European Journal of Applied Mathematics (2024), **35**, pp. 409–429
doi:10.1017/S0956792523000268

PAPER

Generalised solution to a 2D parabolic-parabolic chemotaxis system for urban crime: Global existence and large-time behaviour

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Received: 13 October 2022; **Revised:** 06 August 2023; **Accepted:** 21 August 2023; **First published online:** 25 September 2023

Keywords: Chemotaxis system; generalised solution; global existence; asymptotic behaviour

2020 Mathematics Subject Classification: 35Q91 (Primary); 35K55, 35D99, 35B65, 35B35 (Secondary)

Abstract
We consider a parabolic-parabolic chemotaxis system with singular chemotactic sensitivity and source functions, which is originally introduced by Short et al to model the spatio-temporal behaviour of urban criminal activities with the particular value of the chemotactic sensitivity parameter $\chi = 2$. The available analytical findings for this urban crime model including $\chi = 2$ are restricted either to one-dimensional setting, or to initial data and source functions with appropriate smallness, or to initial data and source functions with some radial symmetry. In the present work, our first result asserts that for any $\chi > 0$ the initial-boundary value problem of this urban crime model possesses a global generalised solution in the two-dimensional setting, without imposing any small or radial conditions on initial data and source functions. Our second result presents the asymptotic behaviour of such solution, under some additional assumptions on source functions.

1. Introduction and main results

This paper is concerned with a class of chemotaxis systems with singular sensitivity of the following form

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + uv + h_2, & x \in \Omega, \ t > 0, \end{cases} \tag{1}$$

in a bounded and smooth spatial domain $\Omega \subset \mathbb{R}^n$, with the parameters $\chi > 0$ and $\kappa > 0$, which is proposed in [33] to describe the propagation of urban criminal activities with the particular value $\chi = 2$. Here, $u(x, t)$ and $v(x, t)$ represent the density of criminals and an abstract so-called attractiveness value at location x and time t, respectively; the given source function h_1 denotes the introduction of criminal agents into the system, and the given function h_2 describes the density of additional criminals, which may exist even in the absence of any criminal agents. We refer to [32, 34] for more details on (1), to [3, 5, 13, 26, 35, 38, 49] for further developments of crime models and to [4, 9] for more comprehensive reviews of related works.

To elucidate our motivation, we first recall some analytical progress related to the system (1). For the classical solvability of the initial-boundary value problem, the local existence and uniqueness of solutions has been obtained in [28] and the global existence of solutions has been established provided that either $n = 1$ [29, 41] or $n \geq 2$ with the additional condition that $\chi < \frac{2}{n}$ [11, 31], or the initial data and the given functions h_1 and h_2 are assumed to be small [1, 37]. It has been shown that these restrictions can be relaxed to $\chi > 0$ when $n = 2$ [44] or to $\chi \in (0, \sqrt{3})$ when $n = 3$ [15], in the sense of radial renormalised solvability. In addition, considering interacting individuals may have attempted to avoid competition, the model corresponding to (1) with the linear criminal diffusion (i.e. Δu) replaced by the nonlinear criminal diffusion (i.e. $\nabla \cdot \left(u^{m-1}\nabla u\right)$ with some $m > 0$) also has been studied by researchers. For this nonlinear model, it is known that its two-dimensional initial-boundary value problem admits a global bounded weak solution provided that either $m > \frac{3}{2}$ [30] or $m > 1$ and $\chi < \frac{\sqrt{3}}{2}$ [47]. On the other hand, if the logistic source term, that is, $au - bu^\alpha$ with $a \in \mathbb{R}$ and $b > 0$, is incorporated into (1), then, the resulting system has a global generalised solution for $n = 2$ and $\alpha = 2$ [14], which is eventually smooth ([27]), and possesses a global classical solution for either $n = 2, 3$ and $\alpha > 2$ [14, 40] or $n \geq 4$ and $\alpha > \frac{n}{4} + 1$ [40]. When $\kappa = 0$, $h_1 \equiv 0$, $h_2 \equiv 0$ and u_t is replaced by τu_t with $\tau \in (0, 1)$, the model (1) arrives at a reduced crime model considered in [12], where a statement on spontaneous emergence of arbitrarily large values of $\|u(\cdot, t)\|_{L^q}$ with $q > \frac{n}{2}$ has been derived for $\chi > 0$ and $n \geq 3$. As to long-time behaviours of solutions of (1), for $\chi < \frac{2}{n}$ with $n \geq 2$ the asymptotic stability of constant steady states has been considered in [31], provided that $h_1 \equiv const.$ and $h_2 \equiv const.$ with certain smallness; under the assumptions that $h_1 \to 0$ and $h_2 \to h_{2,\infty}$ in some sense as $t \to \infty$, the convergence results $(u, v) \to (0, v_\infty)$ have been studied in some appropriate senses in [1, 15, 29, 37, 44], where v_∞ denotes the solution to the boundary value problem

$$\begin{cases} -\Delta v_\infty + v_\infty = h_{2,\infty}, & x \in \Omega, \\ \nabla v_\infty \cdot \nu = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Furthermore, we refer to [6, 8, 16, 23–25, 32, 39] for the mathematical analytical work on related stationary problems, which reveal the possibility of stably spatial heterogeneous behaviour resembling crime hotspot formation and support that the system (1) is adequate to describe the formation of crime hotspots encountered in reality.

Compared the analytical results on (1) mentioned above, we find that the range of the parameter χ which guarantees the global existence of classical solutions of the system (1) seems to become larger when either the spatial domain is one or in multidimensional settings, the initial data and the source functions satisfy some smallness or radial symmetric assumption, or the solution concept is considered under proper generalised frameworks. Thus, we are wondering whether or not there exists an appropriate generalised framework within which for any $\chi > 0$ the corresponding $n(\geq 2)$-dimensional initial-boundary value problem of (1) is solvable without imposing these additional conditions on both the initial data and the source functions h_1 and h_2.

Motivated by this, the first purpose of the present work is to present that for arbitrary $\chi > 0$, the following initial-boundary value problem:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln v) - \kappa uv + h_1, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + uv + h_2, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{3}$$

where ν denotes the exterior normal vector to the boundary $\partial\Omega$, possesses solutions in an appropriate generalised framework in the two-dimensional setting, without imposing any assumption of smallness and radial symmetry of both the initial data and the source functions h_1 and h_2.

Before going further, we shall be precise about the notion of generalised solutions to the initial-boundary value problem (3) considered in this work.

Definition 1. A pair (u, v) is called a global generalised solution to the initial-boundary value problem (3) if for any $T > 0$,

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(1) it holds that

$$\begin{cases} u \in L^1(\Omega \times (0,T)), \quad \nabla \ln (1+u) \in L^2(\Omega \times (0,T)), \\ v \in L^1(\Omega \times (0,T)), \quad \nabla \ln v \in L^2(\Omega \times (0,T)), \\ uv \in L^1(\Omega \times (0,T)), \\ u(x,t) \geq 0, \quad v(x,t) > 0, \quad a.e. \text{ in } \Omega \times [0,T]; \end{cases} \tag{4}$$

(2) it holds that

$$\int_\Omega u(\cdot,t)dx + \kappa \int_0^t \int_\Omega uvdxds \leq \int_\Omega u_0 dx + \int_0^t \int_\Omega h_1 dxds, \quad a.e. \text{ in } [0,T], \tag{5}$$

$$\int_\Omega (u+\kappa v)(\cdot,t)dx + \kappa \int_0^t \int_\Omega vdxds \leq \int_\Omega u_0 + \kappa v_0 dx + \int_0^t \int_\Omega h_1 + \kappa h_2 dxds, \quad a.e. \text{ in } [0,T]; \tag{6}$$

(3) it holds that for some $p, q > 0$ and each non-negative $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$

$$\begin{aligned} &-\int_\Omega (u_0+1)^{-p} v_0^{-q} \varphi|_{t=0} dx - \int_0^T \int_\Omega (u+1)^{-p} v^{-q} \varphi_t dxdt \\ &\leq -p(p+1) \int_0^T \int_\Omega (u+1)^{-p-2} v^{-q} \varphi |\nabla u|^2 dxdt \\ &\quad + \int_0^T \int_\Omega \left(p(p+1)\chi \frac{u}{u+1} - 2pq \right) (u+1)^{-p-1} v^{-q-1} \varphi \nabla u \cdot \nabla v dxdt \\ &\quad + \int_0^T \int_\Omega \left(\chi pq \frac{u}{u+1} - q(q+1) \right) (u+1)^{-p} v^{-q-2} \varphi |\nabla v|^2 dxdt \\ &\quad - p \int_0^T \int_\Omega (u+1)^{-p-1} v^{-q} \left(-\kappa uv + h_1\right) \varphi dxdt - q \int_0^T \int_\Omega (u+1)^{-p} v^{-q-1} \left(-v + uv + h_2\right) \varphi dxdt \\ &\quad + p \int_0^T \int_\Omega (u+1)^{-p-1} v^{-q} \nabla u \cdot \nabla \varphi dxdt - p\chi \int_0^T \int_\Omega (u+1)^{-p-1} v^{-q-1} u \nabla v \cdot \nabla \varphi dxdt \\ &\quad + q \int_0^T \int_\Omega (u+1)^{-p} v^{-q-1} \nabla v \cdot \nabla \varphi dxdt; \end{aligned} \tag{7}$$

(4) it holds that for each non-negative $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$

$$\int_0^T \int_\Omega \left(-\ln v \varphi_t - \frac{|\nabla v|^2 \varphi}{v^2} + \frac{\nabla v \cdot \nabla \varphi}{v} + \varphi - u\varphi - \frac{\varphi h_2}{v} \right) dxdt \geq \int_\Omega \ln v_0 \varphi|_{t=0} dx. \tag{8}$$

We here give a remark on the constants p and q appeared in (7).

Remark 1. For given $\chi > 0$, the admissible (p,q) is that $p \geq 1$ and $q \geq 2$, fulfilling that

$$\frac{p(p+1)\chi^2}{4} - q - \frac{q^2}{p+1} < 0, \tag{9}$$

in our subsequent analyses (see Lemma 3.6 below).

To state the first result on the global existence of such generalised solutions, the initial data (u_0, v_0) are throughout assumed to satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \quad \text{with} \quad u_0 \geq 0 \quad \text{and} \quad u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\overline{\Omega}) \quad \text{with} \quad \inf\limits_{x \in \overline{\Omega}} v_0 > 0, \end{cases} \tag{10}$$

and the source functions h_1 and h_2 are supposed to fulfil

$$0 \leq h_i \in C^1(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)), \quad i = 1,2. \tag{11}$$

With Definition 1 and the assumptions (10)–(11) at hand, the first result reads as follows.

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Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth spatial domain, and (10)–(11) hold. Then, for any $\chi > 0$, the initial-boundary value problem (3) possesses at least one global generalised solution in the sense of Definition 1.*

Remark 2. By a slight adaptation of [45, Lemma 2.1], we can show that if

$$u \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \quad \text{and} \quad v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$$

such that $u \geq 0$ and $v > 0$ in $\overline{\Omega} \times (0, T)$ and (u, v) is a solution of (3) in the sense of Definition 1, then (u, v) also solves (3) in the classical sense. This also supports the interest in our concept of generalised solutions.

Going beyond the global existence statement, we naturally focus on the large-time behaviour of generalised solutions. To achieve it, we additionally assume that

$$\inf_{t>0} \int_\Omega h_2(x, t)dx > 0, \tag{12}$$

$$\int_t^{t+1} \int_\Omega h_1(\cdot, s)dxds \to 0, \quad \text{as } t \to \infty, \tag{13}$$

$$\int_t^{t+1} \int_\Omega |h_2(\cdot, s) - h_{2,\infty}(\cdot)|dxds \to 0, \quad \text{as } t \to \infty \tag{14}$$

with some $h_{2,\infty} \in C^1(\overline{\Omega})$. The second result on the asymptotic behaviour of the generalised solution established in Theorem 1.1 can be stated as follows.

Theorem 1.2. *Let all assumptions in Theorem 1.1 be satisfied, and let (12)–(14) be fulfilled. Under the additional assumption that Ω is convex, for the global generalised solution of the initial-boundary value problem (3) taken from Theorem 1.1, there exists a null set $\mathcal{N} \subset (0, \infty)$ such that*

$$\int_\Omega u(\cdot, t)dx + \int_t^{t+1} \int_\Omega |v(\cdot, s) - v_\infty(\cdot)|dxds \to 0, \quad \text{as } (0, \infty) \setminus \mathcal{N} \ni t \to \infty, \tag{15}$$

where v_∞ denotes the solution of the boundary value problem (2).

Remark 3. In comparison with [20, Theorem 1.2] and [22, Theorem 1.3], where the long-time behaviour of generalised solution to the system that with u instead of uv in the second equation of (3) was obtained in two-dimensional setting and higher dimensional settings, respectively, our result in Theorem 1.2 is weaker due to the presence of the nonlinear production $+uv$; especially, we do not know how to prove the eventual smoothness of the generalised solution established in Theorem 1.1.

1.1. Technical strategy and structure of the article

The first objective of this paper is to present that the initial-boundary value problem (3) possesses a global generalised solution. Usually, to this end, one should seek an appropriate generalised framework and thereby obtain the global existence of generalised solutions via an approximation procedure. Here, our novelty of analysis consists of structuring an appropriate generalised framework, in which the difficulty is to define the solution component v adequately. Although our definition of the solution component u is inspired by the generalised framework introduced in [18, 48] for the logarithmic Keller-Segel system with linear production, our definition of the solution component v is completely different from that in [18, 48] due to the presence of the nonlinear production $+uv$ in the second equation in (3), or, more precisely, we structure the generalised definition of the solution component v by respectively defining the generalised subsolution and supersolution, see (6) and (8) in Definition 1. After this, to get the generalised solution via an appropriate approximation procedure, the key steps are to establish a series of uniform *a priori* estimates, see Lemmas 3.1 and 3.2.

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The second objective of this paper is to perform the large-time behaviour of the generalised solution (u, v) determined in Theorem 1.1, under the additional assumptions (12)–(14). To achieve this, we start to present that for any $\varepsilon \in (0, 1)$

$$\int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon(\cdot, s) dxds \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \varepsilon, \tag{16}$$

see Lemma 4.4. Taking advantage of this and an appropriate approximation procedure invoking the Beppo Levi theorem, for any $t > 0$ we get the key estimate:

$$\int_\Omega |v_\varepsilon - v_\infty|(\cdot, t+1)dx - \int_\Omega |v_\varepsilon - v_\infty|(\cdot, t)dx + \int_t^{t+1} \int_\Omega |v_\varepsilon - v_\infty| dxds$$
$$\leq \int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+1} \int_\Omega |h_2 - h_{2,\infty}| dxds,$$

where v_∞ denotes the solution of the boundary value problem (2), see the proof of Lemma 4.5 for details. In terms of it, setting $z_\varepsilon(t) := \int_t^{t+1} \int_\Omega |v_\varepsilon - v_\infty| dxds$ we have

$$z'_\varepsilon(t) + z_\varepsilon(t) \leq \int_t^{t+1} \int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+1} \int_\Omega |h_2 - h_{2,\infty}| dxds,$$

by which an application of the ODE techniques (see Lemma 4.2) invoking (16) presents the desired decay in Theorem 1.2.

The rest of this paper is arranged as follows. In the following section, we present the global well-posedness for the approximate problem (17). In Section 3, the global existence of generalised solutions to the initial-boundary value problem (3) is established. Section 4 is devoted to showing the large-time behaviour desired in Theorem 1.2 via an appropriate approximation procedure invoking the Beppo Levi theorem.

2. Preliminaries

To construct a generalised solution of the initial-boundary value problem (3) by an approximation procedure, for each $\varepsilon \in (0, 1)$ we shall consider the following approximate problem

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi \nabla \cdot (u_\varepsilon \nabla \ln v_\varepsilon) - \kappa u_\varepsilon v_\varepsilon + h_1, & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + \dfrac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon v_\varepsilon} + h_2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), \quad v_\varepsilon(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{17}$$

An application of the well-known strategy harnessing the contraction mapping principle and the well-known pointwise positivity property of the Neumann heat semigroup, as in [2, 11, 28, 43], ensures the global existence of classical solution to the approximate problems (17).

Lemma 2.1. *Let the assumptions (10)–(11) hold. For each $\varepsilon \in (0, 1)$ and any $\chi > 0$, there exists a unique pair $(u_\varepsilon, v_\varepsilon)$ of positive functions with the properties that for any $T > 0$*

$$\begin{cases} u_\varepsilon \in C^0\big(\overline{\Omega} \times [0, T]\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T]\big), \\ v_\varepsilon \in \bigcap_{p>2} C^0\big([0, T]; W^{1,p}(\overline{\Omega})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T]\big), \end{cases}$$

such that $(u_\varepsilon, v_\varepsilon)$ solves the approximate problem (17) classically in $\Omega \times [0, \infty)$. Moreover, we have

$$v_\varepsilon(\cdot, t) \geq e^{-t} \inf_{x \in \overline{\Omega}} v_0(x), \qquad t > 0 \tag{18}$$

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and

$$\|u_\varepsilon(\cdot,t)\|_{L^1}+\int_0^t\int_\Omega u_\varepsilon v_\varepsilon(\cdot,s)dxds\le C(1+t),\quad t>0, \tag{19}$$

$$\|v_\varepsilon(\cdot,t)\|_{L^1}+\int_0^t\int_\Omega v_\varepsilon(\cdot,s)dxds\le C(1+t),\quad t>0 \tag{20}$$

for some $C>0$, independent of ε.

Proof. Similar to [2, 11, 28, 43], the well-known strategy invoking the contraction mapping principle presents that for each $\varepsilon\in(0,1)$ and any $\chi>0$, there exist a time $T_{\max,\varepsilon}\in(0,\infty]$ and a unique pair $(u_\varepsilon,v_\varepsilon)$ of positive functions with the properties that for any $p>2$

$$\begin{cases} u_\varepsilon\in C^0\big(\overline{\Omega}\times[0,T_{\max,\varepsilon})\big)\cap C^{2,1}\big(\overline{\Omega}\times(0,T_{\max,\varepsilon})\big),\\ v_\varepsilon\in C^0\big([0,T_{\max,\varepsilon});W^{1,p}(\overline{\Omega})\big)\cap C^{2,1}\big(\overline{\Omega}\times(0,T_{\max,\varepsilon})\big)\end{cases}$$

such that $(u_\varepsilon,v_\varepsilon)$ solves the approximate problem (17) classically in $\Omega\times[0,T_{\max,\varepsilon})$. Moreover, if $T_{\max,\varepsilon}<\infty$, then for any $p>2$

$$\limsup_{t\to T_{\max,\varepsilon}}\left(\|u_\varepsilon(\cdot,t)\|_{L^\infty}+\|\nabla v_\varepsilon(\cdot,t)\|_{L^p}+\|v_\varepsilon^{-1}(\cdot,t)\|_{L^\infty}\right)=\infty. \tag{21}$$

To show that $T_{\max,\varepsilon}=\infty$, let us start with the pointwise lower bound for the solution component v_ε. Indeed, we can apply the comparison principle for the Neumann problem associated with the heat equation to the variation-of-constants formula for v_ε, namely

$$v_\varepsilon(\cdot,t)=e^{t(\Delta-1)}v_0+\int_0^t e^{(t-s)(\Delta-1)}\left(\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}+h_2\right)(\cdot,s)ds, \tag{22}$$

and get

$$v_\varepsilon(\cdot,t)\ge e^{t(\Delta-1)}v_0\ge e^{-t}\inf_{x\in\Omega}v_0(x),\quad t\in(0,T_{\max,\varepsilon}), \tag{23}$$

due to the facts that $h_2\ge0$ and $u_\varepsilon,v_\varepsilon>0$. We can also apply the properties of the Neumann heat semigroup (cf. [42, Lemma 1.3], [7, Lemma 2.1]) to (22) and get that for any $q>2$

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q}\le C\|\nabla v_0\|_{L^q}+C\int_0^t\left(1+(t-s)^{-\frac12-(\frac12-\frac1q)}\right)e^{-(t-s)}\left\|\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}+h_2\right\|_{L^2}ds.$$

Since $\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\le\varepsilon^{-1}$ and $-\frac12-(\frac12-\frac1q)=-1+\frac1q>-1$, it follows from (11) that for any $q>2$

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^q}\le C\|\nabla v_0\|_{L^q}+C_\varepsilon,\quad t\in(0,T_{\max,\varepsilon}). \tag{24}$$

We now establish the bound of $\|u_\varepsilon\|_{L^1}$ by integrating the first equation in (17) over Ω

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx+\kappa\int_\Omega u_\varepsilon v_\varepsilon dx=\int_\Omega h_1dx, \tag{25}$$

which, integrating over $[0,t]$, implies that

$$\|u_\varepsilon(\cdot,t)\|_{L^1}+\kappa\int_0^t\int_\Omega u_\varepsilon v_\varepsilon dxds\le\|u_0\|_{L^1}+\|h_1\|_{L^\infty(\Omega\times(0,\infty))}t,\quad t\in(0,T_{\max,\varepsilon}). \tag{26}$$

We next estimate $\|u_\varepsilon\|_{L^\infty}$ by applying the properties of the Neumann heat semigroup (cf. [42, Lemma 1.3], [7, Lemma 2.1]) to the variation-of-constants formula for u_ε, denoted by

$$u_\varepsilon(\cdot,t)=e^{t(\Delta-1)}u_0+\int_0^t e^{(t-s)(\Delta-1)}\left(-\chi\nabla\cdot(u_\varepsilon\nabla\ln v_\varepsilon)-\kappa u_\varepsilon v_\varepsilon+u_\varepsilon+h_1\right)ds, \tag{27}$$

and, due to the maximum principle and the non-negativity of $\kappa u_\varepsilon v_\varepsilon$, conclude that for $r \in (2, q)$

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + C\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{1}{r}}\right) e^{-(t-s)}\|u_\varepsilon \nabla \ln v_\varepsilon\|_{L^r} ds$$
$$+ C\int_0^t \left(1+(t-s)^{-\frac{q-r}{qr}}\right) e^{-(t-s)}\|u_\varepsilon + h_1\|_{L^{\frac{qr}{q-r}}} ds. \tag{28}$$

In view of Hölder's inequality, (23) and (24), we have

$$\|u_\varepsilon \nabla \ln v_\varepsilon\|_{L^r} \le \|u_\varepsilon\|_{L^{\frac{qr}{q-r}}} \|v_\varepsilon^{-1}\|_{L^\infty}\|\nabla v_\varepsilon\|_{L^q} \le C_\varepsilon e^t \|u_\varepsilon\|_{L^{\frac{qr}{q-r}}}, \quad t \in (0, T_{\max,\varepsilon}). \tag{29}$$

By means of the interpolation inequality and (26), we obtain

$$\|u_\varepsilon\|_{L^{\frac{qr}{q-r}}} \le \|u_\varepsilon\|_{L^1}^{\frac{q-r}{qr}} \|u_\varepsilon\|_{L^\infty}^{1-\frac{q-r}{qr}} \le C(1+t^{\frac{q-r}{qr}})\|u_\varepsilon\|_{L^\infty}^{1-\frac{q-r}{qr}}, \quad t \in (0, T_{\max,\varepsilon}). \tag{30}$$

Substituting (29) and (30) into (28), and using (11) we arrive at for $t \in (0, T_{\max,\varepsilon})$

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le C + C_\varepsilon e^t \left(1+t^{\frac{q-r}{qr}}\right)\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{1}{r}} + (t-s)^{-\frac{q-r}{qr}}\right) e^{-(t-s)}\|u_\varepsilon\|_{L^\infty}^{1-\frac{q-r}{qr}} ds.$$

Letting $K(T) := \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty}$ for any $T \in (0, T_{\max,\varepsilon})$, it follows that

$$K(T) \le C + C_\varepsilon e^T \left(1+T^{\frac{1}{2r}}\right) K^{1-\frac{q-r}{qr}}(T).$$

Since $0 < 1 - \frac{q-r}{qr} < 1$, an application of Young's inequality entails that $K(T) \le C_\varepsilon(T)$. Hence for any $T \in (0, T_{\max,\varepsilon})$, we infer that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty} \le C_\varepsilon(T), \quad t \in (0, T).$$

This, combined with (24) and (23), establishes a contradiction to (21) and thereby ensures that actually we must have $T_{\max,\varepsilon} = \infty$.

Finally, using (23) and (26) with $T_{\max,\varepsilon} = \infty$, we can get (18) and (19). After an integration in time, we infer from the second equation in (17) that for any $t > 0$

$$\int_\Omega v_\varepsilon(\cdot,t)dx + \int_0^t\int_\Omega v_\varepsilon(\cdot,s)dxds = \int_\Omega v_0 dx + \int_0^t\int_\Omega \frac{u_\varepsilon v_\varepsilon(\cdot,s)}{1+\varepsilon u_\varepsilon v_\varepsilon(\cdot,s)}dxds + \int_0^t\int_\Omega h_2 dxds, \tag{31}$$

which, together with (11) and (19), ensures (20). □

At the end of this section, we also note a useful consequence of the dominated convergence theorem (see [45, Lemma A.4]).

Lemma 2.2. *Let $M \subset \mathbb{R}^n$ with $n \ge 1$ be measurable, and suppose that $(w_j)_{j\in N} \subset L^\infty(M)$ and $(z_j)_{j\in\mathbb{N}} \subset L^2(M)$ are such that $|w_j| \le C$ in M for all $j \in \mathbb{N}$ and $w_j \to w$ a.e. in M as well as*

$$z_j \to z \quad in \quad L^2(M) \quad \text{as} \quad j \to \infty$$

for some $C > 0$, $w \in L^\infty(M)$, and $z \in L^2(M)$. Then,

$$w_j z_j \to wz \quad \text{in} \quad L^2(M) \quad \text{as} \quad j \to \infty.$$

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3. Global generalised solutions

To construct the global existence of the generalised solution, we will seek some uniform in ε estimates on the approximate solutions $(u_\varepsilon, v_\varepsilon)$ given in Lemma 2.1. To this end, we begin with deriving the spatio-temporal integrability of $\nabla \ln v_\varepsilon$ and some regularity features of the time derivatives.

Lemma 3.1. *Let $(u_\varepsilon, v_\varepsilon)$ be given in Lemma 2.1. For any $T > 0$, there exists $C(T) > 0$, independent of ε, with the property that*

$$\int_0^T \|\ln v_\varepsilon(\cdot, s)\|_{H^1}^2 ds \le C(T), \tag{32}$$

$$\int_0^T \|v_\varepsilon^{-1}(\cdot, s)\|_{H^1}^2 ds \le C(T), \tag{33}$$

$$\int_0^T \|\partial_s \ln v_\varepsilon(\cdot, s)\|_{(H^2)^\star} ds \le C(T), \tag{34}$$

$$\int_0^T \left\|\partial_s v_\varepsilon^{-1}(\cdot, s)\right\|_{(H^2)^\star} ds \le C(T). \tag{35}$$

Proof. Testing the second equation in (17) by $\frac{\varphi}{v_\varepsilon}$ with $\varphi(x) \in C^\infty(\overline{\Omega})$, for $t > 0$ we have

$$\begin{aligned}\int_\Omega \varphi \partial_t \ln v_\varepsilon dx &= \int_\Omega \frac{\varphi}{v_\varepsilon}\left(\Delta v_\varepsilon - v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2\right)\\ &= \int_\Omega \frac{\varphi|\nabla v_\varepsilon|^2}{v_\varepsilon^2}dx - \int_\Omega \frac{\nabla\varphi\cdot\nabla v_\varepsilon}{v_\varepsilon}dx - \int_\Omega \varphi dx + \int_\Omega \frac{u_\varepsilon \varphi}{1+\varepsilon u_\varepsilon v_\varepsilon}dx + \int_\Omega \frac{\varphi h_2}{v_\varepsilon}dx.\end{aligned} \tag{36}$$

By taking $\varphi \equiv 1$ in (36) and using $u_\varepsilon, h_2 > 0$, we arrive at

$$\frac{d}{dt}\int_\Omega \ln v_\varepsilon dx \ge \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}dx - |\Omega|, \quad t > 0,$$

which on integration in time implies

$$\int_\Omega \ln v_\varepsilon(\cdot, t)dx - \int_\Omega \ln v_0 dx \ge \int_0^t \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}dxds - |\Omega|t, \quad t > 0.$$

Since $\ln\zeta \le \zeta$ for any $\zeta > 0$, it follows that

$$\int_0^t \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}dxds \le \int_\Omega v_\varepsilon dx - \int_\Omega \ln v_0 dx + |\Omega|t, \;\; t > 0.$$

Note that (10) ensures

$$-\int_\Omega \ln v_0 dx \le -|\Omega| \ln \inf_{x\in\overline{\Omega}} v_0 < \infty,$$

Hence, it follows from (20) that

$$\int_0^t \|\nabla \ln v_\varepsilon(\cdot, s)\|_{L^2}^2 ds \le C(1+t), \quad t > 0, \tag{37}$$

with $C > 0$ independent of ε. In addition, let $\eta(t) := e^{-t}\inf_{x\in\overline{\Omega}} v_0(x)$; then, we have $\eta^{-1}(t)v_\varepsilon \ge 1$ owing to (18). Thus, from the fact that $\frac{1}{2}\ln^2\zeta \le \zeta$ for any $\zeta \ge 1$ and (20) we infer that

$$\int_\Omega \ln^2\left(\eta^{-1}(t)v_\varepsilon\right) \le 2\int_\Omega \eta^{-1}(t)v_\varepsilon \le Ce^t(1+t), \;\; t > 0.$$

Consequently, we have

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$$\int_\Omega \ln^2 v_\varepsilon \le 2\left\{\int_\Omega \ln^2\left(\eta^{-1}(t)v_\varepsilon\right) + \int_\Omega \ln^2 \eta(t)\right\} \le Ce^t(1+t), \quad t>0.$$

This, combined with (37), entails (32).

Meanwhile, based on (37) and (18), we also have

$$\left\|v_\varepsilon^{-1}(\cdot,t)\right\|_{L^\infty} \le \frac{e^t}{\inf_{x\in\overline{\Omega}} v_0(x)}, \quad t>0 \tag{38}$$

and

$$\begin{aligned}\int_0^t \left\|\nabla v_\varepsilon^{-1}(\cdot,s)\right\|_{L^2}^2 ds = & \int_0^t \left\|-v_\varepsilon^{-1}(\cdot,s)\nabla \ln v_\varepsilon(\cdot,s)\right\|_{L^2}^2 ds\\ \le & \int_0^t \left\|v_\varepsilon^{-1}(\cdot,s)\right\|_{L^\infty}^2 \|\nabla \ln v_\varepsilon(\cdot,s)\|_{L^2}^2 ds\\ \le & C(1+t)e^{2t}, \ \ t>0,\end{aligned}$$

these immediately entail (33) as desired.

On the other hand, taking $\varphi \equiv g(x) \in C^\infty(\overline{\Omega})$ with $\|g\|_{H^2} \le 1$ in (36), it follows Hölder's inequality that

$$\begin{aligned}\left|\int_\Omega g\partial_t \ln v_\varepsilon dx\right| \le & \|\nabla \ln v_\varepsilon\|_{L^2}^2 \|g\|_{L^\infty} + \|\nabla \ln v_\varepsilon\|_{L^2}\|\nabla g\|_{L^2} + \|g\|_{L^1}\\ & + \|u_\varepsilon\|_{L^1}\|g\|_{L^\infty} + \|h_2\|_{L^\infty}\|g\|_{L^1}\|v_\varepsilon^{-1}\|_{L^\infty},\end{aligned}$$

which, with the help of Young's inequality and Sobolev's embedding theorem, ensures

$$\left|\int_\Omega g\partial_t \ln v_\varepsilon dx\right| \le C\left(\|\nabla \ln v_\varepsilon\|_{L^2}^2 + \|u_\varepsilon\|_{L^1} + 1 + \|h_2\|_{L^\infty}\|v_\varepsilon^{-1}\|_{L^\infty}\right)\|g\|_{H^2}.$$

By means of (37), (19), (38), and (11), it in turn ensures (34). Similarly, in view of

$$(v_\varepsilon^{-1})_t = -v_\varepsilon^{-2}\left(\Delta v_\varepsilon - v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2\right)$$

by the same procedure of the proof of (34), we obtain

$$\begin{aligned}\left|\int_\Omega g\partial_t v_\varepsilon^{-1} dx\right| = & \left|\int_\Omega -2v_\varepsilon^{-1}|\nabla \ln v_\varepsilon|^2 g + v_\varepsilon^{-1}\nabla g\cdot\nabla \ln v_\varepsilon + v_\varepsilon^{-1}g - \frac{u_\varepsilon v_\varepsilon^{-1}g}{1+\varepsilon u_\varepsilon v_\varepsilon} - v_\varepsilon^{-2}h_2 g\right|\\ \le & 2\|v_\varepsilon^{-1}\|_{L^\infty}\|\nabla \ln v_\varepsilon\|_{L^2}^2\|g\|_{L^\infty} + \|v_\varepsilon^{-1}\|_{L^\infty}\|\nabla \ln v_\varepsilon\|_{L^2}\|\nabla g\|_{L^2} + \|v_\varepsilon^{-1}\|_{L^\infty}\|g\|_{L^1}\\ & + \|u_\varepsilon\|_{L^1}\|v_\varepsilon^{-1}\|_{L^\infty}\|g\|_{L^\infty} + \|h_2\|_{L^\infty}\|v_\varepsilon^{-2}\|_{L^\infty}\|g\|_{L^1},\\ \le & C\|v_\varepsilon^{-1}\|_{L^\infty}\left(\|\nabla \ln v_\varepsilon\|_{L^2}^2 + 1 + \|u_\varepsilon\|_{L^1} + \|h_2\|_{L^\infty}\|v_\varepsilon^{-1}\|_{L^\infty}\right)\|g\|_{H^2},\end{aligned}$$

which, together with (37), (19), (38) and (11), yields (35) as desired. □

Similar to Lemma 3.1, we focus on deriving the spatio-temporal integrability of $\nabla \ln(u_\varepsilon+1)$.

Lemma 3.2. *Let $(u_\varepsilon, v_\varepsilon)$ be given in Lemma 2.1. For any $T>0$, there exists $C(T)>0$, independent of ε, with the property that*

$$\int_0^T \|\ln(u_\varepsilon+1)(\cdot,s)\|_{H^1}^2 ds \le C(T), \tag{39}$$

$$\int_0^T \|\partial_s \ln(u_\varepsilon(\cdot,s)+1)\|_{(H^2)^\star} ds \le C(T). \tag{40}$$

Proof. Multiplying the first equation in (17) by $\frac{\varphi}{1+u_\varepsilon}$ with $\varphi(x) \in C^\infty(\overline{\Omega})$ and using the integration by parts, we arrive at

$$\int_\Omega \varphi \partial_t \ln(1+u_\varepsilon)dx = \int_\Omega \frac{|\nabla u_\varepsilon|^2 \varphi}{(u_\varepsilon+1)^2}dx - \int_\Omega \frac{\nabla\varphi\cdot\nabla u_\varepsilon}{1+u_\varepsilon}dx - \chi\int_\Omega \frac{u_\varepsilon\varphi\nabla u_\varepsilon\cdot\nabla\ln v_\varepsilon}{(u_\varepsilon+1)^2}dx$$
$$+\chi\int_\Omega \frac{u_\varepsilon\nabla\varphi\cdot\nabla\ln v_\varepsilon}{1+u_\varepsilon}dx - \kappa\int_\Omega \frac{\varphi u_\varepsilon v_\varepsilon}{1+u_\varepsilon}dx + \int_\Omega \frac{h_1\varphi}{1+u_\varepsilon}dx. \tag{41}$$

By taking $\varphi\equiv 1$ in (41), thanks to $h_1 \geq 0$, an application of Young's inequality yields that

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dx \leq \frac{d}{dt}\int_\Omega \ln(1+u_\varepsilon)dx + \frac{\chi^2}{2}\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}|\nabla\ln v_\varepsilon|^2 dx$$
$$+\int_\Omega \frac{\kappa u_\varepsilon v_\varepsilon}{1+u_\varepsilon}dx - \int_\Omega \frac{h_1}{1+u_\varepsilon}dx$$
$$\leq \frac{d}{dt}\int_\Omega \ln(1+u_\varepsilon)dx + \frac{\chi^2}{2}\int_\Omega |\nabla\ln v_\varepsilon|^2 dx + \kappa\int_\Omega v_\varepsilon dx.$$

In view of the non-negativity of $\ln(1+u_\varepsilon)$, on integration in time gives us

$$\frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dxds \leq \int_\Omega \ln(1+u_\varepsilon)(\cdot,t)dx + \frac{\chi^2}{2}\int_0^t\int_\Omega |\nabla\ln v_\varepsilon|^2 dxds + \kappa\int_0^t\int_\Omega v_\varepsilon dxds.$$

Note that $\zeta \geq \ln(1+\zeta) \geq 0$ for any $\zeta \geq 0$. This, together with (19), leads to

$$\int_\Omega \ln(1+u_\varepsilon(\cdot,t))dx \leq \int_\Omega u_\varepsilon(\cdot,t)dx \leq C(1+t).$$

On the basis of this, we infer from (20) and (37) that there exists $C>0$, independent of ε, such that for any $t>0$

$$\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}dxds \leq C(1+t), \quad t>0.$$

On the other hand, since $\frac{1}{2}\ln^2(1+\zeta)\leq\zeta$ for any $\zeta\geq 0$, it follows from (19) that there exists $C>0$, independent of ε, such that

$$\int_0^t \|\ln(1+u_\varepsilon)\|_{L^2}^2 ds \leq 2\int_0^t \|u_\varepsilon\|_{L^1}ds \leq C(1+t), \quad t>0.$$

Combining with the above two inequalities, we obtain (39) as desired.

Now, let $\varphi\in C^\infty(\overline{\Omega})$ with $\|\varphi\|_{H^2}\leq 1$ in (41), Hölder's inequality and Young's inequality imply that

$$\left|\int_\Omega \varphi\partial_t\ln(1+u_\varepsilon)dx\right| \leq \|\varphi\|_{L^\infty}\|\nabla\ln(1+u_\varepsilon)\|_{L^2}^2 + \chi\|\nabla\ln(1+u_\varepsilon)\|_{L^2}\|\nabla\ln v_\varepsilon\|_{L^2}\|\varphi\|_{L^\infty}$$
$$+\|\nabla\ln(1+u_\varepsilon)\|_{L^2}\|\nabla\varphi\|_{L^2} + \chi\|\nabla\ln v_\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^2}$$
$$+\kappa\|v_\varepsilon\|_{L^1}\|\varphi\|_{L^\infty} + \|h_1\|_{L^\infty}\|\varphi\|_{L^1}$$
$$\leq C\|\varphi\|_{H^2}\left(\|\nabla\ln(1+u_\varepsilon)\|_{L^2}^2 + \|\nabla\ln v_\varepsilon\|_{L^2}^2 + \kappa\|v_\varepsilon\|_{L^1} + \|h_1\|_{L^\infty} + 1\right).$$

After an integration in time, we infer from (11), (20), (32), and (39) that (40) holds as desired. □

With the help of Lemmas 2.1, 3.1 and 3.2, we can find a candidate (u, v) for a generalised solution by standard compactness arguments.

Lemma 3.3. *Let $(u_\varepsilon, v_\varepsilon)$ be taken from Lemma 2.1. Then, there exist $u\geq 0$ and $v>0$ defined on $\Omega\times(0,T)$ for any $T>0$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty\subset(0,1)$ such that $\varepsilon_j\to 0$ as $j\to\infty$, with the properties that*

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for any $T > 0$, as $\varepsilon = \varepsilon_j \to 0$,

$$\ln v_\varepsilon \to \ln v \quad \text{in} \quad L^2\big(0,T;L^\sigma(\Omega)\big), \quad \sigma < \infty, \tag{42}$$

$$\ln v_\varepsilon \rightharpoonup \ln v \quad \text{in} \quad L^2\big(0,T;H^1(\Omega)\big), \tag{43}$$

$$v_\varepsilon \to v \quad a.e. \ \text{in} \quad \Omega \times (0,T), \tag{44}$$

$$v_\varepsilon \to v \quad \text{in} \quad L^1\big(\Omega \times (0,T)\big), \tag{45}$$

$$v_\varepsilon^{-1} \to v^{-1} \quad \text{in} \quad L^2\big(0,T;L^\sigma(\Omega)\big), \quad \sigma < \infty, \tag{46}$$

$$\ln(1+u_\varepsilon) \to \ln(1+u) \quad \text{in} \quad L^2\big(0,T;L^\sigma(\Omega)\big), \quad \sigma < \infty, \tag{47}$$

$$\ln(1+u_\varepsilon) \rightharpoonup \ln(1+u) \quad \text{in} \quad L^2\big(0,T;H^1(\Omega)\big), \tag{48}$$

$$u_\varepsilon \to u \quad a.e. \ \text{in} \quad \Omega \times (0,T), \tag{49}$$

$$u_\varepsilon \to u \quad \text{in} \quad L^1\big(\Omega \times (0,T)\big). \tag{50}$$

Proof. Thanks to (32) and (34), the Aubin-Lions compactness theorem [36] implies that there exist a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ (still expressed as $\{\varepsilon_j\}_{j=1}^\infty$) and a function $w \in L^2(0,T;H^1(\Omega))$, with the property that as $\varepsilon = \varepsilon_j \to 0$,

$$\ln v_\varepsilon \to w, \quad \nabla \ln v_\varepsilon \rightharpoonup \nabla w \quad \text{in} \ \ L^2(\Omega \times (0,T))$$

and by Sobolev's inequality

$$\ln v_\varepsilon \to w \quad \text{in} \quad L^2\big(0,T;L^q(\Omega)\big), \quad q < \infty,$$

which, in particular, ensures

$$\ln v_\varepsilon \to w \quad \text{and} \quad v_\varepsilon \to e^w \quad \text{a.e. in} \ \ \Omega \times (0,T).$$

On the basis of these, setting $v = e^w$, we conclude that (42)–(44) hold as desired. Similarly, according to (33) and (35), we can obtain (46). Furthermore, along the lines of the proof of [46, Lemma 2.8] we can establish the uniform integrability of $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$. In view of the Vitali convergence theorem, this together with (44) ensures the validity of (45).

Meanwhile, the assertions (47)–(50) immediately follow from the bounds (19), (39), and (40) and the Sobolev embedding theorem by using a standard subsequence extraction procedure and resorting to the Aubin-Lions compactness theorem [36]. □

Up to now, our knowledge on approximation of (u, v) by $(u_\varepsilon, v_\varepsilon)$ is enough to pass to the limit $\varepsilon = \varepsilon_j \to 0$ in a manner of (8) in Definition 1.

Lemma 3.4. *Let u and v be given in Lemma 3.3. For any $T > 0$, the inequality (8) in Definition 1 is valid for any non-negative $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$.*

Proof. Note that for any non-negative $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$ the identity (36) is also valid. Based on it and using the integration by parts, we have

$$\begin{aligned} -\int_\Omega \ln v_0 \varphi|_{t=0} dx = {} & \int_0^T \int_\Omega \ln v_\varepsilon \varphi_t dx + \int_0^T \int_\Omega \frac{\varphi|\nabla v_\varepsilon|^2}{v_\varepsilon^2} dxds - \int_0^T \int_\Omega \frac{\nabla\varphi \cdot \nabla v_\varepsilon}{v_\varepsilon} dxds \\ & - \int_0^T \int_\Omega \varphi dxds + \int_0^T \int_\Omega \frac{u_\varepsilon \varphi}{(1+\varepsilon u_\varepsilon v_\varepsilon)} dxds + \int_0^T \int_\Omega \frac{\varphi h_2}{v_\varepsilon} dxds. \end{aligned} \tag{51}$$

It follows from (43) that there exists a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ (still expressed as $\{\varepsilon_j\}_{j=1}^\infty$) such that for any $T > 0$, as $\varepsilon = \varepsilon_j \to 0$,

$$-\int_0^T \int_\Omega \frac{\nabla\varphi \cdot \nabla v_\varepsilon}{v_\varepsilon} dxds \to -\int_0^T \int_\Omega \frac{\nabla\varphi \cdot \nabla v}{v} dxds$$

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and from (42) that

$$\int_0^T \int_\Omega \ln v_\varepsilon \varphi_t dx \to \int_0^T \int_\Omega \ln v\varphi_t dx.$$

On the other hand, similar to the proof of [19, Lemma 3.4] we can infer from the Moser-Trudinger inequality that

$$\int_\Omega \left(1+\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\right)^2 dx \leq \int_\Omega (1+u_\varepsilon)^2\, dx$$
$$\leq C_1 \exp\left\{C_2 \int_\Omega |\nabla \ln(1+u_\varepsilon)|^2 dx + C_3 \int_\Omega \ln(1+u_\varepsilon) dx\right\},$$

which, in view of (19) and the fact that $\ln(1+\varsigma) \leq \varsigma$ for any $\varsigma \geq 0$, leads to

$$\ln\left\{\frac{1}{|\Omega|}\int_\Omega \left(1+\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\right)^2 dx\right\} \leq \ln\frac{C_1}{|\Omega|} + C_2\int_\Omega |\nabla \ln(1+u_\varepsilon)|^2 dx + C_4(1+t), \quad t>0.$$

Integrating it in time and using (39), for any $T>0$ we can find $\widehat{C}=\widehat{C}(T)$ such that

$$\int_0^T \ln\left\{\frac{1}{|\Omega|}\int_\Omega \left(1+\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}\right)^2 dxds\right\} \leq \widehat{C}.$$

Invoking this, using (19) again and proceeding along the lines of the proof of [46, Lemma 2.8], for fixed $\eta>0$, we can find $\delta>0$ suitably small such that given an arbitrary measurable $\mathcal{E}\subset\Omega\times(0,T)$ with $|\mathcal{E}|<\delta$,

$$\iint_{\mathcal{E}} \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} dxdt < \eta.$$

Since we already know from (44) and (49) that

$$\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} \to u \quad \text{a.e. in} \quad \Omega\times(0,T) \quad \text{as} \quad \varepsilon=\varepsilon_j\to 0,$$

along with the Vitali theorem this shows that in fact

$$\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} \to u \quad \text{in} \quad L^1\big(\Omega\times(0,T)\big) \quad \text{as} \quad \varepsilon=\varepsilon_j\to 0,$$

and thereby ensures that

$$\int_0^T \int_\Omega \frac{u_\varepsilon \varphi}{1+\varepsilon u_\varepsilon v_\varepsilon} dxds \to \int_0^T \int_\Omega u\varphi dxds.$$

Similarly, in view of (46), we have

$$\int_0^T \int_\Omega \frac{\varphi h_2}{v_\varepsilon} dxds \to \int_0^T \int_\Omega \frac{\varphi h_2}{v} dxds.$$

Moreover, invoking (32) and (43), the weak lower semicontinuity of the norm ensures

$$\int_0^T \int_\Omega \frac{\varphi|\nabla v|^2}{v^2} dxdt \leq \liminf_{\varepsilon=\varepsilon_j\to 0} \int_0^T \int_\Omega \frac{\varphi|\nabla v_\varepsilon|^2}{v_\varepsilon^2} dxds.$$

Substituting these into (51), the functions u and v obtained in Lemma 3.3 satisfy the inequality (8) in Definition 1. □

To show the validity of (7) in Definition 1, we need the following (weak) convergence.

Lemma 3.5. *Let $(u_\varepsilon, v_\varepsilon)$ be described in Lemma 2.1 and let u and v be established in Lemma 3.3. Then, for $p\geq 1$ and $q\geq 2$ there exists a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ (still expressed as $\{\varepsilon_j\}_{j=1}^\infty$) such that for any*

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$T > 0$, *as* $\varepsilon = \varepsilon_j \to 0$,

$$v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p}\nabla \ln (u_\varepsilon + 1) \rightharpoonup v^{-q}(u+1)^{-p}\nabla \ln (u+1) \quad \text{in} \quad L^2\big(\Omega \times (0,T)\big), \tag{52}$$

$$v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p-1}u_\varepsilon\nabla \ln v_\varepsilon \rightharpoonup v^{-q}(u+1)^{-p-1}u\nabla \ln v \quad \text{in} \quad L^2\big(\Omega \times (0,T)\big), \tag{53}$$

$$v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p}\nabla \ln v_\varepsilon \rightharpoonup v^{-q}(u+1)^{-p}\nabla \ln v \quad \text{in} \quad L^2\big(\Omega \times (0,T)\big). \tag{54}$$

Proof. Thanks to (44) and (49); for any $\alpha, \beta \geq 0$, there exists a subsequence of $\{\varepsilon_j\}_{j=1}^\infty$ (still expressed as $\{\varepsilon_j\}_{j=1}^\infty$) such that for any $T > 0$, as $\varepsilon = \varepsilon_j \to 0$,

$$v_\varepsilon^{-\alpha}(u_\varepsilon + 1)^{-\beta} \to v^{-\alpha}(u+1)^{-\beta} \quad a.e. \text{ in } \quad \Omega \times (0,T),$$

which, in particular, ensures that for any $p \geq 1$ and $q \geq 2$

$$v_\varepsilon^{-q+1}(u_\varepsilon + 1)^{-p} \to v^{-q+1}(u+1)^{-p} \quad a.e. \text{ in } \quad \Omega \times (0,T),$$

$$v_\varepsilon^{-q+1}(u_\varepsilon + 1)^{-p-1}u_\varepsilon \to v^{-q+1}(u+1)^{-p-1}u \quad a.e. \text{ in } \quad \Omega \times (0,T).$$

In addition, we can infer from (38) that

$$v^{-1} \in L^\infty,$$

which implies that

$$v^{-q+1}(u+1)^{-p} \in L^\infty.$$

Hence, invoking (46) with $\sigma = 2$ and Lemma 2.2, for $p \geq 1$ and $q \geq 2$ we get that, as $\varepsilon = \varepsilon_j \to 0$,

$$v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p} \to v^{-q}(u+1)^{-p} \quad \text{in} \quad L^2(\Omega \times (0,T)). \tag{55}$$

This, together with (48), entails that, as $\varepsilon = \varepsilon_j \to 0$,

$$v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p}\nabla \ln (u_\varepsilon + 1) \rightharpoonup v^{-q}(u+1)^{-p}\nabla \ln (u+1) \quad \text{in} \quad L^1(\Omega \times (0,T)). \tag{56}$$

Moreover, by means of (38) and (39), for any $T > 0$ there exists $C(T) > 0$, independent of ε, such that

$$\int_0^T \|v_\varepsilon^{-q}(u_\varepsilon + 1)^{-p}\nabla \ln (u_\varepsilon + 1)\|_{L^2}^2 ds \leq C(T),$$

which, combined with (56), implies (52) as desired. Similarly, employing (38), (44), (50), (46), Lemma 2.2, and (32), we conclude that (53) and (54) also hold. □

By means of Lemma 3.5, we present the validity of (7) in Definition 1.

Lemma 3.6. *Let u and v be given in Lemma 3.3. For p and q taken from Lemma 3.5 which satisfy* $\frac{p(p+1)\chi^2}{4} < q + \frac{q^2}{p+1}$, *the inequality (7) in Definition 1 is valid for any non-negative* $\varphi(x,t) \in C_0^\infty(\overline{\Omega} \times [0,T))$.

Proof. By virtue of (18) and the non-negativity of u_ε, for p and q taken from Lemma 3.5 and any $0 \leq \varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u_\varepsilon + 1)^{-p}v_\varepsilon^{-q}\varphi dx - \int_\Omega (u_\varepsilon + 1)^{-p}v_\varepsilon^{-q}\varphi_t dx \\
&= -p\int_\Omega \varphi(u_\varepsilon + 1)^{-p-1}v_\varepsilon^{-q}\partial_t u_\varepsilon dx - q\int_\Omega \varphi(u_\varepsilon + 1)^{-p}v_\varepsilon^{-q-1}\partial_t v_\varepsilon dx \\
&= -p\int_\Omega (u_\varepsilon + 1)^{-p-1}v_\varepsilon^{-q}\left(\Delta u_\varepsilon - \chi\nabla \cdot (u_\varepsilon\nabla \ln v_\varepsilon) - \kappa u_\varepsilon v_\varepsilon + h_1\right)\varphi dx \\
&\quad - q\int_\Omega (u_\varepsilon + 1)^{-p}v_\varepsilon^{-q-1}\left(\Delta v_\varepsilon - v_\varepsilon + \frac{u_\varepsilon v_\varepsilon}{1 + \varepsilon u_\varepsilon v_\varepsilon} + h_2\right)\varphi dx,
\end{aligned}$$

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which, by the integration by parts, leads to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q}\varphi dx-\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q}\varphi_t dx\\
&=-p(p+1)\int_\Omega (u_\varepsilon+1)^{-p-2}v_\varepsilon^{-q}\varphi|\nabla u_\varepsilon|^2dx\\
&\quad+\int_\Omega\left(p(p+1)\chi\frac{u_\varepsilon}{u_\varepsilon+1}-2pq\right)(u_\varepsilon+1)^{-p-1}v_\varepsilon^{-q-1}\varphi\nabla u_\varepsilon\cdot\nabla v_\varepsilon dx\\
&\quad+\int_\Omega\left(\chi pq\frac{u_\varepsilon}{u_\varepsilon+1}-q(q+1)\right)(u_\varepsilon+1)^{-p}v_\varepsilon^{-q-2}\varphi|\nabla v_\varepsilon|^2dx\\
&\quad-p\int_\Omega (u_\varepsilon+1)^{-p-1}v_\varepsilon^{-q}\left(-\kappa u_\varepsilon v_\varepsilon+h_1\right)\varphi dx\\
&\quad-q\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q-1}\left(-v_\varepsilon+\frac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon}+h_2\right)\varphi dx\\
&\quad+p\int_\Omega (u_\varepsilon+1)^{-p-1}v_\varepsilon^{-q}\nabla u_\varepsilon\cdot\nabla\varphi dx-p\chi\int_\Omega (u_\varepsilon+1)^{-p-1}v_\varepsilon^{-q-1}u_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi dx\\
&\quad+q\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q-1}\nabla v_\varepsilon\cdot\nabla\varphi dx\\
&=:\int_\Omega\sum_{i=1}^{8}P_i^\varepsilon(\cdot,t)dx.
\end{aligned}
$$

Integrating it in time arrives at

$$
-\int_\Omega (u_0+1)^{-p}v_0^{-q}\varphi|_{t=0}dx-\int_0^T\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q}\varphi_t dxdt=\int_0^T\int_\Omega\sum_{i=1}^{8}P_i^\varepsilon(t)dxdt.
$$

A straightforward rearrangement entails

$$
\begin{aligned}
\sum_{i=1}^{3}P_i^\varepsilon(x,t)=&-p(p+1)\left|(u_\varepsilon+1)^{-\frac{p}{2}-1}v_\varepsilon^{-\frac{q}{2}}\nabla u_\varepsilon-\frac{1}{2}\left(\frac{\chi u_\varepsilon}{u_\varepsilon+1}-\frac{2q}{p+1}\right)(u_\varepsilon+1)^{-\frac{p}{2}}v_\varepsilon^{-\frac{q}{2}-1}\nabla v_\varepsilon\right|^2\varphi\\
&+\left(\frac{p(p+1)\chi^2u_\varepsilon^2}{4(u_\varepsilon+1)^2}+\frac{pq^2}{p+1}-q(q+1)\right)(u_\varepsilon+1)^{-p}v_\varepsilon^{-q-2}|\nabla v_\varepsilon|^2\varphi.
\end{aligned}
$$

Note that the assumption $\frac{p(p+1)\chi^2}{4}<q+\frac{q^2}{p+1}$ implies that

$$
\frac{p(p+1)\chi^2u_\varepsilon^2}{4(u_\varepsilon+1)^2}+\frac{pq^2}{p+1}-q(q+1)<\frac{p(p+1)\chi^2}{4}-q-\frac{q^2}{p+1}<0;
$$

thus, it follows that

$$
-\int_0^T\int_\Omega\sum_{i=1}^{3}P_i^\varepsilon(t)dxdt=\left\|\sqrt{-\sum_{i=1}^{3}P_i^\varepsilon}\right\|_{L^2(\Omega\times(0,T))}^2.
$$

Subsequently, based on Lemmas 3.3 and 3.5, a lower semicontinuity argument entails that there exists a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ (still expressed as $\{\varepsilon_j\}_{j=1}^{\infty}$) such that for any $T>0$, as $\varepsilon=\varepsilon_j\to 0$,

$$\begin{aligned}
&\liminf_{\varepsilon=\varepsilon_j\to 0}\int_0^T\int_\Omega -\sum_{i=1}^{3}P_i^\varepsilon dxdt\\
&\geq p(p+1)\int_0^T\int_\Omega (u+1)^{-p-2}v^{-q}\varphi|\nabla u|^2dxdt\\
&\quad -\int_0^T\int_\Omega\left(p(p+1)\chi\frac{u}{u+1}-2pq\right)(u+1)^{-p-1}v^{-q-1}\varphi\nabla u\cdot\nabla vdxdt\\
&\quad -\int_0^T\int_\Omega\left(\chi pq\frac{u}{u+1}-q(q+1)\right)(u+1)^{-p}v^{-q-2}\varphi|\nabla v|^2dxdt.
\end{aligned}$$

In addition, similar to (55), we can prove that, as $\varepsilon=\varepsilon_j\to 0$.

$$v_\varepsilon^{-q+1}(u_\varepsilon+1)^{-p-1}u_\varepsilon\to v^{-q+1}(u+1)^{-p-1}u \quad \text{in} \quad L^2(\Omega\times(0,T))$$

and

$$v_\varepsilon^{-q}(u_\varepsilon+1)^{-p-1}\to v^{-q}(u+1)^{-p-1} \quad \text{in} \quad L^2(\Omega\times(0,T)).$$

Hence, there exists a subsequence of $\{\varepsilon_j\}_{j=1}^{\infty}$ (still expressed as $\{\varepsilon_j\}_{j=1}^{\infty}$) such that for any $T>0$, as $\varepsilon=\varepsilon_j\to 0$,

$$\int_0^T\int_\Omega P_4^\varepsilon dxdt\to -p\int_0^T\int_\Omega (u+1)^{-p-1}v^{-q}\left(-\kappa uv+h_1\right)\varphi dxdt.$$

Similarly, we have

$$\int_0^T\int_\Omega P_5^\varepsilon dxdt\to -q\int_0^T\int_\Omega (u+1)^{-p}v^{-q-1}\left(-v+uv+h_2\right)\varphi dxdt$$

and

$$-\int_0^T\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon^{-q}\varphi_t dxdt\to -\int_0^T\int_\Omega (u+1)^{-p}v^{-q}\varphi_t dxdt.$$

Finally, in view of Lemma 3.5, we obtain

$$\begin{aligned}
\int_0^T\int_\Omega\sum_{i=6}^{8}P_i^\varepsilon dxdt\to p&\int_0^T\int_\Omega (u+1)^{-p-1}v^{-q}\nabla u\cdot\nabla\varphi dxdt\\
&-p\chi\int_0^T\int_\Omega (u+1)^{-p-1}v^{-q-1}u\nabla v\cdot\nabla\varphi dxdt\\
&+q\int_0^T\int_\Omega (u+1)^{-p}v^{-q-1}\nabla v\cdot\nabla\varphi dxdt.
\end{aligned}$$

Hence, by collecting these (7) holds as desired. □

We are now able to proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. Invoking Lemmas 3.6 and 3.4, we only need to verify the validity of (5) and (6). In fact, according to (25) we have

$$\int_\Omega u_\varepsilon(\cdot,t)dx+\kappa\int_0^t\int_\Omega u_\varepsilon v_\varepsilon dxds=\int_\Omega u_0dx+\int_0^t\int_\Omega h_1dxds \tag{57}$$

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for any $t > 0$ and each $\varepsilon \in (0, 1)$. Hence, (5) is a consequence from (44), (49) and Fatou's lemma. Moreover, combined with (31) and (57), we arrive at

$$\int_\Omega (\kappa v_\varepsilon + u_\varepsilon)(\cdot, t)dx + \int_0^t \int_\Omega \kappa v_\varepsilon(\cdot, s)dxds = \int_\Omega \kappa v_0 + u_0 dx + \int_0^t \int_\Omega \kappa h_2 + h_1 dxds,$$

which, together with (44), (49) and Fatou's lemma, ensures that (6) holds as desired. Therefore, (u, v) is a global generalised solution to the initial-boundary value problem (3) in the sense of Definition 1. This finishes the proof of Theorem 1.1. □

4. Large-time behaviour

This section is devoted to the large-time behaviour of the generalised solution (u, v) determined in Theorem 1.1, under the additional assumptions (12)–(14). We start with the result on the solvability of the boundary value problem (2), which directly follows from [17].

Lemma 4.1. *For any given $h_{2,\infty} \in C^1(\overline{\Omega})$, the problem (2) possesses a unique classical solution v_∞ fulfilling that $v_\infty \in C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$.*

We are also concerned with the decay in a linear differential inequality, which is an extended version of [10, Lemma 4.6] (see also [20, Lemma 2.5], [21, Lemma 2.6]).

Lemma 4.2. *Let $\varepsilon > 0$, $y_\varepsilon \in C^1([0, \infty))$ be non-negative functions satisfying*

$$y_\varepsilon(0) = m \tag{58}$$

with some positive constant m independent of ε. If there exist a positive constant μ and a non-negative function $g_\varepsilon(t) \in C([0, \infty)) \cap L^\infty([0, \infty))$ which satisfy

$$\lim_{t\to\infty} \int_t^{t+1} g_\varepsilon(s)ds = 0 \quad \textit{uniformly in} \quad \varepsilon, \tag{59}$$

and

$$\|g_\varepsilon\|_{L^\infty(0,\infty)} \le \mu \quad \textit{uniformly in} \quad \varepsilon, \tag{60}$$

such that for each $\varepsilon > 0$ and some $\lambda > 0$,

$$y'_\varepsilon(t) + \lambda y_\varepsilon(t) \le g_\varepsilon(t) \quad \textit{for all} \quad t > 0, \tag{61}$$

then

$$y_\varepsilon(t) \to 0 \quad \textit{as} \quad t \to \infty \quad \textit{uniformly in} \quad \varepsilon. \tag{62}$$

Proof. Based on (58), an integration of (61) shows that

$$y_\varepsilon(t) \le y_\varepsilon(0)e^{-at} + e^{-at} \int_0^t e^{as} g_\varepsilon(s)ds = me^{-at} + e^{-at} \int_0^t e^{as} g_\varepsilon(s)ds, \quad t > 0.$$

Therefore, we only need to show that

$$\lim_{t\to\infty} e^{-at} \int_0^t e^{as} g_\varepsilon(s)ds = 0. \tag{63}$$

Similar to [10, Lemma 4.6], thanks to (60), for any $\varsigma > 0$ we may fix some k (independent of ε) enough large such that $\frac{\mu e^{-ak}}{a} < \frac{\varsigma}{2}$. For such k, we further take $\sigma > 0$ (independent of ε) fulfilling that $k\sigma < \frac{\varsigma}{2}$. Subsequently, due to (59) we can find t_0, independent of ε, sufficiently large in the sense that

$$\int_t^{t+1} g_\varepsilon(s)ds < \sigma, \quad t \ge t_0 - k.$$

Consequently, we have

$$e^{-at}\int_0^t e^{as}g_\varepsilon(s)ds = e^{-at}\int_0^{t-k} e^{as}g_\varepsilon(s)ds + e^{-at}\sum_{j=0}^{j=k-1}\int_{t-k+j}^{t-k+j+1} e^{as}g_\varepsilon(s)ds$$
$$\le \frac{\mu e^{-ak}}{a} + k\sigma$$
$$< \varsigma, \quad t > t_0.$$

Hence, we have (63) as a desired result and (62) has been established simultaneously. □

Next, we focus on the pointwise lower bound for the solution component v_ε, which plays a key role in the sequel.

Lemma 4.3. *Let $(u_\varepsilon, v_\varepsilon)$ come from Lemma 2.1, and let (12) be in force. Under the additional assumption that Ω is convex, then there exists $c_1 > 0$, independent of t and ε, fulfilling that*

$$v_\varepsilon(x,t) \ge c_1, \quad x \in \Omega, \ t > 0. \tag{64}$$

Proof. Thanks to the convexity of Ω, a slight adaptation of the proof of [20, Corollary 3.1] is easy to show that (64) holds as desired. □

A straightforward consequence of Lemma 4.3 is the following L^1-decay on the component u_ε.

Lemma 4.4. *Let (12)–(13) hold, and let all assumptions in Lemma 4.3 be fulfilled. Then, the solution $(u_\varepsilon, v_\varepsilon)$ fulfils that for some $C > 0$ independent of (ε, t)*

$$\int_\Omega u_\varepsilon(\cdot,t)dx + \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon(\cdot,s)dxds \le C, \quad t > 0, \tag{65}$$

and that

$$\int_\Omega u_\varepsilon(\cdot,t)dx \to 0 \quad \text{as} \quad t\to\infty \quad \text{uniformly in} \quad \varepsilon, \tag{66}$$

and

$$\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon(\cdot,s)dxds \to 0 \quad \text{as} \quad t\to\infty \quad \text{uniformly in} \quad \varepsilon. \tag{67}$$

Proof. If (12) holds, invoking (25) and (64), we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon dx + \frac{1}{2}\kappa c_1\int_\Omega u_\varepsilon dx + \frac{1}{2}\kappa\int_\Omega u_\varepsilon v_\varepsilon dx \le \int_\Omega h_1 dx. \tag{68}$$

This, invoking (11) and a standard ODE technique, ensures that for some $C > 0$ independent of (ε, t)

$$\int_\Omega u_\varepsilon(\cdot,t)dx \le C, \quad t > 0. \tag{69}$$

In addition, thanks to (13) and Lemma 4.2, using (68) again we infer that the decay (66) holds as desired.

We now integrate (68) over $[t, t+1]$ to get

$$\int_\Omega u_\varepsilon(\cdot,t+1)dx + \frac{1}{2}\kappa\int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds \le \int_\Omega u_\varepsilon(\cdot,t)dx + \int_t^{t+1}\int_\Omega h_1 dxds, \quad t > 0.$$

Recalling (13) and (66), we arrive at (67); moreover, due to (69) and (11), using (69) again, we obtain (65). □

In the sequel, we will track the time evolution of $\|v_\varepsilon(\cdot,t) - v_\infty(\cdot)\|_{L^1}$, where v_∞ is the classical solution of the boundary value problem (2). For convenience, we set $\widehat{v}_\varepsilon := v_\varepsilon - v_\infty$. Thanks to (2) and (17), it is

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clear that for $(u_\varepsilon, v_\varepsilon)$ given in Lemma 2.1, the initial-boundary value problem

$$\begin{cases} \widehat{v}_{\varepsilon t} = \Delta \widehat{v}_\varepsilon - \widehat{v}_\varepsilon + \dfrac{u_\varepsilon v_\varepsilon}{1+\varepsilon u_\varepsilon v_\varepsilon} + h_2 - h_{2,\infty}, & x \in \Omega,\ t>0, \\ \nabla \widehat{v}_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ t>0, \\ \widehat{v}_\varepsilon(x,0) = v_0(x) - v_\infty(x), & x \in \Omega \end{cases} \tag{70}$$

admits a unique classical solution $\widehat{v}_\varepsilon$.

Lemma 4.5. *Let all assumptions in Theorem 1.2 be in force. Then, we have*

$$\int_t^{t+1}\int_\Omega |\widehat{v}_\varepsilon| dxds \to 0 \quad \text{as} \quad t\to\infty \quad \text{uniformly in} \quad \varepsilon, \tag{71}$$

where $\widehat{v}_\varepsilon$ is a unique classical solution of (70).

Proof. For any $\sigma > 0$, we multiply the first equation in (70) by $\frac{\widehat{v}_\varepsilon}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}$ and integrate by parts to obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \sqrt{\sigma+|\widehat{v}_\varepsilon|^2}dx + \int_\Omega \frac{|\nabla\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx - \int_\Omega \frac{|\nabla\widehat{v}_\varepsilon|^2|\widehat{v}_\varepsilon|^2}{(\sigma+|\widehat{v}_\varepsilon|^2)^{\frac{3}{2}}}dx \\ &= -\int_\Omega \frac{|\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx + \int_\Omega \frac{u_\varepsilon v_\varepsilon \widehat{v}_\varepsilon}{(1+\varepsilon u_\varepsilon v_\varepsilon)\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx + \int_\Omega \frac{\widehat{v}_\varepsilon(h_2-h_{2,\infty})}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx. \end{aligned}$$

Thanks to the non-negativity of $u_\varepsilon v_\varepsilon$ and the fact that

$$\int_\Omega \frac{|\nabla\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx - \int_\Omega \frac{|\nabla\widehat{v}_\varepsilon|^2|\widehat{v}_\varepsilon|^2}{(\sigma+|\widehat{v}_\varepsilon|^2)^{\frac{3}{2}}}dx = \sigma\int_\Omega \frac{|\nabla\widehat{v}_\varepsilon|^2}{(\sigma+|\widehat{v}_\varepsilon|^2)^{\frac{3}{2}}}dx,$$

we arrive at

$$\frac{d}{dt}\int_\Omega \sqrt{\sigma+|\widehat{v}_\varepsilon|^2}dx + \int_\Omega \frac{|\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dx \le \int_\Omega u_\varepsilon v_\varepsilon dx + \int_\Omega |h_2-h_{2,\infty}|dx.$$

Integrating it over $[t, t+1]$, we have

$$\begin{aligned} &\int_\Omega \sqrt{\sigma+|\widehat{v}_\varepsilon|^2}(\cdot,t+1)dx - \int_\Omega \sqrt{\sigma+|\widehat{v}_\varepsilon|^2}(\cdot,t)dx + \int_t^{t+1}\int_\Omega \frac{|\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dxds \\ &\le \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+1}\int_\Omega |h_2-h_{2,\infty}|dxds. \end{aligned}$$

Using the Beppo Levi theorem, as $\sigma\to 0$, it follows that

$$\int_t^{t+1}\int_\Omega \frac{|\widehat{v}_\varepsilon|^2}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}}dxds \to \int_t^{t+1}\int_\Omega |\widehat{v}_\varepsilon|dxds.$$

A straightforward calculation shows that

$$\left|\sqrt{\sigma+|\widehat{v}_\varepsilon|^2} - |\widehat{v}_\varepsilon|\right| = \frac{\sigma}{\sqrt{\sigma+|\widehat{v}_\varepsilon|^2}+|\widehat{v}_\varepsilon|} \le \sigma^{\frac{1}{2}},$$

which leads to that, as $\sigma\to 0$,

$$\int_\Omega \sqrt{\sigma+|\widehat{v}_\varepsilon|^2}(\cdot,\varsigma)dx \to \int_\Omega |\widehat{v}_\varepsilon|(\cdot,\varsigma)dx, \quad \varsigma\in\{t,t+1\}.$$

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Collecting these, we conclude that

$$\int_\Omega |\widehat{v}_\varepsilon|(\cdot,t+1)dx - \int_\Omega |\widehat{v}_\varepsilon|(\cdot,t)dx + \int_t^{t+1}\int_\Omega |\widehat{v}_\varepsilon|dxds$$
$$\le \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+1}\int_\Omega |h_2 - h_{2,\infty}|dxds.$$

By setting $z_\varepsilon(t) := \int_t^{t+1}\int_\Omega |\widehat{v}_\varepsilon|dxds$, we have

$$z'_\varepsilon(t) = \int_\Omega |\widehat{v}_\varepsilon|(\cdot,t+1)dx - \int_\Omega |\widehat{v}_\varepsilon|(\cdot,t)dx,$$

and thereby obtain

$$z'_\varepsilon(t) + z_\varepsilon(t) \le g_\varepsilon(t) := \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+1}\int_\Omega |h_2 - h_{2,\infty}|dxds.$$

To use Lemma 4.2, we shall need to verify that $g_\varepsilon(t)$ is uniformly in (ε, t) bounded and that $\int_t^{t+1} g_\varepsilon(s)ds$ uniformly in ε converges to 0, as $t \to \infty$. In fact,

$$g_\varepsilon(t) \le \int_t^{t+1}\int_\Omega u_\varepsilon v_\varepsilon dxds + \|h_2 - h_{2,\infty}\|_{L^\infty(\Omega\times(0,\infty))}|\Omega|, \quad t > 0,$$

which, together with (65) and (11), entails

$$g_\varepsilon(t) \le C, \quad t > 0,$$

for some $C > 0$ independent of (ε, t). Thanks to the non-negativity of $g_\varepsilon(t)$, this is enough to present the uniform in ε bound of $\|g_\varepsilon(t)\|_{L^\infty(0,\infty)}$. On the other hand, due to the definition of $g_\varepsilon(t)$ we arrive at

$$\int_t^{t+1} g_\varepsilon(s)ds \le \int_t^{t+2}\int_\Omega u_\varepsilon v_\varepsilon dxds + \int_t^{t+2}\int_\Omega |h_2 - h_{2,\infty}|dxds, \quad t > 0,$$

which, combined with (14) and (67), implies that $\int_t^{t+1} g_\varepsilon(s)ds \to 0$ as $t \to \infty$ uniformly in $\varepsilon \in (0, 1)$. Hence, it follows from Lemma 4.2 that

$$z_\varepsilon(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \varepsilon.$$

This finishes the proof. □

As a consequence of Lemmas 4.4 and 4.5, the large-time behaviour of the generalised solution featured in Theorem 1.2 is now almost immediate.

Proof of Theorem 1.2. On the basis of Lemma 3.3 and the Fubini-Tonelli theorem, there evidently exist $(\varepsilon_j)_{j\in N} \subset (0, 1)$ and a null set $\mathcal{N} \subset (0, \infty)$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \ \text{ and } \ v_\varepsilon(\cdot,t) \to (\cdot,t) \ \ a.e. \text{ in } \Omega \text{ for all } t \in (0,\infty)\setminus\mathcal{N}$$

as $\varepsilon = \varepsilon_j \to 0$. This, by virtue of Fatou's lemma and Lemmas 4.4 and 4.5, gives us the desired large-time behaviour of the generalised solution in Theorem 1.2. □

Acknowledgements. The authors are very grateful to the referees for their detailed comments and valuable suggestions, which greatly improved the manuscript.

Financial support. The research of BL is supported by Natural Science Foundation of Ningbo Municipality (No. 2022J147). The research of LX is supported by Chongqing Science and Technology Commission Project (No. CSTB2023NSCQ-MSX0411) and Research Project of Chongqing Education Commission (No. CXQT21014).

Competing interests. None.

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Cite this article: Li B. and Xie L. (2024). Generalised solution to a 2D parabolic-parabolic chemotaxis system for urban crime: Global existence and large-time behaviour. *European Journal of Applied Mathematics*, **35**, 409–429. https://doi.org/10.1017/S0956792523000268