




Tangle equations, the Jones conjecture, slopes of surfaces in tangle complements, and q -deformed rationals

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Abstract. We study systems of two-tangle equations

$$\begin{cases} N(X + T_1) = L_1, \\ N(X + T_2) = L_2, \end{cases}$$

which play an important role in the analysis of enzyme actions on DNA strands.

We show that every system of framed tangle equations has at most one framed rational solution. Furthermore, we show that the Jones unknot conjecture implies that if a system of tangle equations has a rational solution, then that solution is unique among all two-tangles. This result potentially opens a door to a purely topological disproof of the Jones unknot conjecture.


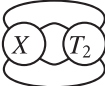
We introduce the notion of the Kauffman bracket ratio $\{T\}_q \in \mathbb{Q}(q)$ of any two-tangle T and we conjecture that for $q = 1$ it is the slope of meridionally incompressible surfaces in $D^3 - T$. We prove that conjecture for algebraic T . We also prove that for rational T , the brackets $\{T\}_q$ coincide with the q -rationals of Morier-Genoud and Ovsienko.

Additionally, we relate systems of tangle equations to the cosmetic surgery conjecture and the nugatory crossing conjecture.

1 Introduction

1.1 Systems of tangle equations

A system of tangle equations has the form

(1.1)  = L_1 and  = L_2 ,

where $T_1 \neq T_2$ are rational two-tangles and L_1, L_2 are links and X is an unknown two-tangle. (From now on, two-tangles will be referred to as “tangles.”) Such systems play an important role in the analysis of entanglement of DNA molecules (see Section 2.3). We study them in the settings of both unframed and framed links, by combining

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topological methods (surgery theory on double branched covers of links and tangles) and algebraic methods involving the Jones polynomial and the Kauffman bracket.

We prove foundational results in Sections 2.2 and 2.3. In particular, we show that every tangle equation has infinitely many solutions when L is not the unlink of two components. We also exhibit systems (1.1) which have an arbitrarily large numbers of solutions.

The *Kauffman bracket* of a framed two-tangle T is $[T] = \begin{pmatrix} [T]_0 \\ [T]_\infty \end{pmatrix}$, where $[T]_0, [T]_\infty \in \mathbb{Z}[A^{\pm 1}]$ are determined by applying the Kauffman bracket skein relations reducing T to a linear combination

$$T = [T]_0 \cdot \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + [T]_\infty \cdot \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array},$$

(cf. [10, 27, 41]).

A rational solution of (1.1) is one given by a rational tangle. We prove the following.

Proposition (Proof in Section 8) *Every framed system (1.1) has at most one-framed rational solution.*

We propose, more generally, the following.

Conjecture *If a framed system (1.1) has a rational solution X , then it is unique (among all solutions X).*

The Jones unknot conjecture, henceforth, called the Jones conjecture, states that the Jones polynomial distinguishes all nontrivial knots from the trivial one (see [19]). One of the main results of this paper relates the above two conjectures.

Theorem 1.3 (Proof in Section 9) *The Jones conjecture implies Conjecture 1.2.*

A stronger version of this statement, formulated in Section 9, shows that because the Jones conjecture holds up to 24 crossings (see [42]), Conjecture 1.2 holds for tangles up to 11 crossings.

The topological meaning of the Jones polynomial is rather obscure, despite much research devoted to it in the last decades. Hence, we find this purely topological consequence of the Jones conjecture intriguing. In particular, this result opens a door to disproving the Jones conjecture through purely topological methods.

We will show in Proposition 2.1 that there is no upper bound on the number of solutions of systems (1.1) (framed and unframed ones).

Conjecture 1.2 does not hold for unframed systems. However, every unframed system has at most two rational solutions (see [11]). Conjecture 1.2 suggests the following unframed version of it.

Conjecture *If an unframed system (1.1) has a rational solution X , then its every solution is rational.*

The importance of this conjecture stems from the fact that a counterexample to it may lift to a framed counterexample of Conjecture 1.2; thus, disproving the Jones conjecture. We will prove cases of this conjecture and connect it with the theory of strongly invertible knots in an upcoming work.

1.2 Approach through the surgery theory

Taking double branched covers of tangles and of links translates equations (1.1) into the language of surgery theory of three-manifolds. Apart from the Kauffman bracket, this is the second main source of methods utilized in this paper. In particular, we show the following.

Proposition (Proof in Section 3) *If $L_1 = L_2$ is either a nontrivial rational knot or a nontrivial torus knot, then system (1.1) has no algebraic tangle (framed or unframed) solutions for any T_1, T_2 .*

(This result settles, in particular, Conjecture 1.2 when $L_1 = L_2$ is a nontrivial rational knot or a torus knot.) We will relate systems of tangle equations to cosmetic surgery conjecture in Section 3.

1.3 Connection with q -rationals and slopes of essential surfaces

Theorem 1.6 (Proof in Section 7) *For any tangle T ,*

$$\{T\}_q := A^{-2}[T]_\infty/[T]_0$$

is well defined (i.e., not 0/0) and independent of the framing of T . Furthermore,

$$\{T\}_q \in \widehat{\mathbb{Q}(q)} = \mathbb{Q}(q) \cup \{\infty\},$$

where $q = -A^4$.

We call $\{T\}_q$ the *Kauffman bracket ratio* of T or the *KB ratio* for short.

Theorem 1.7 (Proof in Section 7) *For any $x \in \widehat{\mathbb{Q}}$, $\{\{x\}\}_q$ is the q -rational of [33].*

q -Rationals are q -deformations of rational numbers, in the sense that $\{\{x\}\}_q$ evaluates to x at $q = 1$. (They were independently discovered by us in the first version of this paper on arXiv.) Interestingly, much of the theory of continued fractions extends to these q -rational numbers. We believe that our approach to q -rational numbers provides a useful new geometric intuition.

Finally, we relate the Kauffman bracket ratios to slopes of dividing surfaces in tangle complements in Section 10. A surface S properly embedded in the complement of a tangle, $B^3 - T$, is *meridionally incompressible* if every two-disk D^2 in B^3 with boundary in S intersecting T transversely precisely once can be deformed to a disk in $S \cup T$. A surface S is *m -essential* (for “meridionally essential”) if it is incompressible, meridionally incompressible and not boundary-parallel in $D^3 \setminus T$ (see [34, 35]).

We say that a loop γ in S_4^2 is *dividing* if it separates the punctures of S_4^2 into two groups of two. Such loops are classified by their slope $s(\gamma) \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ which can be defined by lifting γ to the torus being the double cover of S^2 branched over the four ends of T (see Section 10).

If the boundary of an m -essential surface contains a dividing loop γ , we call S *dividing* and we call $s(\gamma)$ the *slope* $s(S)$ of S . (Then it is easy to see that all other dividing loops of ∂S must be parallel.) This definition is inspired by that in [35], where the slopes are defined for algebraic tangles.



Figure 1: The $-1, 0, 1$ and ∞ tangles and the tangle addition $T + T'$. (We follow here Conway's notation [7]. Kauffman's and his collaborators' papers use opposite signs, e.g., [24].)

Conjecture Every dividing m -essential surface in the complement of any tangle T has slope $\{T\}_1$.

For the above reason, we call $\{T\}_1 \in \widehat{\mathbb{Q}}$ the algebraic slope of T . This conjecture is related to [35, Question 3.4] which asks more broadly if the slopes of such surfaces are determined by T .

Theorem 1.9 (Proof in Section 10) Our algebraic slope coincides with Ozawa's slope for algebraic tangles. In other words, the above conjecture holds for algebraic tangles.

2 Preliminaries

2.1 Rational and algebraic tangles and links

Throughout, two-tangles will be called *tangles* for brevity. They are proper tame embeddings of one-manifolds into $D^3 = D^2 \times (-1, 1)$ with ends at NE, SE, SW, NW points of ∂D^2 , where D^2 is a compact disk identified with $D^2 \times \{0\}$. Tangles are considered up to isotopies fixing ∂D^2 . A *two-string tangle* is one which consists of two strings only (no loops). The $\langle -1 \rangle, \langle 0 \rangle, \langle 1 \rangle$ and $\langle \infty \rangle$ tangles and the *tangle addition* are depicted in Figure 1.

The result of adding n tangles $\langle 1 \rangle$ (resp. $\langle -1 \rangle$) together is denoted by $\langle n \rangle$ (resp. $\langle -n \rangle$), for $n = 1, 2, 3, \dots$. These tangles, together with $\langle 0 \rangle$, are called *integral*.

The *mirror image* $-T$ of T is obtained by switching all crossings of T . The *tangle rotation* $R(T)$ is the 90° clockwise rotation and the *tangle inversion* is the tangle rotation followed by the mirror image.

All tangles obtained from the integral ones by the operations of addition and rotation are called *algebraic*. (This class is closed under the mirror image and inversion.) Among them are rational tangles defined as follows:

By $\langle a_n, \dots, a_1 \rangle$, we denote the tangle obtained from $\langle 0 \rangle$ by adding $\langle a_1 \rangle$ followed by the inversion, then by adding $\langle a_2 \rangle$ followed by the inversion, and so on, until this construction is finished by adding a_n at the end, as in Figure 2 (left). Tangles of this form, for $a_1, \dots, a_n \in \mathbb{Z}$, are called *rational* because as observed by Conway [7] (and proved in [6, 24, 32])

$$\langle a_n, \dots, a_1 \rangle \rightarrow a_n + \frac{1}{a_{n-1} + \frac{1}{\ddots + \frac{1}{a_1}}}$$

defines a bijection between rational tangles (up to tangle isotopy) and $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Under this bijection, tangle inversion corresponds to the fraction inversion, $x \rightarrow 1/x$, and the mirror image operation corresponds to the negation, $x \rightarrow -x$.

This association does not preserve addition.

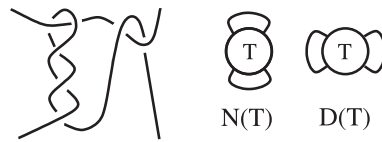


Figure 2: Rational tangle $\langle -2, -3, 2 \rangle$, and the numerator, and the denominator closures.



Figure 3: A satellite of a long trefoil.

By applying the numerator or the denominator closure (defined in Figure 2) to a rational tangle, we obtain a *rational link*, also referred to as a two-bridge link or four-plat.

2.2 Tangle equations

A *tangle equation* has the form

$$(2.1) \quad N(X + T) = L,$$

where a rational tangle T and a link L are given and the tangle X is unknown. Finding all solutions of (2.1) is usually exceedingly difficult.

Let us denote by B and D the three-balls in which X and T lie, and let S be the four-punctured sphere $S_4^2 = \partial B - X = \partial D - T$. For any ϕ in the mapping class group, $MCG(S_4^2)$, the action of ϕ on S_4^2 extends to maps of $B - X$ and of $D - T$ to $B - X'$ and $D - T'$, respectively, such that $N(X + T) = N(X' + T')$. In other words, $MCG(S_4^2)$ acts on all pairs (X, T) while preserving $N(X + T)$. (The orbits of that action on tangles are the isotopy classes of tangles moving their endpoints around.) We call pairs (X, T) and $\phi(X, T)$, for $\phi \in MCG(S_4^2)$ *equivalent*. Since all rational tangles belong to a single orbit of $MCG(S_4^2)$, this equivalence defines a 1–1 correspondences between solutions of tangle equations (2.1) for different T 's.

Recall that a tangle is two-string if it has no loops.

Proposition (1) *If L is the unlink of two components, then Eq. (2.1) has infinitely many solutions but only one two-string one.*

(2) *If L is a knot or a nontrivial link then Eq. (2.1) has infinitely many two-string solutions.*

Proof Since every tangle equation is equivalent to one with $T = \langle 0 \rangle$, we assume that for the proof. Hence, clearly it has at least one solution, X . For L the unlink, $X = \langle 0 \rangle$.

Assume that L is not the unlink now. For any long knot K , let $Sat_X(K)$ be its satellite with X as a companion tangle, as in Figure 3. Then $N(Sat_X(K)) = N(X)$. Hence, if X is a solution of Eq. (2.1), then $Sat_X(K)$ is a solution as well. Since $Sat_X(K)$ and

$Sat_X(K')$ are non-isotopic for different K and K' (here, we assume that $X \neq \langle 0 \rangle$), there are infinitely many different tangles of this form. ■

For a nontrivial rational L [11, Theorem 2.2] shows infinitely many rational solutions to Eq. (2.1).

Finally, we remark that Proposition 2.1 does not hold if T is allowed to be nonrational. For example, Eq. (2.1) has no solutions for certain pretzel tangles T and L the unknot, since one can show that some pretzel tangles do not embed into the unknot. (That fact is an easy application of Proposition 5.1.)

2.3 Systems of tangle equations

A system of tangle equations has the form (1.1) or, equivalently,

$$\begin{cases} N(X + T_1) = L_1, \\ N(X + T_2) = L_2, \end{cases}$$

where $T_1 \neq T_2$.

Systems of tangle equations play a crucial role in the analysis of entanglement of DNA molecules. The reason for that is that certain enzymes (called recombinase) separate circular DNA substrate molecules into two tangles: T_1 , consisting of the part of the DNA molecule bound to the enzyme, and the other part, X , not bound to the enzyme. Then the enzyme replaces T_1 by a tangle T_2 . The substrate knot L_1 is controlled by the experiment. The link L_2 is observable in the experiment. Since tangles T_1 and T_2 are known, DNA recombination processes lead to systems (1.1), where X is the unknown (see, e.g., [1, 5, 8, 9, 11, 38, 39, 43, 44] for further discussion of such systems).

The action $MCG(S_4^2)$ on tangles of the previous section extends onto an action on triples (X, T_1, T_2) . By analogy, we call triples (X, T_1, T_2) and $\phi(X, T_1, T_2)$ *equivalent* for any $\phi \in MCG(S_4^2)$. This equivalence defines a 1-1 correspondences between solutions of systems of tangle equations (1.1) for equivalent pairs (T_1, T_2) and (T'_1, T'_2) .

Proposition For every pair (T_1, T_2) not equivalent to $(\langle 0 \rangle, \langle k \rangle)$ for $k \in \mathbb{Z} \cup \{\infty\}$ and for every $n > 0$ there are links L_1 and L_2 such that (1.1) has at least n solutions.

A Montesinos tangle is a sum of rational ones, $R_1 + \dots + R_k$. The numerator closure of a Montesinos tangle is a *Montesinos link*. In the above proposition, one can further assume that L_1, L_2 and the solutions X are Montesinos.

Proof of Proposition 2.3 By utilizing an equivalence, we can assume that equations (1.1) have form

$$N(X) = L_1, \quad N(X + T) = L_2,$$

for some L_1, L_2 and $T = \langle x_0 \rangle$, for some $x_0 \in \widehat{\mathbb{Q}} - \mathbb{Z}$. Let

$$X_i = T + \dots + T + \langle x_i \rangle + T + \dots + T,$$

for some $x_i \in \widehat{\mathbb{Q}}$, where there are iT 's on the left and $2n - iT$'s on the right. By a cyclic symmetry, $N(X_i)$ coincide for $i = 0, \dots, n - 1$ and also $N(X_i + T)$ coincide for $i = 0, \dots, n - 1$.

We claim that choosing $x_1 \in \widehat{\mathbb{Q}}$ so that $x_1 \neq x_0 \pmod{1}$ guarantees that all X_1, \dots, X_n are distinct. Indeed, the links $N(X_i + \langle x_0 \rangle)$ are Montesinos. By the classification of Montesinos links [4], they are all distinct if $x_0 \neq x_1 \pmod{1}$.

In the case of systems of tangle equations, the conditions for the existence and uniqueness or finiteness of solutions seem unknown in general. We propose the following.

Conjecture Every system (1.1) have finitely many solutions X only.

We are going to see later (e.g., in Section 8) that the uniqueness of solutions is easier to analyze in the context of framed tangles and links.

3 Surgery methods, cosmetic surgery conjecture

We denote by $\Sigma(T)$ the double cover of the three-ball B^3 branched along a tangle T in it. Similarly, $\Sigma(L)$ denotes the double cover of S^3 branched along a link L . Since the double branched cover of a rational tangle T is a solid torus, every solution X of (1.1) defines a knot (the core of $\Sigma(T)$) in $\Sigma(L)$.

As an application of surgery methods, we obtain Proposition 1.5 which we recall here.

Proposition 1.5 If $L_1 = L_2$ is either a nontrivial rational knot or a nontrivial torus knot, then system (1.1) has no algebraic tangle (framed or unframed) solutions for any T_1, T_2 .

Proof Suppose that X is an algebraic tangle solution to

$$(3.1) \quad N(X + T_1) = L = N(X + T_2),$$

for $T_1 \neq T_2$. By the above discussion, there is a nontrivial surgery of $\Sigma(L)$ yielding $\Sigma(L)$. If L is rational then $\Sigma(L)$ is a lens space and if L is a torus knot then $\Sigma(L)$ is Seifert with base the two-sphere and three exceptional fibers (see, e.g., [18, Section 3.1]). Either way, it is irreducible and atoroidal. Furthermore, $\Sigma(X)$ is a graph manifold and, hence, non-hyperbolic. Hence, [31, Corollary 1.4] implies that $\Sigma(T_i)$ lies in a three-ball. Since $\Sigma(L) \neq S^3$, the boundary of that ball is incompressible in $\Sigma(X)$, making it reducible. Since X is rational or prime, that is impossible by [29, Theorem 5]. ■

As observed in the proof above, every solution of (3.1) yields a cosmetic surgery on $\Sigma(L)$. Such surgeries are rare and are subject of the cosmetic surgery conjecture, (see [15, 25]). A special case of that conjecture is the nugatory crossing conjecture, asserting that if a change of a sign of a crossing c in a knot diagram D results in an isotopic knot then c is nugatory, that is K has form below (see [2, 3, 21, 30, 40]).

We say that a tangle $X \subset B^3$ is *unlinked* if it is a two-string tangle whose components can be isotoped so that they do not intersect each other. (They can be all obtained by knotting the arcs of $\langle 0 \rangle$ or of $\langle \infty \rangle$.) The following is straightforward.

Proposition Nugatory crossing conjecture is equivalent to the following statement: If $N(X + \langle 1 \rangle) = N(X + \langle -1 \rangle)$ is a knot, then X is unlinked.



Figure 4: A nugatory crossing in knot. Disks denote one-tangles.

Proof (1) \Rightarrow Suppose the nugatory crossing conjecture holds and

$$N(X + \langle 1 \rangle) = N(X + \langle -1 \rangle)$$

is a knot K . Then the crossing in $\langle \pm 1 \rangle$ is nugatory in K . By the nugatory crossing conjecture, K consists of two arcs, each in a disjoint ball, as in Figure 4. Since these balls can be placed in the three-disk B^3 to form X , the statement follows.

\Leftarrow Any knot with a nugatory crossing can be realized as $N(X + \langle 1 \rangle)$ where the nugatory crossing is in $\langle 1 \rangle$. Since X is unlinked the statement follows. ■

It is worth noting that $N(X + \langle 1 \rangle) = N(X + \langle -1 \rangle)$ cannot be a link for a two-string X .

Finally, the following example shows that the assumption of nontriviality of $L_1 = L_2$ in Proposition 1.5 is essential. It will be useful later.

Lemma 3.2 $X = \langle \infty \rangle$ and $\langle -1/2 \rangle$ are the only solutions to

$$(3.2) \quad N(X) = U = N(X + \langle 1 \rangle),$$

where U denotes the unknot.

Proof The numerator closure $N(\cdot)$ and the operation $N(\cdot + \langle 1 \rangle)$, on the level of the double covers correspond to two different Dehn fillings of $\Sigma(X)$. Each of them yields the double cover of S^3 branched along U , i.e., S^3 . By a theorem of Gordon and Luecke [16, Theorem 2], $\Sigma(X)$ must be a solid torus and, hence, X is rational, $X = \langle p/q \rangle$.

By a theorem of Schubert,

$$N(\langle p/q \rangle) = U = N(\langle 1 \rangle)$$

only if $p = 1$ (see [24, 37]). Since $N(\langle \frac{1}{q} \rangle + \langle 1 \rangle) = U$, it is easy to see that q is either 0 or $-1/2$. ■

4 Framed tangles and systems of equations

Framed links are tame embeddings of annuli $S^1 \times I \cup \dots \cup S^1 \times I$ into \mathbb{R}^3 , where I is an open interval. Similarly, *framed tangles* are proper tame embeddings

$$J_1 \times I \cup J_2 \times I \hookrightarrow D^2 \times (-1, 1),$$

where J_1, J_2 are closed intervals and the end arcs $\partial J_1 \times I, \partial J_2 \times I$ lie disjointly in $\partial D^2 = \partial(D^2 \times \{0\})$ each containing a different point from among NE, SE, SW, NW in ∂D^2 . Clearly, every link diagram and tangle diagram defines a framed link or tangle with its framing parallel to the page. We require that every framed link and tangle can be represented in that way. (Hence, components with a half-twist framing are not allowed.)

For our purposes, it will be convenient to consider framed links and tangles up to *balanced isotopy* given by the balanced Reidemeister moves, presented in Figure 5.



Figure 5: Balanced Reidemeister moves. (Diagrams have blackboard framing.)

Note that this is a somewhat more flexible isotopy than the standard isotopy of framed links or tangles.

A *framed rational tangle* is a rational tangle with an arbitrary framing. Given a diagram D of a tangle or a link, we will denote by D^n the framed diagram obtained from the page framing of D by adding $|n|$ positive or negative twists, depending on the sign of n . Note that location of these twists does not matter up to balanced Reidemeister moves.

Definitions of tangle addition and of numerator and denominator closures generalize immediately to framed tangles. Consequently, systems (1.1) can be considered in the context of framed tangles and links.

4.1 Systems of framed tangle equations

Framed links and tangles seem more appropriate for modeling DNA molecules which are double stranded. Additionally, we will see soon, that it sometimes easier to solve tangle equations in the framed setting and framed solutions inform on unframed ones.

Let us assume that T_1, T_2 are framed rational tangles now. As before, we assume that T_1, T_2 are unequal as unframed tangles. (Note that if T_1 and T_2 in (1.1) differ by framing twists only, then by adjusting the framing of L_2 , one can reduce (1.1) to a form in which $T_1 = T_2$ as framed tangles. In that form, (1.1) is either inconsistent or reduces to a single equation.)

Note that Proposition 2.1 easily extends to framed equations.

Given an unframed system (1.1), its *framing* is a choice of a framing for L_1, L_2, T_1, T_2 . Let us consider the following framed version of system (3.2) which will be useful later:

$$(4.1) \quad N(X) = U^n \text{ and } N(X + \langle 1 \rangle) = U^m,$$

for some $n, m \in \mathbb{Z}$, where as defined above, U^n denotes the unknot with $|n|$ twists of framing, positive or negative, depending on the sign of n .

- Lemma 4.1** (1) If $m = n + 1$, then $\langle \infty \rangle^n$ is the only solution to (4.1).
 (2) If $m = n - 3$, then $\langle -1/2 \rangle^{n+2}$ is the only solution to (4.1).
 (3) If $m - n$ is neither 1 or -3 , then (4.1) has no framed solution.

Proof Suppose that X satisfies the above equations. Then stripped of its framing, it is either $X = \langle \infty \rangle$ or $\langle -1/2 \rangle$, by Lemma 3.2. Since

$$\text{Diagram 1} = U^f \text{ and } \text{Diagram 2} = U^{f+1}$$

where the dashed square marks the tangle $\langle \infty \rangle^f$, for some $f \in \mathbb{Z}$, the tangle $X = \langle \infty \rangle$ lifts to a framed solution only iff $m - n = 1$ and that solution is $\langle \infty \rangle^f$, where $f = n$. (For reader's convenience, we marked the signs of crossings in the diagrams.) Similarly,

$$\begin{array}{c} \text{---} \\ \diagup \\ \diagdown \\ \text{---} \end{array} = U^{f+2} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \diagdown \\ \diagup \\ \text{---} \end{array} = U^{f-1}$$

shows that $X = \langle -1/2 \rangle$ lifts to a framed solution only iff $m - n = -3$ and that solution is $\langle -1/2 \rangle^f$ for $f = n - 2$.

For $m - n \neq 1, -3$, these equations are contradictory. ■

Note that every framed system (1.1) defines an unframed one and every framed solution descends to an unframed one. Lemmas 3.2 and 4.1 show that not every solution to the unframed system can be lifted to a framed one. In fact, there may be no framing of (1.1) for which all unframed solutions can be framed.

5 The Kauffman bracket of a tangle

The Kauffman bracket $[L] \in \mathbb{Z}[A^{\pm 1}]$ is an invariant of framed unoriented links L up to isotopy, satisfying the skein relations

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = A \begin{array}{c} \diagdown \\ \diagup \end{array} + A^{-1} \begin{array}{c} \diagdown \\ \diagdown \end{array}, \quad \bigcirc = \delta$$

(with the blackboard framing), where $\delta = -A^2 - A^{-2}$, normalized so that the bracket of the trivially framed unknot U^0 is $[U^0] = 1$.

By the above skein relations, each framed tangle can be expressed as

$$T = [T]_0 \cdot \begin{array}{c} \diagdown \\ \diagdown \end{array} + [T]_\infty \cdot \begin{array}{c} \diagdown \\ \diagup \end{array}$$

where $[T]_0, [T]_\infty \in \mathbb{Z}[A^{\pm 1}]$ are uniquely defined. Recall that we call the vector $[T] = \begin{pmatrix} [T]_0 \\ [T]_\infty \end{pmatrix}$ the Kauffman bracket of T .

Kauffman brackets of tangles provide a criterium for embedding them into links.

The *KB determinant* of T denoted by $det_{KB}(T) \in \mathbb{Q}[A^{\pm 1}]$ is $gcd([T]_0, [T]_\infty)$ in $\mathbb{Q}[A^{\pm 1}]$. Note that it is well defined for unframed tangles up to a multiplicative factor of A^n , for $n \in \mathbb{Z}$.

Proposition *A necessary condition for a tangle T embedding into a link L is that $det_{KB}(T)$ divides $[L]$ in $\mathbb{Q}[A^{\pm 1}]$.*

This result generalizes the main result of [26] which treats $A = e^{\pi i/4}$ case (see also [36]). (This idea was utilized for the trivial L in [41].) More generally, a necessary condition is that $[L] \in \mathbb{Z}[A^{\pm 1}]$ lies in the ideal $\langle [T]_0, [T]_\infty \rangle \triangleleft \mathbb{Z}[A^{\pm 1}]$.



Figure 6: 1 and -1 tangle types oriented vertically (on the left) and horizontally (on the right).

Proof of Proposition 5.1

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} \text{---} = [T]_0[S]_0 \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} + [T]_0[S]_\infty \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} + [T]_\infty[S]_0 \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} + [T]_\infty[S]_\infty \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---}$$

and, therefore, a tangle T embeds into a link L only if $\gcd([T]_0, [T]_\infty)$ divides $[L]$ in $\mathbb{Q}[A^{\pm 1}]$. (Formally, one needs framed tangles and links for that, but since framing affects the bracket by a multiplicative factor of $(-A^3)^n$ which is a unit in $\mathbb{Q}[A^{\pm 1}]$, that statement makes sense in the unframed setting as well.)

Note that $[T]_0, [T]_\infty$ are integers for that $A = e^{\pi i/4}$. The topological meaning of $\gcd([T]_0, [T]_\infty)$ for that A seems to be an interesting question. It appears to be related to the order of the torsion of the double branched cover of T , however, it is not always equal to it, as observed in [36].

The following discussion will be useful later. We say that an unframed two-string tangle T is of type 0, ± 1 , or ∞ if it can be reduced to one of these rational tangles through crossing changes. Note that each tangle is of precisely one of these three types, depending on connections between its endpoints. Type ± 1 has two subtypes: $+1$ and -1 , depending on whether one can get from the original tangle T to tangle $+1$ in an even or odd number of crossing changes between different components. For the statement below, we will need to consider oriented tangles. The orientation will be considered up to the total orientation reversal, hence

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{---}$$

Every oriented tangle of type ± 1 is either oriented *vertically* or *horizontally* depending on whether one can reduce it to the vertically or horizontally oriented tangles of Figure 6, by crossing changes which involve an even number of changes between different components.

If we denote the writhe of a framed oriented link L by $w(L)$, then $(-A)^{-3w(L)}[L]$ is the Jones polynomial $J(L)$ for $t = A^{-4}$. Since $J(L) \in t^{(|L|-1)/2}\mathbb{Z}[t^{\pm 1}]$, where $|L|$ denotes the number of components of L , the bracket $[L]$ can be written with all exponents of A congruent to

$$3w(L) + 2(|L| - 1) \pmod{4}.$$

We extend this property to Kauffman brackets of tangles.

Let $|T|$ denote the number of connected components of a tangle T .

Proposition For any oriented tangle T , the bracket $[T]_\tau$, for τ is 0 or ∞ , can be written with all exponents of A congruent to

$$3w(T) + 2|T| + 2\eta(T) + \delta_\tau \pmod 4,$$

where $\delta_\tau = 0$ or 2 depending on whether τ is 0 or ∞ and $\eta(T) = 0$ for T of type ± 1 vertically oriented (see Figure 6) and for type 0. Otherwise, $\eta(T) = 1$; that is for type ± 1 oriented horizontally and for T of type ∞ .

Proof We prove it for all oriented tangle diagrams D by induction on the crossing number $c(D)$ of D . The statement is obvious for $c(D) = 0$. Assume that the statement holds for all diagrams with $c(D) \leq C$. We are going to prove the statement for diagrams D with $C + 1$ crossings by the induction on the *undescendedness* $ud(D)$, of D , which measures the failure of being a descending diagram.

It is defined as follows: We will refer to the connected components of the tangle represented by D as the components of D for simplicity. Let us also orient and order all these components and let us choose a base point on each loop component of D away from crossings. Next, let us follow all components of D according to their order, and according to their orientations, starting with their beginning (if they are arcs) or their base points (if they are loops). A crossing of D is *improper* if its underpass is transversed before its overpass is. (The improperness of a crossing will depend on the above choices.) The undescendedness, $ud(D)$, of D is the minimum of the numbers of improper crossings in D over all choices of basepoints and choices of orientations and orderings of the components of the tangle represented by D . In particular,

$$0 \leq ud(D) \leq c(D).$$

If $ud(D) = 0$ then D is isotopic to $\langle 0 \rangle$, $\langle \pm 1 \rangle$ or $\langle \infty \rangle$ depending on the type of the tangle represented by D , with possibly some additional unlinked trivial loop components. It is easy to see that the statement follows in this case. Assume that the statement holds for all diagrams D with $c(D) = C + 1$ and $ud(D) \leq u$. Let D be a diagram with $c(D) = C + 1$ and $ud(D) = u + 1$, and let ν be one of the improper crossings counted in $ud(D)$ (according to a certain choice of base points, orientations, and of ordering). Assume that this crossing has sign $\varepsilon = \pm$. Then

$$A[D_+] - A^{-1}[D_-] = (A^2 - A^{-2})[D_0],$$

where D_ε is D , and $D_{-\varepsilon}$ is D with the crossing ν reversed and D_0 is D with ν smoothed out. Consequently,

$$[D] = A^{-2\varepsilon}[D_{-\varepsilon}] + \varepsilon \cdot A^{-\varepsilon}(A^2 - A^{-2})[D_0].$$

Since

$$ud(D_{-\varepsilon}) < ud(D), \quad c(D_0) < c(D),$$

and

$$w(D_{-\varepsilon}) = w(D) - 2\varepsilon \text{ and } w(D_0) = w(D) - \varepsilon,$$

by the inductive assumption, the exponents of $A^{-2\varepsilon}[D_{-\varepsilon}]_\tau$ are

$$-2\varepsilon + 3(w(D) - 2\varepsilon) + 2|D| + 2\eta(D) + \delta_\tau \pmod 4$$

which equals

$$(5.1) \quad 3w(D) + 2|D| + 2\eta(D) + \delta_\tau \pmod 4.$$

Similarly, the exponents of $A^{-\varepsilon}(A^2 - A^{-2})[D_0]_\tau$ are

$$-\varepsilon \pm 2 + 3(w(D) - \varepsilon) + 2|D_0| + 2\eta(D_0) + \delta_\tau \pmod 4$$

which equals

$$(5.2) \quad \pm 2 + 3w(D) + 2|D_0| + 2\eta(D_0) + \delta_\tau \pmod 4.$$

To complete the argument, let us assume first that v is a single component crossing. Then $D, D_{-\varepsilon}, D_0$ have the same type and

$$|D_\varepsilon| = |D_{-\varepsilon}| = |D_0| \pm 1,$$

Hence, the quantities in (5.1) and (5.2) coincide and imply that the exponents of $[D]_\tau$ are

$$3w(D) + 2|D| + 2\eta(D) + \delta_\tau \pmod 4,$$

as claimed in the statement, thus completing the proof of the inductive step in the single-component crossing case.

If v is a two-component crossing, then $D, D_{-\varepsilon}$ have the same type and

$$|D_\varepsilon| = |D_{-\varepsilon}| = |D_0|.$$

A two-component crossing smoothing transforms tangle types and orientations in one of the following three ways:



(Since these are tangle types, no crossing signs are indicated.)

Consequently, $\eta(D) \neq \eta(D_0)$. Hence, the quantities in (5.1) and (5.2) coincide and imply that the the exponents of $[D]_\tau$ are

$$3w(D) + 2|D| + 2\eta(D) + \delta_\tau \pmod 4,$$

as claimed in the statement, thus completing the proof of the inductive step in the two-component crossing case. ■

6 The KB ratio of a tangle

Let

$$Q(T) = [T]_\infty / [T]_0 \in \mathbb{Q}(A) \cup \{\infty\}.$$

It is well defined, because $[T]_\infty$ and $[T]_0$ cannot both vanish. (This follows, for example, from Proposition 5.1 and the fact that $[N(T)] \neq 0$, because $[N(T)] = (-2)^{|N(T)|}$ for $A = \pm 1$.)

Note that $Q(T)$ is preserved by the first Reidemeister move and, hence, it is an invariant of unframed tangles.

Let

$$q = -A^4 \text{ and } \{T\}_q = A^{-2}Q(T).$$

Proposition 5.2 implies

$$\{T\}_q \in \mathbb{Q}(q) \cup \{\infty\},$$

thus proving Theorem 1.6. We call it the *Kauffman bracket ratio*, or the *KB-ratio* of T , for short. KB-ratios will play an important role in the remainder of the paper.

Proposition For any tangle T ,

(1) $\{-T\}_q = -q^{-1}\{T\}_{q^{-1}}$, where the subscript on the right-hand side indicates q substituted by q^{-1} . (Recall that $-T$ is the mirror image of T .)

(2) $\{R(T)\}_q = -q^{-1}/\{T\}_q$, where R denotes the rotation operation, as before.

Additionally, for any T' ,

(3) $\{T + T'\}_q = \{T\}_q + \{T'\}_q + (q - 1)\{T\}_q\{T'\}_q$.

Proof (1) $\{-T\}_q = A^{-2}Q(-T) = A^{-2}Q(T)_{A^{-1}} = A^{-4}\{T\}_{q^{-1}}$.

(2) Since $Q(R(T)) = 1/Q(T)$,

$$\{R(T)\}_q = A^{-2}Q(R(T)) = A^{-2}/Q(T) = A^{-4}/(A^{-2}Q(T)) = -q^{-1}/\{T\}_q.$$

(3) By the following identity

$$\langle \text{Diagram of } T \text{ and } T' \text{ with crossing} \rangle = [T]_0[T']_0 \langle \text{Diagram of } T \text{ and } T' \text{ with crossing} \rangle + ([T]_0[T']_\infty + [T]_\infty[T']_0 + [T]_\infty[T']_\infty \delta) \langle \text{Diagram of } T \text{ and } T' \text{ with crossing} \rangle,$$

$$Q(T + T') = Q(T) + Q(T') + \delta Q(T)Q(T'),$$

and, hence,

$$\{T + T'\}_q = \{T\}_q + \{T'\}_q + A^2 \delta \{T\}_q \{T'\}_q. \quad \blacksquare$$

7 q -Deformed rationals

Quantum integers,

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{for } n \neq 0, \\ 0, & \text{for } n = 0, \end{cases}$$

for $n \in \mathbb{Z}$, appear already in the 1808 work of Gauss on binomial coefficients [14, 17]. They are at the foundation of quantum calculus (see [20]), and are indispensable in quantum algebra and in quantum topology (see, e.g., [23]).

Quantum integers were extended in [33] to “ q -deformed rationals” $[x]_q$ for $x \in \mathbb{Q}$. They satisfy the following identities:

$$(7.1) \quad [x + 1]_q = q[x]_q + 1, \quad [-x]_q = -q^{-1}[x]_{q^{-1}}, \quad [1/x]_q = \frac{1}{[x]_{q^{-1}}}$$

(cf. [28]).

They can be defined also through our bracket $\{\cdot\}_q$ and, in fact, in this way, they have been discovered independently by us in the first version of this paper on arXiv.

Note that the first two identities of Eq. (7.1) are satisfied by $\{\cdot\}_q$ by Proposition 6.1. The last one is satisfied as well: According to Conway's correspondence $\langle 1/x \rangle = R(-\langle x \rangle)$ and, hence,

$$\{\langle 1/x \rangle\}_q = \{R(-\langle x \rangle)\}_q = -q^{-1}/\{-\langle x \rangle\}_q = -q^{-1}/(-q^{-1}\{\langle x \rangle\}_q)_{q \rightarrow q^{-1}} = 1/\{\langle x \rangle\}_{q^{-1}}.$$

By the theory of continued fractions, the above identities determine the values of $[x]_q$ and of $\{x\}_q$ for all $x \in \mathbb{Q}$. (In fact, the formulas for $x + 1$ and $-1/x$ are sufficient.) This implies the following statement from the Introduction.

Theorem 1.7 For every $x \in \widehat{\mathbb{Q}}$, $[x]_q = \{\langle x \rangle\}_q$.

Let $\{\langle x \rangle\}_1$ denote $\{\langle x \rangle\}_q$ evaluated at $q = 1$. By the above theorem and Eq. (7.1), we have the following.

Corollary 7.1 ([33]) For every $x \in \widehat{\mathbb{Q}}$, $\{\langle x \rangle\}_1 = x$.

Corollary 7.2 The Kauffman bracket distinguishes all framed tangles among rational ones.

Proof Any two framed rational tangles T, T' with $\{T\}_q = \{T'\}_q$ must represent the same rational number and, hence, may differ by framing only. However, if T' is obtained from T by adding n twists to it (where negative n means $|n|$ negative twists) then $[T'] = (-A)^{3n}[T]$. Hence, $n = 0$. ■

8 The uniqueness of the KB of a solution of a system of tangle equations

Theorem 8.1 For any system of framed tangle equations (1.1), $[X]$ is unique.

Proof Let X be a solution of framed (1.1). Then



and analogously for T_2 . Hence,

$$\begin{pmatrix} [N(X + T_1)] \\ [N(X + T_2)] \end{pmatrix} = B \cdot \begin{pmatrix} [X]_0 \\ [X]_\infty \end{pmatrix},$$

where

$$B = \begin{pmatrix} [T_1]_0 \delta + [T_1]_\infty & [T_1]_0 + [T_1]_\infty \delta \\ [T_2]_0 \delta + [T_2]_\infty & [T_2]_0 + [T_2]_\infty \delta \end{pmatrix}.$$

Note that

$$\det B = (\delta^2 - 1) \cdot \det \begin{pmatrix} [T_1]_0 & [T_1]_\infty \\ [T_2]_0 & [T_2]_\infty \end{pmatrix}$$

is nonzero, even for $q = 1$ by Corollary 7.1.

Given a system of equations (1.1), let

$$q_\mu = \frac{\det \begin{pmatrix} [L_1] & [T_1]_0 + \mu[T_1]_\infty \\ [L_2] & [T_2]_0 + \mu[T_2]_\infty \end{pmatrix}}{(\delta - \mu) \cdot \det \begin{pmatrix} [T_1]_0 & [T_1]_\infty \\ [T_2]_0 & [T_2]_\infty \end{pmatrix}} \text{ for } \mu = \pm 1.$$

Then, by Cramer’s rule, one can verify that

$$[X]_0 = \det \begin{pmatrix} [L_1] & [T_1]_0 + [T_1]_\infty \delta \\ [L_2] & [T_2]_0 + [T_2]_\infty \delta \end{pmatrix} / \det B = \frac{1}{2}(q_1 - q_{-1})$$

and

$$[X]_\infty = \det \begin{pmatrix} [T_1]_0 \delta + [T_1]_\infty & [L_1] \\ [T_2]_0 \delta + [T_2]_\infty & [L_2] \end{pmatrix} / \det B = \frac{1}{2}(q_1 + q_{-1}). \quad \blacksquare$$

The above equations provide necessary algebraic conditions for the existence of a framed solution of (1.1). In particular, we have the following.

Corollary 8.2 *A necessary condition for the existence of a framed solution to (1.1) is that*

$$p_0 := \frac{1}{2}(q_1 - q_{-1}), \quad p_\infty := \frac{1}{2}(q_1 + q_{-1}) \in \mathbb{Z}[A^{\pm 1}]$$

and that $A^{-2}p_\infty/p_0 \in \mathbb{Q}(A^4)$.

Note that Theorem 8.1 together with Corollary 7.2 implies Proposition 1.1 stating that every framed system (1.1) has at most one-framed rational solution.

9 Jones conjecture for tangles

Recall that U^n denotes the unknot with framing $n \in \mathbb{Z}$.

Lemma 9.1 (Kauffman bracket version of Jones conjecture)

The Jones conjecture (JC) is equivalent to its Kauffman bracket version (KBJC): if $[K] = r \cdot A^k$, for some $r, k \in \mathbb{Z}$, then $K = U^n$ for some $n \in \mathbb{Z}$.

Proof KBJC \Rightarrow JC: Suppose that the Jones polynomial of K is $J(K) = 1$ for some knot K . Then K with some framing has its Kauffman bracket equal to $(-A^3)^n$ for some $n \in \mathbb{Z}$. By KBJC, $K = U^n$. Hence, K is trivial as an unframed knot.

JC \Rightarrow KBJC: Suppose that $[K] = r \cdot A^k$, for some $r, k \in \mathbb{Z}$. Then $J(K) = r(-A)^{-3w(K)}A^k$. By [13, Corollary 3], $J(K) = 1$. Hence, K is (unframed) trivial, by the Jones conjecture. \blacksquare

Now, we can formulate three versions of the Jones conjecture for tangles.

Theorem 9.2 *The Jones conjecture is equivalent to each of the following statements:*

- (a) *For any framed tangle T , if $[T] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $T = \langle 0 \rangle^0$.*
- (b) *If $[T] = \begin{pmatrix} r \cdot A^n \\ 0 \end{pmatrix}$ for some $r, n \in \mathbb{Z}$, then $T = \langle 0 \rangle$ as an unframed tangle.*
- (c) *If $[T'] = [T]$ and T is rational, then $T' = T$ as framed tangles.*

It is worth noting that statement (c) above does not hold for rational knots. That is, there are examples of distinct rational knots with coinciding Jones polynomials [22].

Proof of Theorem 9.2 *JC implies (a): Assume the Jones conjecture holds and that $[T] = (1, 0)$. (For convenience, in this proof we will write all vertical vectors horizontally.) Then $[R(T)] = (0, 1)$ and it is easy to check that*

$$[N(R(T))] = 1 \text{ and } [N(R(T) + \langle 1 \rangle)] = -A^3.$$

By Lemma 9.1,

$$N(R(T)) = U^0 \text{ and } N(R(T) + \langle 1 \rangle) = U^1.$$

Now, by Lemma 4.1, $R(T) = \langle \infty \rangle$ and, hence, $T = \langle 0 \rangle$.

(a) implies (c): Suppose that T is rational and $[T'] = [T]$. Then T can be transformed

$$T = T_1 \rightarrow \dots \rightarrow T_k = \langle 0 \rangle$$

by the operations of rotation, $R(\cdot)$, of addition of one, $P(T) = T + \langle 1 \rangle$, its inverse, $P^{-1}(T)$, and adding a positive or negative kink, $F^{\pm 1}(T)$, (for framing changes). Let us apply the same operations to T' :

$$(9.1) \quad T' = T'_1 \rightarrow \dots \rightarrow T'_k = T''.$$

Since

$$[T + \langle 1 \rangle] = \left[\begin{array}{c} \text{two crossings} \end{array} \right] = [T]_0 \left[\begin{array}{c} \text{crossing} \end{array} \right] + [T]_{\infty} \left[\begin{array}{c} \text{crossing with kink} \end{array} \right] = [T]_0 \begin{pmatrix} A^{-1} \\ A \end{pmatrix} + [T]_{\infty} \begin{pmatrix} 0 \\ -A^3 \end{pmatrix},$$

the operations R and P induce linear transformations of the Kauffman brackets:

$$[R(T)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} [T], \quad [P(T)] = [T + \langle 1 \rangle] = \begin{pmatrix} A^{-1} & 0 \\ A & -A^3 \end{pmatrix} \cdot [T].$$

Since R, P and F are invertible, $[T_i] = [T'_i]$ for every i and, hence, $[T''] = (1, 0)$. By (a), $T'_k = \langle 0 \rangle$, implying that $T' = T$.

(c) implies (b): Assume that $[T] = (r \cdot A^n, 0)$, for some $r, n \in \mathbb{Z}$. Then $[D(T)] = r \cdot A^n$ and by [13, Corollary 3] (as in the proof of Lemma 9.1), $J(D(T)) = 1$. That implies that $[T] = ((-A^3)^k, 0)$ has the bracket of $\langle 0 \rangle$ with some framing $k \in \mathbb{Z}$. Hence, by (c), $T = \langle 0 \rangle$, as unframed tangle.

(b) implies (a): Suppose $[T] = (1, 0)$. Then by (b), $T = \langle 0 \rangle$ as an unframed tangle. Since $[T] = (1, 0)$, the framing of T must be trivial.

(a) implies JC: Let K be a knot with trivial Jones polynomial. Let us frame it so that $[K] = 1$. Let $K\#\langle 0 \rangle$ be the connected sum of K with the lower strand of $\langle 0 \rangle$. Then $[K\#\langle 0 \rangle] = (1, 0)$ and $K\#\langle 0 \rangle = \langle 0 \rangle$ as unframed tangles, by (b) (which we proved is implied by (a)), implying that K is trivial.

Theorem 9.2 can be further refined to consider its statements up to a certain numbers of crossings. The crossing number, $c(T)$ of a framed or unframed tangle T is the minimal number of crossings in its unframed isotopy class. Then we can strengthen the implication $JC \Rightarrow (c)$ above as follows.

Proposition Assume that the Jones conjecture holds for knots up to N crossings. If $[T'] = [T]$, and $c(T) + c(T') < N$ and T is rational then $T' = T$ (as framed tangles).

Proof follows the above proof for $JC \Rightarrow (a) \Rightarrow (c)$. Note that the tangle T'' in Eq. (9.1) has at most $c(T) + c(T')$ crossings. Now, observe that the Jones conjecture up to N crossings implies statement (a) up to $N - 1$ crossings. ■

Note that Theorems 8.1 and 9.2 imply Theorem 1.3.

10 Slopes of tangles

Recall that we call $\{T\}_1 \in \widehat{\mathbb{Q}}$ the algebraic slope of T and that we postulate in Conjecture 1.8 that if an m -essential surface in the complement of T has a slope, that slope is $\{T\}_1$. In this section, we define the slope of a surface and we prove our conjecture for algebraic tangles.

Let us denote by \mathcal{T} the double cover of the boundary S^2 of the three-ball containing T , branched around $T \cap S^2$, and let $\theta : \mathcal{T} \rightarrow \mathcal{T}$ be the Deck transformation of that covering. The arc μ of the denominator closure arc connecting SW and NW and its image $\theta(\mu)$ under θ form a loop which we call the meridian of \mathcal{T} , cf. Figure 7. (By considering the numerator arc SE–NE, we get a parallel loop in \mathcal{T} .) Similarly, the arc SW–SE of the numerator closure together with its θ -image forms a loop which we call the longitude. We orient the meridian and longitude as in Figure 7.

Recall that a loop γ in S^2_4 is dividing if it separates the punctures of S^2_4 into two groups of two. Its slope $s(\gamma)$ is the slope of its lift to \mathcal{T} in the longitude–meridian basis of \mathcal{T} , so that the slopes of the longitude and the meridian in \mathcal{T} are of the slope 0 and ∞ , respectively. It is well defined by [12, Proposition 2.6].

As before, consider an m -essential surface S properly embedded in $D^3 \setminus T$. Recall that if its boundary contains a dividing loop γ we say that S is dividing and has slope $s(\gamma)$. (Then it is easy to see that all other dividing loops of ∂S must be parallel.) This definition is inspired by that in [35], where the slopes are defined for algebraic tangles.

For example, $T = \langle 0 \rangle$ has a horizontal disk $S = D^2$ in its complement which is m -essential. Since ∂D^2 lifts to the longitude in \mathcal{T} , the slope of $\langle 0 \rangle$ is 0. One can construct any rational tangle from $\langle 0 \rangle$ by the operations of rotation and addition of ± 1 which modify S accordingly. Consequently, it is easy to see that the slope of any rational tangle $\langle x \rangle$ is x .

Ozawa proves that for every algebraic tangle T , every m -essential surface in $B^3 - T$ is dividing and its slope depends on T only (see [35]). This is the slope of T , $s(T) \in \widehat{\mathbb{Q}}$.

Recall Theorem 1.9.

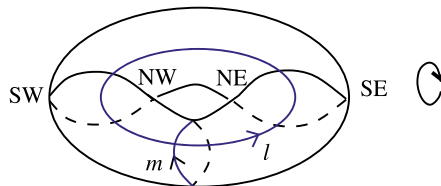


Figure 7: The double cover of a sphere ∂B^3 branched along the endpoints of a tangle in B^3 .

Theorem 1.9 For every algebraic tangle T , $\{T\}_1 = s(T)$.

Proof By [35, Proposition 2.4],

$$s(T_1 + T_2) = s(T_1) + s(T_2), \quad s(-T) = -s(T), \quad s(R(T)) = -1/s(T)$$

for algebraic tangles. Furthermore, $s(\langle n \rangle) = n$, for $n \in \mathbb{Z}$. Since $\{\cdot\}_1$ satisfies the same properties by Proposition 6.1 and since these properties determine the values of $s(T)$ and of $\{T\}_1$ for algebraic tangles, the statement follows. ■

The following provides an interpretation of the algebraic slope in terms of algebraic topology.

Proposition $|\{T\}_1| = \det(N(T))/\det(D(T))$ where \det denotes the link determinant.

Proof The determinant of $N(T)$ is

$$|J(N(T), -1)| = |(-A^3)^{-w(N(T))}[N(T)]| = |[N(T)]|,$$

where $A = e^{\pi i/4}$, $w(N(T))$ is the writhe of $N(T)$ (with some orientation) and

$$[N(T)] = \left(\text{link diagram of } [T]_\infty \right) + \left(\text{link diagram of } [T]_0 \right) = [T]_\infty.$$

Taking an analogous formula for $[D(T)]$, we have

$$\det(N(T))/\det(D(T)) = |[T]_\infty/[T]_0| = |Q(T)_1| = |A^2\{T\}_1| = |\{T\}_1|. \quad \blacksquare$$

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