\infty} \|x\|_{\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)}$$

are well defined and have the desired form. □

We now want to show T_e and T_o are not scalar multiples of each other. In fact, the following lemma will allow us to show that T_e and T_o are the extreme points of the simplex $\mathbb{P}\mathcal{D}(T)$.

Lemma 9.2. *There exist two sequences α_i and β_i of conjugacy classes of elements of $\mathbb{F}_7$ such that the following holds. For any positive length vector $\ell = \ell_0$ on τ_0, let T_e and T_o be the respective limiting trees in ∂CV_7 for $\left(\tau_{2m}, \frac{\ell_{2m}}{c_{2m}}\right)$ and $\left(\tau_{2m+1}, \frac{\ell_{2m+1}}{c_{2m+1}}\right)$. Then*

$$\frac{||\alpha_i||_{T_o}}{||\alpha_i||_{T_e}} \xrightarrow{i\to\infty} \infty, \qquad \textit{and} \qquad \frac{||\beta_i||_{T_o}}{||\beta_i||_{T_e}} \xrightarrow{i\to\infty} 0.$$

Proof. Take the letter $e \in \mathbb{F}_7$ and recall the automorphisms Φ_i used to define the folding and unfolding sequences. Set $x_i = \Phi_i(e)$. For each i, x_i is legal in τ_i' and is represented by the vector $e_5 = (0,0,0,0,1,0,0)^T$ in τ_i'.

Using notation from Lemma 9.1, set $c_i^e = c_{x_i}^e$ and $c_i^o = c_{x_i}^e$. Note here i is the smallest index such that x_i is legal in τ_i'. We compare the ratio of c_i^o and c_i^e. Since $r_{i+1} - r_i \to \infty$, we have

$$\frac{c_{2i}^e}{c_{2i}^o} = \frac{\lambda_C^{r_2} \cdots \lambda_C^{r_{2i}}}{\lambda_C^{r_1} \cdots \lambda_C^{r_{2i-1}}} \xrightarrow{i\to\infty} \infty, \qquad \text{while} \qquad \frac{c_{2i+1}^e}{c_{2i+1}^o} = \frac{\kappa_C}{\lambda_C^{r_1}} \frac{\lambda_C^{r_2} \cdots \lambda_C^{r_{2i}}}{\lambda_C^{r_3} \cdots \lambda_C^{r_{2i+1}}} \xrightarrow{i\to\infty} 0.$$

Recall that both ℓ_e and ℓ_o are positive and by Lemma 7.1 the sequence Z_i converges to Z. Since Ze_5 is the zero vector, by continuity of the dot product,

$$\lim_{i\to\infty} \ell_e^T Z_{2i+1} e_5 = \ell_e^T Z e_5 = 0 \qquad \text{and} \qquad \lim_{i\to\infty} \ell_o^T Z_{2i+1} e_5 = \ell_o^T Z e_5 = 0.$$

Next, let $N_\infty = \lim_{i\to\infty} \frac{N_{r_i}}{\lambda_C^{r_i}}$ and recall by Lemma 4.7 that the vector $ZN_\infty e_5 = (\star, \star, \star, 0,0,0,0)$ is nonnegative. Thus, there are positive constants A and B such that

$$\lim_{i\to\infty} \ell_e^T \left(Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 = \ell_e^T Z N_\infty e_5 = A > 0,$$

and

$$\lim_{i\to\infty} \ell_o^T \left(Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5 = \ell_o^T Z N_\infty e_5 = B > 0.$$

Combining the above observations and the formulas for length of x_i in T_e and T_o obtained in Lemma 9.1 we get

$$\frac{\|x_{2i}\|_{T_o}}{\|x_{2i}\|_{T_e}} = \frac{\ell_o^T \left(Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5}{\ell_e^T \left(Z_{2i+1} \right) e_5} \frac{c_{2i}^e}{c_{2i}^o} \quad \xrightarrow{i\to\infty} \quad \frac{A}{0} \cdot \infty$$

$$\frac{\|x_{2i+1}\|_{T_o}}{\|x_{2i+1}\|_{T_e}} = \frac{\ell_o^T \left(Z_{2i+1} \right) e_5}{\ell_e^T \left(Z_{2i+2} \frac{N_{r_{2i+1}}}{\lambda_C^{r_{2i+1}}} \right) e_5} \frac{c_{2i+1}^e}{c_{2i+1}^o} \quad \xrightarrow{i\to\infty} \quad \frac{0}{B} \cdot 0.$$

Setting $\alpha_i = x_{2i}$ and $\beta_i = x_{2i+1}$ finishes the proof. □

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Corollary 9.3. *For a sequence $(r_i)_{i\geq 1}$ with $r_{i+1} - r_i \geq i$, if the folding sequence $(\tau_i')_i$ converges to an arational tree T, then for any positive length vector ℓ_0 on τ_0, the limit set in ∂CV_7 of the rescaled unfolding sequence (τ_i, ℓ_i) is always the 1-simplex $\mathbb{P}\mathcal{D}(T)$.*

Proof. Since the folding $(\tau_i)_i'$ and the unfolding sequence $(\tau_i)_i$ are equal as marked graphs for all $i \geq 0$, no matter the metric, they both visit the same sequence of simplices in CV_7. In particular, they both project to the same quasigeodesic in $\mathcal{FF}_7$. Thus, the two limiting trees T_e and T_o of the even and odd sequences of (τ_i, ℓ_i) are length measures on T.

Recall $\mathbb{P}\mathcal{D}(T)$ is a 1-simplex by Corollary 7.2. If neither T_e nor T_o are the extreme points of this simplex, then there exist constants $c, c' > 0$ such that any $x \in \mathbb{F}_n$,

$$c' \leq \frac{\|x\|_{T_o}}{\|x\|_{T_e}} \leq c.$$

On the other hand, if one of them, say T_o, is an extreme point but T_e is not, then we have a constant $c > 0$ such that for any $x \in \mathbb{F}_n$, $\frac{\|x\|_{T_o}}{\|x\|_{T_e}} \leq c$. In both the cases, we get a contradiction to Lemma 9.2. □

10. Conclusion

Recall $\phi \in \mathrm{Aut}(\mathbb{F}_7)$ is the automorphism

$$a \mapsto b, b \mapsto c, c \mapsto ca, d \mapsto d, e \mapsto e, f \mapsto f, g \mapsto g$$

and $\rho \in \mathrm{Aut}(\mathbb{F}_7)$ is the rotation by four clicks:

$$a \mapsto e, b \mapsto f, c \mapsto g, d \mapsto a, e \mapsto b, f \mapsto c, g \mapsto d.$$

For any integer r, let $\phi_r = \rho\phi^r$. To each sequence $(r_i)_{i\geq 0}$ of positive integers, we have an unfolding sequence $(\tau_i)_i$ with train track map $\phi_{r_i}\colon \tau_i \to \tau_{i-1}$, and a folding sequence $(\tau_i')_i$ with train track map $\phi_{r_i}^{-1}\colon \tau_{i-1}' \to \tau_i$. By the limit set of the unfolding sequence $(\tau_i)_i$ in $\partial\mathrm{CV}_n$ we mean the limit set of (τ_i, ℓ_i) with respect to some (any) positive length vector ℓ_i on τ_i.

Main Theorem. Given a strictly increasing sequence $(r_i)_{i\geq 1}$ satisfying $r_i \equiv i \mod 7$ and $r_i \equiv 0 \mod 3$, then the folding sequence $(\tau_i')_i$ converges to a nongeometric arational tree T.

If $(r_i)_i$ grows fast enough, that is, if $r_{i+1} - r_i \geq i$, then T is both nonuniquely ergometric and nonuniquely ergodic. Both $\mathbb{P}\mathcal{D}(T)$ and $\mathbb{P}\mathrm{Curr}(T)$ are one-dimensional simplices.

Furthermore, the limit set in $\partial\mathrm{CV}_7$ of the unfolding sequence $(\tau_i)_i$ is always the 1-simplex spanned by the two ergodic metrics on T.

Proof. A sequence as in the statement exists by the Chinese remainder theorem. The first statement follows from Corollary 5.8 and Proposition 5.10. Nonunique ergometricity of T follows from Proposition 3.1 and Corollary 7.2. Nonunique ergodicity of T is Corollary 8.5. Finally, the last statement is Corollary 9.3. □

A. Appendix

A.1. Convergence lemma

Let $\|\cdot\|$ denote the operator norm. Thus, $\|Y\| \geq 1$ for a nontrivial idempotent matrix Y.

Lemma A.1. *Let Y be an idempotent matrix and Δ_i, $i \geq 1$, a sequence of matrices with $\|\Delta_i\| \leq \frac{\epsilon}{2^i}$ for some $\epsilon > 0$. Assume also that $\epsilon\|Y\| \leq 1/2$. Then the infinite product*

$$\prod_{i=1}^{\infty}(Y+\Delta_i)$$

converges to a matrix X with $\|X-Y\| \leq 2\epsilon\Big(\|Y\|+\|Y\|^2\Big)$. Moreover, the kernel of Y is contained in the kernel of X.

Proof. Write

$$Y+\Sigma_k = \prod_{i=1}^{k}(Y+\Delta_i)$$

Then $(Y+\Sigma_k)(Y+\Delta_{k+1}) = Y+\Sigma_{k+1}$ and since $Y^2 = Y$ it follows that

$$\Sigma_{k+1} = Y\Delta_{k+1} + \Sigma_k(Y+\Delta_{k+1}). \tag{1}$$

Multiplying on the right by Y and using $Y^2 = Y$, we get

$$\Sigma_{k+1}Y = Y\Delta_{k+1}Y + \Sigma_k Y + \Sigma_k\Delta_{k+1}Y$$

and applying the norm

$$\|\Sigma_{k+1}Y\| \leq \|\Sigma_k Y\| + \frac{\epsilon}{2^{k+1}}\|Y\|^2 + \|\Sigma_k\|\frac{\epsilon}{2^{k+1}}\|Y\|.$$

By adding these for $k = 1, 2, \cdots, m-1$ and using $\Sigma_1 = \Delta_1$, we have

$$\begin{aligned}\|\Sigma_m Y\| &\leq \|\Sigma_1 Y\| + \epsilon\|Y\|^2\left(\frac{1}{4}+\cdots+\frac{1}{2^m}\right) + \epsilon\|Y\|\left(\frac{\|\Sigma_1\|}{4}+\cdots+\frac{\|\Sigma_{m-1}\|}{2^m}\right)\\ &\leq \epsilon\Big(\|Y\|+\|Y\|^2\Big) + \epsilon\|Y\|\left(\frac{\|\Sigma_1\|}{4}+\cdots+\frac{\|\Sigma_{m-1}\|}{2^m}\right).\end{aligned}$$

So the norms of $\Sigma_m Y$ are bounded by norms of Σ_i with $i < m$. From Equation 1, we also see that the norm of Σ_{k+1} is bounded by the norms of $\Sigma_k Y$. Putting this together, we have

$$\begin{aligned}\|\Sigma_{k+1}\| &\leq \|Y\|\|\Delta_{k+1}\| + \|\Sigma_k Y\| + \|\Sigma_k\|\|\Delta_{k+1}\|\\ &\leq \frac{\epsilon}{2^{k+1}}\|Y\| + \|\Sigma_k Y\| + \frac{\epsilon}{2^{k+1}}\|\Sigma_k\|\\ &\leq \frac{\epsilon}{2^{k+1}}\|Y\| + \frac{\epsilon}{2}\Big(\|Y\|+\|Y\|^2\Big) + \epsilon\|Y\|\left(\frac{\|\Sigma_1\|}{4}+\cdots+\frac{\|\Sigma_{k-1}\|}{2^k}\right) + \frac{\epsilon}{2^{k+1}}\|\Sigma_k\|\\ &\leq \epsilon\Big(\|Y\|+\|Y\|^2\Big) + \epsilon\|Y\|\left(\frac{\|\Sigma_1\|}{4}+\cdots+\frac{\|\Sigma_{k-1}\|}{2^k}+\frac{\|\Sigma_k\|}{2^{k+1}}\right).\end{aligned}$$

Thus, we have an inequality of the form

$$\|\Sigma_{k+1}\| \leq a + b\left(\frac{\|\Sigma_1\|}{4} + \cdots + \frac{\|\Sigma_k\|}{2^{k+1}}\right)$$

for $a = \epsilon\left(\|Y\| + \|Y\|^2\right)$ and $b = \epsilon\|Y\|$.

Set $c = 2\epsilon\left(\|Y\| + \|Y\|^2\right)$. Then $c \geq \epsilon$, $a \leq c/2$ and $b \leq 1/2$ by assumption. Easy induction then shows for all $k \geq 1$,

$$\|\Sigma_k\| \leq c. \tag{2}$$

This obtains the inequality $\|X - Y\| \leq c$ from the statement, once we establish convergence.

To see convergence, we argue that the sequence of partial products forms a Cauchy sequence. For $1 < k < m$,

$$\prod_{i=1}^{m}(Y + \Delta_i) - \prod_{i=1}^{k}(Y + \Delta_i) = \prod_{i=1}^{k-1}(Y + \Delta_i)\left(\prod_{i=k}^{m}(Y + \Delta_i) - (Y + \Delta_k)\right).$$

By Equation 2, the norm of $\prod_{i=1}^{k-1}(Y + \Delta_i) = Y + \Sigma_{k-1}$ is bounded by $c + \|Y\|$. We can apply the same estimate to the sequence starting with $Y + \Delta_k$ and with ϵ replaced with $\frac{\epsilon}{2^{k-1}}$ to see that

$$\left\|\prod_{i=k}^{m}(Y + \Delta_i) - Y\right\| \leq \frac{2\epsilon(\|Y\| + \|Y\|^2)}{2^{k-1}} \leq \frac{c}{2^{k-1}}$$

and so

$$\left\|\prod_{i=k}^{m}(Y + \Delta_i) - (Y + \Delta_k)\right\| \leq \frac{c}{2^{k-1}} + \frac{1}{2^k}$$

which proves the sequence is Cauchy.

For the second statement, set $X_k = \prod_{i=k}^{\infty}(Y + \Delta_i)$ for $k \geq 1$. By the same estimate as above with ϵ replaced with $\frac{\epsilon}{2^{k-1}}$, we know that X_k exists and

$$\|X_k - Y\| \leq \frac{2\epsilon}{2^{k-1}}\left(\|Y\| + \|Y\|^2\right) = \frac{c}{2^{k-1}}.$$

By definition, $X = (Y + \Sigma_k)X_{k+1}$. Suppose v is a unit vector with $Yv = 0$. Then

$$\begin{aligned} \|Xv\| &\leq \|Y + \Sigma_k\|\,\|X_{k+1}v\| \\ &= \|Y + \Sigma_k\|\,\|X_{k+1}v - Yv\| \\ &\leq \|Y + \Sigma_k\|\,\|X_{k+1} - Y\| \\ &\leq (\|Y\| + c)\frac{c}{2^k}. \end{aligned}$$

Since this is true for all $k \geq 0$, letting $k \to \infty$ yields $Xv = 0$. □

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A.2. Sage code

The following is the Sage code used to check Lemma 4.2, Lemma 4.3 and Lemma 4.4.

```
from train_track import*

#Lemma 4.2

A=AlphabetWithInverses(['a','b','c'])
F3=FreeGroup(A)
theta = FreeGroupAutomorphism('a->b,b->c,c->ca')
vartheta = theta.inverse()
theta_tt = theta.train_track()
vartheta_tt = vartheta.train_track()
print("================= theta/vartheta ====================")

print("----------theta----------")
print(theta_tt)
print("gates:", theta_tt.gates(0))
print("INP:", theta_tt.indivisible_nielsen_paths())
print("pNp:", theta_tt.periodic_nielsen_paths())

print("----------vartheta----------")
print(vartheta_tt)
print("gates:", vartheta_tt.gates(0))
print("INP:", vartheta_tt.indivisible_nielsen_paths())
print("pNp:", vartheta_tt.periodic_nielsen_paths())

#Lemma 4.3

A=AlphabetWithInverses(['a','b','c','d','e','f','g'])
F=FreeGroup(A)
print("================= phi_r =============================")
phi = FreeGroupAutomorphism('a->b,b->c,c->ca,d->d,e->e,f->f,g->g')
rho = FreeGroupAutomorphism('a->e,b->f,c->g,d->a,e->b,f->c,g->d')

for r in range(3,6):
  phi_r=rho*phi^r
  phi_r_tt = phi_r.train_track()
  print("----------r=", r, "----------")
  print(phi_r_tt)
  print(phi_r_tt.gates(0))

#Lemma 4.4

A=AlphabetWithInverses(['a','b','c','d','e','f','g'])
F=FreeGroup(A)
print("================= psi_r =============================")

for r in range(3,6):
  psi_r=phi.inverse()^r*rho.inverse()
  psi_r_tt = psi_r.train_track()
  print("----------r=", r, "----------")
  print(psi_r_tt)
  print(psi_r_tt.gates(0))
```

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