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LAW OF LARGE NUMBERS FOR DYNAMIC BARGAINING MARKETS

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Abstract

We describe the random meeting motion of a finite number of investors in markets with friction as a Markov pure-jump process with interactions. Using a sequence of these, we prove a functional law of large numbers relating the large motions with the finite market of the so-called continuum of agents.

Keywords: Functional law of large numbers; dynamic bargaining market; market makers

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1. Introduction

To study how asset prices in over-the-counter markets are affected by illiquidity associated with counterparties search and bargaining, Duffie *et al.* (2005) developed a model for markets with friction. They assumed that there was a continuum of interacting agents and derived a quadratic system of differential equations with constraints for the fractions of investors of different types. But in the intuitive argument behind their derivation, they reasoned as if, in fact, such a market was composed of a large finite number of investors performing random meetings. Here we build the random motion of that large finite set and obtain, as the number of investors increases, an associated quadratic system of ordinary differential equations (ODEs) through a functional law of large numbers. (In the applied probability literature this kind of result bears different names; see, for example, Dawson (1985), Ferland (1994), Perthame and Pulverenti (1995), Feng (1997), Clark and Katsouros (1999), and McDonald and Reynier (2006).)

One interest for the functional law is that it gives an algorithm to obtain the solution of the quadratic system of ODEs with constraints. It also shows that, for a large number of agents, the quadratic system provides a reasonable approximation of the probabilistic behavior of the agents and can therefore be used as an alternate modeling tool.

2. A functional law of large numbers

2.1. The random motion

We consider a large finite set of agents and model the random encounters using a continuous-time pure-jump Markov process. For the moment, we do so in a quite abstract setup, where the state of an agent belongs to a finite set S. We shall show in Section 3 how to apply our result to the special case of Duffie *et al.* (2005). We imagine a random motion for the agents which

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is driven by two effects:

1. change of agents states on their own (exogenous effect);
2. change of agents states via binary interactions (endogenous effect).

More precisely, each agent, independently of the others, changes its state according to a continuous-time Markov chain, on S, whose intensity matrix is denoted by $\Gamma = (\gamma(x, y))_{x,y \in S}$. In addition, agents meet each other at rate λ and, when such a meeting occurs, the pair (x, y) of agent states is replaced by a new pair (u, v) with probability $Q(x, y; u, v)$. We assume that

$$Q(x, y; u, v) = Q(y, x; v, u)$$

and

$$\sum_{u,v \in S} Q(x, y; u, v) = 1 \quad \text{for all } x, y \in S.$$

Let $(Z_1^n(t), \dots, Z_n^n(t))$ be the random vector giving the states of the agents at time t. Then $\{Z^n(t),\ t \geq 0\}$ is an S^n-valued Markov process, which may be described either by its generator or by its predictable compensator ν^n:

$$\begin{aligned}\nu^n(\mathrm{d}t \times (z_1, \dots, z_n)) = {} & \sum_{k=1}^{n} \mathbf{1}_{\{Z_k^n(t-) \neq z_k\}} \gamma(Z_k^n(t-), z_k)\, \mathrm{d}t \\ & + \frac{\lambda}{n} \sum_{k \neq j} \mathbf{1}_{\{(Z_k^n(t-), Z_j^n(t-)) \neq (z_k, z_j)\}} Q(Z_k^n(t-), Z_j^n(t-); z_k, z_j)\, \mathrm{d}t\end{aligned}$$

(see Last and Brandt (1995, pp. 113–154) for more details).

For $x \in S$ and $t \geq 0$, we denote by

$$\mu_t^n(x) = \frac{1}{n} \operatorname{card}\{k \colon Z_k^n(t) = x\}$$

the average number of agents in state x at time t. Since there are binary interactions, the weak convergence of $\{\mu_t^n(x)\}_{n \geq 2}$ does not follow readily from the classical law of large numbers; but it does occur and, furthermore, can be proved. The identification of the limit is of major interest, and this can be acheived by showing the weak convergence, as n increases, of the *processes* $\{\mu_t^n,\ t \geq 0\}$, where

$$\mu_t^n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k^n(t)}$$

is the empirical measure of $Z^n(t)$. This is a functional law of large numbers and it relates μ_t^n to the solution of the following quadratic system of differential equations:

$$\begin{aligned}\frac{\partial \mu_t(x)}{\partial t} = {} & \sum_{y \in S} \mu_t(y) \gamma(y, x) \\ & + \sum_{y \in S} \mu_t(y) \Big(2\lambda \sum_{u,v \in S} \mu_t(u) (Q(y, u; x, u) - \delta_{(y,u)}(x, v)) \Big), \qquad x \in S,\end{aligned} \tag{1}$$

with $\mu_t \in \mathcal{P}(S)$, the set of probability measures on S.

2.2. Convergence to the master equation

Let $\langle\cdot,\cdot\rangle$ be the natural duality bracket between a probability measure $\mu \in \mathcal{P}(S)$ and a function $\varphi\colon S \to \mathbb{R}$, that is, $\langle\mu,\varphi\rangle = \sum_{z\in S}\mu(z)\varphi(z)$. Then writing (1) in integral form, it is not difficult to show that

$$\langle\mu_t,\varphi\rangle = \langle\mu_0,\varphi\rangle + \int_0^t \langle\mu_s\otimes\mu_s, \Lambda_1\varphi\rangle\,\mathrm{d}s + \int_0^t \langle\mu_s, \Gamma_1\varphi\rangle\,\mathrm{d}s, \tag{2}$$

where

$$\Lambda_k\varphi(x,y) = \lambda \sum_{u,v\in S}(\varphi(u)+\varphi(v)-\varphi(x)-\varphi(y))^k Q(x,y;u,v),$$
$$\Gamma_k\varphi(x) = \sum_{v\in S}(\varphi(v)-\varphi(x))^k\gamma(x,v).$$

Equation (2) is just (1) written as a master equation on $\mathcal{P}(S)$ along the 'test' functions φ. Of course, we can obtain (1) from (2) by choosing

$$\varphi(z) = \begin{cases} 1 & \text{if } z = x,\\ 0 & \text{otherwise,}\end{cases}$$

and taking the derivative on both sides of the resulting equation. To obtain the master equation from the large finite motion of random matching agents, we shall prove the convergence of the empirical measure processes. These are random elements in $D_\infty(\mathcal{P}(S))$, the Skorohod space of càdlàg functions from $[0,\infty)$ to $\mathcal{P}(S)$, i.e. those which are right continuous and have left limits.

Theorem 1. *Suppose that the empirical measures $\{\mu_0^n\}$ converge weakly to a probability measure $\mu_0 \in \mathcal{P}(S)$, then the empirical measure processes $\{\mu_t^n,\ t\ge 0\}$ converge weakly on $D_\infty(\mathcal{P}(S))$ to a deterministic process $\{\mu_t,\ t\ge 0\}$ which is the unique solution of (2).*

Proof. We closely follow the proof of Theorem 2.1 of Bezandry *et al.* (1994).

Step 1. First we show that (2) has at most one solution. Let μ and $\bar\mu$ be two solutions with the same initial value $\mu_0 = \bar\mu_0$. For $\varphi\colon S\to\mathbb{R}$, define

$$\|\varphi\| := \max_{z\in S}|\varphi(z)|.$$

We know that

$$\|\mu_t - \bar\mu_t\| = \sup_{\|\varphi\|\le 1}|\langle\mu_t-\bar\mu_t,\varphi\rangle|.$$

Using (2), we may write

$$|\langle\mu_t-\bar\mu_t,\varphi\rangle| \le \int_0^t |\langle\mu_s\otimes\mu_s - \bar\mu_s\otimes\bar\mu_s, \Lambda_1\varphi\rangle|\,\mathrm{d}s + \int_0^t |\langle\mu_s-\bar\mu_s,\Gamma_1\varphi\rangle|\,\mathrm{d}s. \tag{3}$$

On the one hand, we have $\|\Gamma_1\varphi\| \le \kappa(\gamma)\|\varphi\|$, where $\kappa(\gamma) = \max_{x\in S}\sum_{y\in S}|\gamma(x,y)|$; therefore, $|\langle\mu_s-\bar\mu_s,\Gamma_1\varphi\rangle| \le \kappa(\gamma)\|\mu_s-\bar\mu_s\|$ whenever $\|\varphi\|\le 1$. On the other hand, we have

$$\begin{aligned}|\langle\mu_s\otimes\mu_s - \bar\mu_s\otimes\bar\mu_s, \Lambda_1\varphi\rangle| &\le |\langle\mu_s\otimes(\mu_s-\bar\mu_s),\Lambda_1\varphi\rangle| + |\langle(\mu_s-\bar\mu_s)\otimes\bar\mu_s,\Lambda_1\varphi\rangle|\\ &= |\langle\mu_s-\bar\mu_s,\psi_s\rangle| + |\langle\mu_s-\bar\mu_s,\bar\psi_s\rangle|,\end{aligned}$$

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where

$$\psi_s(y) = \sum_{x\in S} \mu_s(x)\Lambda_1\varphi(x, y), \qquad \bar{\psi}_s(x) = \sum_{y\in S} \bar{\mu}_s(y)\Lambda_1\varphi(x, y).$$

Both $\|\psi_s\|$ and $\|\bar{\psi}_s\|$ are bounded by $4\lambda\|\varphi\|$. Hence, $|\langle\mu_s - \bar{\mu}_s, \psi_s\rangle| \le 4\lambda\|\mu_s - \bar{\mu}_s\|$ whenever $\|\varphi\| \le 1$, with a similar inequality for $\bar{\psi}_s$. Combining all these in (3), we obtain

$$|\langle\mu_t - \bar{\mu}_t, \varphi\rangle| \le (8\lambda + \kappa(\gamma)) \int_0^t \|\mu_s - \bar{\mu}_s\|\,\mathrm{d}s \quad \text{for } \|\varphi\| \le 1.$$

Taking the supremum over φ yields

$$\|\mu_t - \bar{\mu}_t\| \le (8\lambda + \kappa(\gamma)) \int_0^t \|\mu_s - \bar{\mu}_s\|\,\mathrm{d}s, \qquad t \ge 0,$$

and the result follows from Gronwall's lemma.

Step 2. Now we show that the processes $\{\mu_t^n,\ t \ge 0\}$ are tight. Since $\mathcal{P}(S)$ is compact, it suffices to prove that the real-valued processes $\{\langle\mu_t^n, \varphi\rangle,\ t \ge 0\}$ are tight for any function $\varphi\colon S \to \mathbb{R}$. We apply a well-known criterion (see Billingsley (1968, Theorem 15.5, p. 127)). We show that

(i) $\displaystyle\lim_{R\uparrow\infty}\Big(\sup_n \Pr\Big\{\sup_{t\ge 0} |\langle\mu_t^n, \varphi\rangle| > R\Big\}\Big) = 0;$

(ii) for all $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \ge 1$ such that

$$\sup_{n\ge n_0} \Pr\Bigg\{\sup_{\substack{s,t\ge 0\\ |t-s|<\delta}} |\langle\mu_t^n, \varphi\rangle - \langle\mu_s^n, \varphi\rangle| \ge \varepsilon\Bigg\} \le \varepsilon.$$

Condition (i) is easy to prove. Since μ_t^n is a probability measure, we find that $|\langle\mu_t^n, \varphi\rangle| \le \|\varphi\|$. Therefore, $\Pr\{\sup_{t\ge 0} |\langle\mu_t^n, \varphi\rangle| > R\} = 0$ whenever $R > \|\varphi\|$. To prove condition (ii), we use the modulus V'' defined on $D_\infty(\mathbb{R})$ by

$$V''(f, \delta) = \sup\{|f(t) - f(r)| \wedge |f(r) - f(s)|;\ 0 \le s \le r \le t,\ |t - s| < \delta\}.$$

It is well known that

$$\sup_{\substack{s,t\ge 0\\ |t-s|<\delta}} |f(t) - f(s)| \le 2V''(f, \delta) + \sup_{t\ge 0} |f(t) - f(t-)|. \tag{4}$$

Since $\{Z^n(t),\ t \ge 0\}$ is a pure-jump process for which at most two components can change at the time of a jump, we see that $|\langle\mu_t^n, \varphi\rangle - \langle\mu_{t-}^n, \varphi\rangle|$ is bounded by $4\|\varphi\|/n$. Let us fix $\varepsilon > 0$ and then choose $n_0 \ge 1$ large enough to have $\|\varphi\| < n\varepsilon/8$ for all $n \ge n_0$. For these n, we then have

$$\Pr\Big\{\sup_{t\ge 0} |\langle\mu_t^n, \varphi\rangle - \langle\mu_{t-}^n, \varphi\rangle| > \frac{\varepsilon}{2}\Big\} = 0,$$

and (4) gives

$$\Pr\Bigg\{\sup_{\substack{s,t\ge 0\\ |t-s|<\delta}} |\langle\mu_t^n, \varphi\rangle - \langle\mu_s^n, \varphi\rangle| > \varepsilon\Bigg\} \le \Pr\Big\{V''(\langle\mu^n, \varphi\rangle, \delta) > \frac{\varepsilon}{4}\Big\}.$$

To obtain condition (ii), it remains to prove that

$$\lim_{\delta\downarrow 0}\sup_{n\geq n_0}\Pr\left\{V''(\langle\mu^n,\varphi\rangle,\delta)>\frac{\varepsilon}{4}\right\}=0.$$

The latter limit is a consequence of the following inequality:

$$\mathrm{E}[(\langle\mu_t^n,\varphi\rangle-\langle\mu_r^n,\varphi\rangle)^2(\langle\mu_r^n,\varphi\rangle-\langle\mu_s^n,\varphi\rangle)^2]\leq C(t-s)^2, \tag{5}$$

where $0\leq s\leq r\leq t$ (see Billingsley (1968, Theorem 15.6, p. 128)). To prove (5), we need some martingales related to $\{\mu_t^n,\ t\geq 0\}$. Let $(\mathcal{F}_t^n)$ be the natural filtration of $\{Z^n(t),\ t\geq 0\}$. Using the compensator ν^n, we can show that the processes

$$M_t^n=\langle\mu_t^n,\varphi\rangle-\int_0^t\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_1\varphi\rangle\,\mathrm{d}s-\int_0^t\langle\mu_s^n,\Gamma_1\varphi\rangle\,\mathrm{d}s,$$
$$S_t^n=(M_t^n)^2-\frac{1}{n}\int_0^t\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_2\varphi\rangle\,\mathrm{d}s-\frac{1}{n}\int_0^t\langle\mu_s^n,\Gamma_2\varphi\rangle\,\mathrm{d}s,$$

are $(\Pr,\mathcal{F}_t^n)$-martingales. In the above we have set

$$\mu_s^n\dot{\otimes}\mu_s^n=\frac{1}{n^2}\sum_{i\neq j}\delta_{Z_i^n(t)}\otimes\delta_{Z_j^n(t)}.$$

We use these martingales to obtain an upper bound for $\mathrm{E}[(\langle\mu_t^n,\varphi\rangle-\langle\mu_r^n,\varphi\rangle)^2\mid\mathcal{F}_r^n]$. Given the definition of M_t^n, the conditional expectation is almost surely bounded by

$$3\,\mathrm{E}[(M_t^n)^2-(M_r^n)^2\mid\mathcal{F}_r^n]+3(t-r)\,\mathrm{E}\left[\int_r^t(|\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_1\varphi\rangle|^2+|\langle\mu_s^n,\Gamma_1\varphi\rangle|^2)\,\mathrm{d}s\mid\mathcal{F}_r^n\right].$$

But $|\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_1\varphi\rangle|$ and $|\langle\mu_s^n,\Gamma_1\varphi\rangle|$ are bounded by $4\lambda\|\varphi\|$ and $2\kappa(\gamma)\|\varphi\|$, respectively. Therefore, the second expectation in the above sum is bounded by $(16\lambda^2+4\kappa(\gamma)^2)\|\varphi\|^2(t-r)$. The first expectation is handled in the same way, using the martingale S_t^n instead. Indeed, the martingale property gives

$$\mathrm{E}[(M_t^n)^2-(M_r^n)^2\mid\mathcal{F}_r^n]=\frac{1}{n}\,\mathrm{E}\left[\int_r^t(\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_2\varphi\rangle+\langle\mu_s^n,\Gamma_2\varphi\rangle)\,\mathrm{d}s\mid\mathcal{F}_r^n\right].$$

This time, $\langle\mu_s^n\dot{\otimes}\mu_s^n,\Lambda_2\varphi\rangle$ and $\langle\mu_s^n,\Gamma_2^N\varphi\rangle$ are bounded by $16\lambda\|\varphi\|^2$ and $4\kappa(\gamma)\|\varphi\|^2$, respectively, so

$$\mathrm{E}[(M_t^n)^2-(M_r^n)^2\mid\mathcal{F}_r^n]\leq\frac{16\lambda+4\kappa(\gamma)}{n}\|\varphi\|^2(t-r). \tag{6}$$

Combining the two upper bounds yields

$$\mathrm{E}[(\langle\mu_t^n,\varphi\rangle-\langle\mu_r^n,\varphi\rangle)^2\mid\mathcal{F}_r^n]\leq C_1(t-r).$$

Taking the expectation, we also have

$$\mathrm{E}[(\langle\mu_t^n,\varphi\rangle-\langle\mu_r^n,\varphi\rangle)^2]\leq C_1(t-r),$$

and, consequently,

$$\begin{aligned}
&\mathrm{E}[(\langle \mu_t^n, \varphi\rangle - \langle \mu_r^n, \varphi\rangle)^2 (\langle \mu_r^n, \varphi\rangle - \langle \mu_s^n, \varphi\rangle)^2] \\
&\quad = \mathrm{E}[(\langle \mu_r^n, \varphi\rangle - \langle \mu_s^n, \varphi\rangle)^2\, \mathrm{E}[(\langle \mu_t^n, \varphi\rangle - \langle \mu_r^n, \varphi\rangle)^2 \mid \mathcal{F}_r^n]] \\
&\quad \le C_1(t-r)\,\mathrm{E}[(\langle \mu_r^n, \varphi\rangle - \langle \mu_s^n, \varphi\rangle)^2] \\
&\quad \le C_1^2 (t-r)(r-s) \\
&\quad \le C(t-s)^2.
\end{aligned}$$

Step 3. We prove that $\{\mu_t^n,\ t \ge 0\}$ converges weakly by characterizing the limiting process as the unique solution of (2). We denote by P^n the probability measure induced by $\{\mu_t^n,\ t \ge 0\}$ on $D_\infty(\mathcal{P}(S))$, we denote by E^n the corresponding expectation, and we denote by $\{U(t),\ t \ge 0\}$ the canonical projection process on $D_\infty(\mathcal{P}(S))$. We now show that any limit point P^∞ of $\{\mathrm{P}^n\}$ is concentrated on a specific path in $D_\infty(\mathcal{P}(S))$, namely the solution of (2).

For $\varphi\colon S \to \mathbb{R}$, we define the following process:

$$M_t = \langle U(t), \varphi\rangle - \int_0^t \langle U(s) \otimes U(s), \Lambda_1 \varphi\rangle \,\mathrm{d}s - \int_0^t \langle U(s), \Gamma_1 \varphi\rangle \,\mathrm{d}s. \tag{7}$$

Let $\{\mathrm{P}^{n_k}\}$ be a subsequence of $\{\mathrm{P}^n\}$ that converges weakly to P^∞. We first prove that, for all t,

$$\lim_k \mathrm{E}^{n_k}[(M_t - M_0)^2] = \mathrm{E}[(M_t - M_0)^2]. \tag{8}$$

Let $g\colon D_\infty(\mathcal{P}(S)) \to \mathbb{R}$ be defined by

$$g(w) = \int_0^t \langle w(s) \otimes w(s), \Lambda_1 \phi\rangle \,\mathrm{d}s.$$

This function is bounded and measurable. It is also continuous on any w in $C_\infty(\mathcal{P}(S))$ (the continuous functions from $[0,\infty)$ to $\mathcal{P}(S)$). Indeed, suppose that $\{w^n\}$ converges to w in $D_\infty(\mathcal{P}(S))$, then $\{w^n(s)\}$ weakly converges to $w(s)$ for all s (since w is continuous). Hence, $\langle w^n(s) \otimes w^n(s), \Lambda_1\phi\rangle \to \langle w(s) \otimes w(s), \Lambda_1 \phi\rangle$, and the continuity of g follows from the bounded convergence theorem. In addition, the criterion used in step 2 not only shows that $\{\mathrm{P}^n\}$ is tight but also that $\mathrm{P}^\infty(C_\infty(\mathcal{P}(S))) = 1$. As a result, the set of discontinuities of g is a P-null set. The continuous mapping theorem then gives $\lim_k \mathrm{E}^{n_k}[g] = \mathrm{E}^\infty[g]$. A similar argument works for the functions

$$w \mapsto \int_0^t \langle w(s), \Gamma_1 \phi\rangle \,\mathrm{d}s, \qquad w \mapsto \langle w(t), \phi\rangle,$$

and (8) follows.

Next we prove that

$$\lim_k \mathrm{E}^{n_k}[(M_t - M_0)^2] = 0. \tag{9}$$

Let

$$M_t^{(n)} = M_t - \frac{1}{n}\int_0^t \langle U(s), \overline{\Lambda}_1 \varphi\rangle \,\mathrm{d}s,$$

where $\overline{\Lambda}_1 \varphi(x) = \Lambda_1 \varphi(x,x)$. It immediately follows that

$$\mathrm{E}^{n_k}[(M_t - M_0)^2] \le 2\,\mathrm{E}^{n_k}[(M_t^{(n_k)} - M_0^{(n_k)})^2] + \frac{2}{{n_k}^2}\,\mathrm{E}^{n_k}\biggl[\biggl(\int_0^t \langle U(s), \overline{\Lambda}\varphi\rangle \,\mathrm{d}s\biggr)^2\biggr].$$

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By (6), the first term on the right-hand side is bounded by $(16\lambda + 4\kappa(\gamma))\|\varphi\|^2 t/n_k$, while the expectation for the second term is clearly bounded by a constant that depends only on φ and t. Taking the limit on k, we obtain (9).

The equality $\mathrm{E}^\infty[(M_t - M_0)^2] = 0$ means that M_t is equal to M_0 almost surely for P^∞. But $\{M_t, t \geq 0\}$ is right continuous. Hence, there exists a set $\Omega_0 \subset D_\infty(\mathcal{P}(S))$ with $\mathrm{P}^\infty(\Omega_0) = 1$ and $M_t(w) = M_0(w)$ for any $t \geq 0$ and $w \in \Omega_0$. Looking at the definition of M_t, it precisely means that any $w \in \Omega_0$ is a solution of (2). Finally, because μ_0^n weakly converges to μ, we find that $\mathrm{P}^\infty(U(0) = \mu) = 1$. By step 1, $\Omega_0 \cap \{U(0) = \mu\}$ is a singleton $\{w_0\}$, and P^∞ must be a Dirac measure on w_0. All the limit points of $\{\mathrm{P}^n\}$ being the same, $\{\mathrm{P}^n\}$ weakly converges to P^∞. The proof is complete.

3. Dynamic bargaining markets

3.1. Direct bargaining

Duffie *et al.* (2005) imagined a continuum of agents in which, along time and randomly, the agents switch their level of interest or meet each other for possible exchange of the asset. They developed a dynamic asset-pricing model and derived the equilibrium allocation of assets and the price negotiated. To do so they used a quadratic system of ODEs for the fractions of investors of different types. We will show here that this system is a particular case of (1) and, therefore, the intuitive argument for the ODEs stated in an 'abstract' continuum setup can be replaced by a more formal validation. Indeed, according to Theorem 1, when the number of investors is large, the random fractions μ_t^n of investor's types is approximately given by the solution of (1).

To see how the ODE system of Duffie *et al.* (2005) is a special case of (1) (or (2)), we just have to specify the appropriate triple (S, Γ, Q). In their model an investor was characterized by whether he/she owned the asset or not, and by an intrinsic type that is 'high' or 'low'. Therefore, the full set of investor types is $S = \{\text{ho}, \text{hn}, \text{lo}, \text{ln}\}$, where the letters 'h' and 'l' designate the investor's intrinsic (liquidity) state, and 'o' and 'n' respectively indicate whether the investor owns the asset or not. Duffie *et al.* (2005) assumed that an investor switched from a low to a high type with rate λ_u, and from a high to a low type with rate λ_d. So Γ is given by

$$\begin{array}{c} \\ \text{ho} \\ \text{hn} \\ \text{lo} \\ \text{ln} \end{array} \begin{array}{c} \begin{array}{cccc} \text{ho} & \text{hn} & \text{lo} & \text{ln} \end{array} \\ \begin{pmatrix} -\lambda_d & 0 & \lambda_d & 0 \\ 0 & -\lambda_d & 0 & \lambda_d \\ \lambda_u & 0 & -\lambda_u & 0 \\ 0 & \lambda_u & 0 & -\lambda_u \end{pmatrix} \end{array}.$$

In addition, investors meet each other at rate λ, but an exchange of the asset occurs only if an investor of type lo (owns the asset but has low interest for it) meets one of type hn (does not own the asset but has high interest for it). This behavior is properly described by the kernel Q below:

$$Q(x, y; u, v) = \begin{cases} 1 & \text{if } (x, y) = (\text{lo}, \text{hn}) \text{ and } (u, v) = (\text{ln}, \text{ho}), \\ 1 & \text{if } (x, y) = (\text{hn}, \text{lo}) \text{ and } (u, v) = (\text{ho}, \text{ln}), \\ 1 & \text{if } (x, y) = (u, v) \text{ and } (x, y) \notin \{(\text{lo}, \text{hn}), (\text{hn}, \text{lo})\}, \\ 0 & \text{otherwise.} \end{cases}$$

For this triple (S, Γ, Q), (2) becomes quite simple and intuitive. For example, let φ be the indicator function of the state hn. Then it is easy to see that $\Lambda_1\varphi(x, y)$ is 0 except when

$(x, y) \in \{(\text{lo}, \text{hn}), (\text{hn}, \text{lo})\}$ and, for the latter case,

$$\Lambda_1\varphi(\text{lo}, \text{hn}) = \Lambda_1\varphi(\text{hn}, \text{lo}) = -\lambda,$$

so $\langle \mu_s \otimes \mu_s, \Lambda_1\varphi\rangle = -2\lambda\mu_s(\text{hn})\mu_s(\text{lo})$. Moreover, we have

$$\langle \mu_s, \Gamma_1\varphi\rangle = -\lambda_d\mu_s(\text{hn}) + \lambda_u\mu_s(\text{ln}).$$

Substituting into (2) and taking the derivative, we obtain

$$\dot{\mu}_t(\text{hn}) = -2\lambda\mu_s(\text{hn})\mu_s(\text{lo}) - \lambda_d\mu_s(\text{hn}) + \lambda_u\mu_s(\text{ln}),$$

which is Equation (4) of Duffie *et al.* (2005) (with $\rho = 0$). Therefore, Theorem 1 applies to their model when bargaining is done without intermediaries ($\rho = 0$). When bargaining is eventually done through an intermediary, it is still possible to prove a law of large numbers. This is done in the next subsection.

3.2. Market makers

Because searching for counterparties can reduce liquidity, some over-the-counter markets have intermediaries, the so-called market makers. Duffie *et al.* (2005) also studied a market model where asset exchange occurred through market makers, in addition to direct bargaining. But in their model intermediaries had to be searched too. They assumed a rate ρ for meeting with a market maker and that exchange occurred provided investors were present. The quadratic ODE system changes accordingly. For instance, for state *hn* the new ODE becomes

$$\dot{\mu}_t(\text{hn}) = -(2\lambda\mu_s(\text{hn})\mu_s(\text{lo}) + \rho\min\{\mu_s(\text{hn}), \mu_s(\text{lo})\}) - \lambda_d\mu_s(\text{hn}) + \lambda_u\mu_s(\text{ln}),$$

with a similar modification for the other states. This equation is not a special case of (1), but we can modify Theorem 1 to cover the situation. We did not do so in Section 2.2 because the changes to be made are quite specific to their model.

The state space S remains the same. However, the predictable compensator of the Markov chain Z^n is replaced by

$$\nu^n(\mathrm{d}t \times (z_1, \dots, z_n)) + \mathbf{1}_{\{Z^n(t-)\neq(z_1,\dots,z_n)\}}\,\hat{\rho}(Z^n(t-), (z_1, \dots, z_n))\,\mathrm{d}t.$$

The intensity $\hat{\rho}$ is 0 except when $Z^n(t-)$ and $(z_1, \dots, z_n)$ are ‘compatible’, in which case

$$\hat{\rho}(Z^n(t-), (z_1, \dots, z_n)) = n\rho\min\{\mu^n_{t-}(\text{hn}), \mu^n_{t-}(\text{lo})\}.$$

The vectors $Z^n(t-)$ and $(z_1, \dots, z_n)$ are said to be compatible if the latter results from the former via a meeting with a market maker. More precisely, it means that

(a) $\mu^n_{t-}(\text{hn})$ and $\mu^n_{t-}(\text{lo})$ are both nonzero (there are investors of proper types for an exchange to occur through market makers);

(b) $z = (z_1, \dots, z_n)$ is obtained from $Z^n(t-)$ by replacing the first hn of the latter by ho and the first lo by ln (that is, z is the market configuration obtained from $Z^n(t-)$ after the asset exchange occurred through the market maker).

With this new kind of transition, the master equation (2) becomes

$$\langle \mu_t, \varphi\rangle = \langle \mu_0, \varphi\rangle + \int_0^t \langle \mu_s \otimes \mu_s, \Lambda_1\varphi\rangle\,\mathrm{d}s + \int_0^t \langle \mu_s, \Gamma_1\varphi\rangle\,\mathrm{d}s + \int_0^t \rho\bar{\mu}_s\Delta(\varphi)\,\mathrm{d}s, \qquad (10)$$

where $\bar{\mu}_s = \min\{\mu_s(\text{hn}), \mu_s(\text{lo})\}$ and $\Delta(\varphi) = \varphi(\text{hn}) + \varphi(\text{lo}) - \varphi(\text{ho}) - \varphi(\text{ln})$.

The proof of Theorem 2.1 can be easily adapted. Uniqueness for (10) (step 1) has been proved by Duffie *et al.* (2005). For step 2, we use the martingales

$$\overline{M}_t^n = \langle \mu_t^n, \varphi \rangle - \int_0^t \langle \mu_s^n \dot{\otimes} \mu_s^n, \Lambda_1 \varphi \rangle \,\mathrm{d}s - \int_0^t \langle \mu_s^n, \Gamma_1 \varphi \rangle \,\mathrm{d}s - \int_0^t \rho \bar{\mu}_s^n \Delta(\varphi) \,\mathrm{d}s,$$
$$\overline{S}_t^n = (\overline{M}_t^n)^2 - \frac{1}{n} \int_0^t \langle \mu_s^n \dot{\otimes} \mu_s^n, \Lambda_2 \varphi \rangle \,\mathrm{d}s - \frac{1}{n} \int_0^t \langle \mu_s^n, \Gamma_2 \varphi \rangle \,\mathrm{d}s - \frac{1}{n} \int_0^t \rho \bar{\mu}_s^n \Delta(\varphi)^2 \,\mathrm{d}s,$$

with $\bar{\mu}_s^n = \min\{\mu_s^n(\mathrm{hn}), \mu_s^n(\mathrm{lo})\}$. Following the previous computation to obtain (5), we just have to prove that there exist constants $C_2, C_3 > 0$ such that

$$\mathrm{E}\bigg[\int_r^t |\rho \bar{\mu}_s^n \Delta(\varphi)|^2 \,\mathrm{d}s \;\bigg|\; \mathcal{F}_r^n \bigg] \le C_2 (t - r) \tag{11}$$

and

$$\mathrm{E}\bigg[\frac{1}{n} \int_r^t \rho \bar{\mu}_s^n \Delta(\varphi)^2 \,\mathrm{d}s \;\bigg|\; \mathcal{F}_r^n \bigg] \le \frac{C_3}{n} (t - r). \tag{12}$$

This is immediate, since $\bar{\mu}_s^n = \min\{\mu_s(\mathrm{hn}), \mu_s(\mathrm{lo})\} \le 1$ and $|\Delta(\varphi)| \le 4\|\varphi\|$. Finally, for step 3, we use the process

$$\overline{M}_t = M_t - \int_0^t \rho \overline{U}(s) \Delta(\varphi) \,\mathrm{d}s,$$

with $\overline{U}(s) = \min\{U(s, \mathrm{hn}), U(s, \mathrm{lo})\}$ and M given by (7). Equality (8) for $\overline{M}$ is proved as before because, on $C_\infty(\mathcal{P}(S))$, the mapping

$$w \mapsto \int_0^t \rho \overline{w}(s) \Delta(\varphi) \,\mathrm{d}s$$

is continuous. As for equality (9), we use

$$\overline{M}_t^{(n)} = M_t^{(n)} - \int_0^t \rho \overline{U}(s) \Delta(\varphi) \,\mathrm{d}s,$$

and observe that

$$\mathrm{E}^{n_k}[(\overline{M}_t^{(n_k)} - \overline{M}_0^{(n_k)})^2] \le \frac{C_4 t}{n_k},$$

as a consequence of (11) and (12). All the other arguments of the proof carry on and, therefore, the result follows.

3.3. Concluding remarks

We have shown a finite-agent limit result even with market makers. We believe this could be a fruitful line of work connecting probability theory and economics. Of course ODEs are also important objects in their own right, and from them we obtain, in many cases, some simpler results.

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