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RESEARCH ARTICLE

Strategic underreporting and optimal deductible insurance*

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Abstract
This paper proposes a theoretical insurance model to explain well-documented loss underreporting and to study how strategic underreporting affects insurance demand. We consider a utility-maximizing insured who purchases a deductible insurance contract and follows a barrier strategy to decide whether she should report a loss. The insurer adopts a bonus-malus system with two rate classes, and the insured will move to or stay in the more expensive class if she reports a loss. First, we fix the insurance contract (deductibles) and obtain the equilibrium reporting strategy in semi-closed form. A key result is that the equilibrium barriers in both rate classes are strictly greater than the corresponding deductibles, provided that the insured economically prefers the less expensive rate class, thereby offering a theoretical explanation to underreporting. Second, we study an optimal deductible insurance problem in which the insured strategically underreports losses to maximize her utility. We find that the equilibrium deductibles are strictly positive, suggesting that full insurance, often assumed in related literature, is *not* optimal. Moreover, in equilibrium, the insured underreports a positive amount of her loss. Finally, we examine how underreporting affects the insurer's expected profit.

1. Introduction

Underreporting losses is prevalent in insurance markets as confirmed by numerous empirical studies.[1] Yet, the majority of classical insurance models either completely ignore underreporting (see Arrow, 1963 and Borch, 1962) or, even worse, lead to the contradictory conclusion that there is no incentive for accident underreporting (see Dionne and Lasserre, 1985). To offer a theoretical explanation for underreporting, Cao *et al.* (2023b) propose a multi-period insurance model under a bonus-malus system (BMS) and formulate a stochastic control problem in which the insured chooses a barrier strategy to report or to hide losses. However, a restrictive assumption in that work is that the insured has *full* insurance coverage, which raises two significant issues. First, insurance policies in real life often come with a strictly positive deductible (copay). Second, and more importantly, by assuming full insurance, we have also assumed a *fixed* contract and consequently cannot investigate how underreporting affects the insured's

*This paper was previously circulated under the title "Equilibrium Loss Reporting for a Risk-Averse Insured of Deductible Insurance"; see https://ssrn.com/abstract=4549720.

[1]For automobile insurance, Cohen (2005), Abbring *et al*. (2008), Robinson and Zheng (2010), and Gong (2017), all confirm that a significant portion of accidents is not reported, using data from Israel, the Netherlands, Canada, and China, respectively. For underreporting in workers' compensation insurance, see Petitta *et al*. (2017) and Probst *et al*. (2019).

Table 1. *Summary statistics on the deductibles of BC policies (in million USD).*

Min	25%-quantile	Median	Mean	75%-quantile	Max
0	6.215	6.908	7.165	7.824	11.513

insurance demand and contracting decision. In order to address these issues, we extend the model in Cao *et al.* (2023b) to incorporate deductible insurance and we study the following two questions:

- Question 1. Given a contract of deductible insurance, what is the equilibrium reporting strategy for the insured?
- Question 2. Knowing how she would strategically underreport, what is the optimal deductible insurance contract for the insured?

As hinted from the above description, the essential difference in modeling between Cao *et al.* (2023b) and this paper is that the insured here purchases a deductible insurance contract. For this reason, we start with providing empirical evidence to justify the consideration of deductible insurance over full insurance. Using the publicly available Wisconsin Local Government Property Insurance Fund (LGPIF) dataset,[2] we examine the deductible information on all building and contents (BC) policies. Among the total observed 6265 BC policies during 2006–2011, 6248 of them have a positive deductible, equivalent to a proportion of 99.73%; we report summary statistics about the range of deductibles in Table 1. Both the proportion with deductibles and the size of the deductibles in the BC policies strongly support the extension from full insurance to deductible insurance.

Once a loss occurs (assuming it is covered and larger than the contract deductible), the insured faces a trade-off when it comes to deciding whether she should report this loss to or hide it from the insurer. On the one hand, reporting the loss will yield a positive reimbursement from the insurer but likely cause an increase in future premiums;[3] on the other hand, if the insured hides the loss, she bears the entire cost herself but may enjoy a discount on future premiums due to her "clean claim record." As is obvious, such a trade-off exists for all insurance lines that apply experience rating to compute premiums. A typical example of such a rating system is a BMS, frequently adopted in nonlife insurance; see Lemaire (1995) for its application in automobile insurance. Under a BMS, a reported claim often downgrades the insured's rate class to a worse one, resulting in a higher premium for the next period; conversely, adding one more year (period) to the clean record improves the insured's rate class, leading to additional discounts on her premium. In this work, we follow Cao *et al*. (2023b) and consider a BMS insurance model with two rate classes: rate class 1 is the "good" one with lower premiums and rate class 2 is the "bad" one with higher premiums. If the insured does not file a claim in the current period, she will move to or remain in rate class 1 for the next period; otherwise, if she reports a claim in the current period, she will move to or remain in rate class 2 for the next period.[4] As such, the difference in premium between the two rate classes might provide a monetary incentive to the insured to hide certain losses.

Based on the described pros and cons of both reporting and hiding a loss from the insured's perspective, it is, then, natural to study a decision-making problem allowing the insured to decide whether

[2]Please see https://sites.google.com/a/wisc.edu/local-government-property-insurance-fund for information on the LGPIF dataset and Jeong and Zou (forthcoming) for a recent application of this dataset.

[3]According to Progressive automobile insurance data, one at-fault accident can raise your premium rates by up to 28% on average; see https://www.progressive.com/answers/what-is-accident-forgiveness/.

[4]The transition in our two-class BMS is memoryless (Markov) because the insured's rate class in the next period only depends on the reporting status of the immediate previous period, but not on the entire history over multiple periods, which we plan to investigate in a future work. The same Markov transition is also adopted in Zacks and Levikson (2004) and Charpentier *et al*. (2017).

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she should report or hide an incurred loss. However, the extant literature largely ignores this important question,[5] and only a few theoretical works attempt to offer an answer, which we summarize in chronological order as follows. Zacks and Levikson (2004) study the claim-reporting problem in a multi-class BMS for a risk-neutral insured with deductible insurance. The insured seeks an optimal barrier reporting strategy (i.e., reporting only losses above a threshold) to minimize her discounted aggregate expenses. By a standard dynamic programming approach, they characterize the optimal barrier for each rate class as "deductible + reduction in expenses from underreporting"; however, such a characterization is implicit because the reduction in expenses is related to the value functions in different rate classes. In the special case of three rate classes, they numerically solve the problem and obtain the optimal barriers. Ludkovski and Young (2010) propose a two-period model with asymmetric information, in which the insured follows a randomized reporting strategy (a Bernoulli random variable), and the insurer applies a Bayesian approach to update its belief on the insured's risk type based on her reporting. They find that the optimal reporting strategy of a risk-neutral insured varies by her risk type and may be total nonreporting, full reporting, or mixed strategies. Robinson and Zheng (2010) show that a *barrier* reporting strategy arises endogenously in equilibrium when the insurance market is competitive. Charpentier *et al.* (2017) build upon the work of Zacks and Levikson (2004) and provide a rigorous, mathematical formulation of the claim-reporting problem in a discrete-time BMS setup. In a numerical study assuming the Spanish BMS with five rate classes, they not only compute the optimal barriers but also conduct sensitivity analysis on the optimal barriers and the probability of underreporting. Cao *et al.* (2023b) aim to answer Question 1 for an insured who has purchased full insurance and follows a barrier reporting strategy. They show that the insured's optimal (equilibrium) barrier is strictly positive in both rate classes, and the two equilibrium barriers are equal due to model symmetry. Although the aforementioned papers offer theoretical justification for an insured to hide certain losses, their findings all suffer from the same limitation that the insured has purchased *full* insurance coverage.[6]

We proceed to elaborate how we tackle Question 1 under deductible insurance in detail. As introduced earlier, the underlying insurance model is a multi-period BMS with two rate classes, and we consider a representative insured with deductible insurance and denote the deductible amount by $d_i \geq 0$ when the insured is in rate class i, $i = 1, 2$. The decision horizon of the insured is *random* and independent of the covered loss.[7] Following Zacks and Levikson (2004), Charpentier *et al.* (2017), and Cao *et al.* (2023b), we assume the insured follows a *barrier* strategy to make her reporting decision. Let (b_1, b_2) denote a barrier reporting strategy, with b_i being the barrier when the insured is in rate class i, $i = 1, 2$; then, such a strategy dictates the insured in rate class i will report a loss if and only if it is greater than the barrier b_i, $i = 1, 2$. Following standard references in decision-making (see, e.g., Arrow, 1963, 1974 and Borch, 1962), we model the insured's preferences by expected utility theory and assume, for tractability reasons, that the insured's utility is given by an exponential function.[8] Regardless of her current rate

[5]This question is of obvious significance to insureds; we further argue that it also concerns insurers and regulators. Underreporting is a source of information asymmetry because insureds thereby strategically manipulate the distribution of losses to their advantage (see, e.g., Ludkovski and Young, 2010). Failing to accurately capture insureds' behavior related to moral hazard might lead to imprecise ratemaking, ultimately resulting in adverse selection and financial losses for insurers. For public policy-makers, recognizing underreporting is crucial for "public policy evaluation of spending on highway safety, driver education, and accident reduction measures" (see Robinson and Zheng, 2010). Thus, it is important to understand insureds' underreporting behavior to prevent its adverse effects.

[6]Several related empirical papers also assume full insurance in the study; see, for instance, Gong (2017).

[7]Such an assumption allows us to focus on stationary (time-invariant) reporting strategies; see Cao *et al*. (2023b) for additional motivation. In addition, the random terminal time τ may be interpreted as the insured's surrender time. Under such context, our assumptions on τ specify that the surrender behavior is *exogenous*, following a geometric distribution (equivalent to an exponential distribution in continuous time). If surrender is endogenous and constitutes part of the insured's decision, then we face a complex optimal stopping-and-control problem, which adds an extra layer of maximization over the surrender time τ on top of the equilibrium reporting and deductible problem we solve in this paper. Such a problem is likely untractable and lies beyond the scope of this paper.

[8]Exponential utility is a popular choice in the study of optimal insurance problems; see Ghossoub *et al*. (2023) and Meng *et al*. (2022) for recent examples.

class, the insured may transit into the other rate class with a strictly positive probability; therefore, the two barriers b_1 and b_2 are intertwined and affect each other. We treat such a potential conflict in decision-making by a game-theoretical approach (see Björk *et al*., 2014 and 2017 for references). To be precise, we formulate a noncooperative Nash game with two players who both maximize their own expected exponential utility, in which player i represents the incarnation (or version) of the insured when she is in rate class i, $i = 1, 2$. Then, Question 1 can be stated in a more precise form:

> Question 1. For a fixed pair of deductibles (d_1, d_2), what is the Nash equilibrium barrier strategy $\left(b_1^*, b_2^*\right)$ that maximizes the insured's expected exponential utility of terminal wealth?

We obtain a complete answer to Question 1 in Theorems 3.1 and 3.2. Theorem 3.1 provides an equivalent condition (3.8) of (d_1, d_2) under which the insured prefers rate class 2 in terms of "utility," which implies that it is optimal for her to report all losses in excess of the deductible. As noted in the discussion following the theorem, this condition defeats rate class 1 as being the preferable rating class. By contrast to the condition in Theorem 3.1, when rate class 1 is preferred, Theorem 3.2 shows that the equilibrium barrier strategy is fully characterized by the unique root of a nonlinear function ($\hbar$ in (3.11)), which can be easily computed once the loss distribution and model parameters are specified. Denoting such a root by $\hbar_0$, we obtain

$$b_1^* = \hbar_0 > d_1 \geq 0 \quad \text{and} \quad b_2^* = b_1^* - d_1 + d_2 > d_2 \geq 0. \tag{1.1}$$

Based on the relationship between b_1^* and b_2^* in (1.1), we see that $b_1^* - d_1 = b_2^* - d_2$. When the insured prefers rate class 2, as in Theorem 3.1, then $b_1^* - d_1 = b_2^* - d_2 = 0$; otherwise, when the insured prefers rate class 1, as in Theorem 3.2, both are positive. We call this quantity the *amount of hidden losses* because losses greater than the deductible but less than the reporting barrier are those that the insured strategically hides.

Several key remarks about the above results are due. First, because both $b_1^* > d_1$ and $b_2^* > d_2$ hold when rate class 1 is preferred to rate class 2, our results in (1.1) confirm that the insured of *any* deductible insurance will hide a strictly positive percentage of losses in both rate classes (i.e., $\mathbb{P}(Z \leq b_i^* | Z > d_i) > 0$, $i = 1, 2$, in which Z denotes the per-period loss). This finding immediately generalizes the one in Cao *et al*. (2023b) from full insurance (with $d_1 = d_2 = 0$) to deductible insurance (when rate class 1 is preferred) and is consistent with the implicit result in Zacks and Levikson (2004) and Charpentier *et al*. (2017). Second, the two equilibrium barriers b_1^* and b_2^* are *not* equal unless the two deductibles are equal. In comparison, because of $d_1 = d_2 = 0$ in Cao *et al*. (2023b), their equilibrium barriers are the same for the two rate classes.

The above discussion mainly focuses on the economic interpretation of the answer (1.1) to Question 1. Next, we explain how solving Question 1 under deductible insurance is more challenging than under full insurance, as in Cao *et al*. (2023b), which can be seen as a special case of ours by setting $d_1 = d_2 = 0$. The first mathematical issue we face here is that the candidate solution to the objectives, $J_i = -A_i e^{-\gamma x}$ in (3.1) for $i = 1, 2$, require A_i to be strictly positive, but this is *not* guaranteed for any pair of deductibles d_1 and d_2. Therefore, our first task is to establish sufficient and necessary condition for $A_1 > 0$ and $A_2 > 0$, which is achieved in Lemma 3.1 (see the inequality condition in (3.5)). This step is completely skipped in Cao *et al.* (2023b) because they only need to verify the desired inequality holds for *the* equilibrium reporting strategy under $(d_1, d_2) = (0, 0)$. However, we must derive (3.5) for *all* $(d_1, d_2) \in \mathbb{R}_+^2$, *not* just one particular pair of deductibles (d_1, d_2). We also point out that the associated verification step of $D\left(b_1^*, b_2^*\right) > 0$ (D is defined by (3.4)) is technical here, as seen in proof of Theorem 3.2, but the same step is simple in Cao *et al*. (2023b) (see their proof of Theorem 2). Furthermore, both papers need to show that the equilibrium is *not* achieved at the boundary; that is, $b_1^* \neq 0$ or $b_2^* \neq 0$, as in Cao *et al*. (2023b), or $b_1^* \neq d_1$ or $b_2^* \neq d_2$ for any pair of deductibles when rate class 1 is preferred, as in this paper. This step is more involved here than in Cao *et al*. (2023b).

The solution to Question 1 in (1.1) reveals that the equilibrium barriers b_1^* and b_2^* depend on the fixed contract deductibles d_1 and d_2. For this reason, write them as $b_1^*(d_1, d_2)$ and $b_2^*(d_1, d_2)$; we remark that the game-theoretic feature of our model is the reason both b_1^* and b_2^* depend on both d_1 and d_2. Upon understanding such a dependence relation, a rational insured should and will take into account underreporting when she chooses her insurance contract. Therefore, under the same setup as Question 1, we state Question 2 in a more precise form:

> Question 2. Knowing that the insured will strategically underreport losses by the barrier strategy $(b_1^*(d_1, d_2), b_2^*(d_1, d_2))$, what is the Nash equilibrium deductible insurance contract $\left(d_1^*, d_2^*\right)$ that maximizes the insured's expected exponential utility of terminal wealth?

To the best of our knowledge, the optimal insurance problem formulated in Question 2 is *new* to the actuarial literature.[9] As explained in the opening paragraph, it is essential to consider deductible insurance in order to study Question 2; recall that Cao *et al*. (2023b) only obtain the equilibrium reporting strategy $(b_1^*(0, 0), b_2^*(0, 0))$ under the *fixed* full insurance contract. To understand the impact of underreporting on insurance demand, consider an insured in rate class i, $i = 1, 2$, who has a current wealth of X_t and faces a random loss Z, and assume a zero income for simplicity; to reduce the risk exposure, she purchases deductible insurance with deductible d_i and pays premium π_i. If we ignore underreporting, her wealth at the end of the next period, X_{t+1}, is given by:

$$X_{t+1} = X_t - \pi_i - Z + (Z - d_i)_+. \tag{1.2}$$

However, if the insured follows the equilibrium barrier strategy for reporting, then we have

$$X_{t+1} = X_t - \pi_i - Z + (Z - d_i)_+ \cdot \mathbb{1}_{\{Z > b_i^*(d_1, d_2)\}}, \tag{1.3}$$

in which $b_i^*(d_1, d_2)$ is the equilibrium solution to Question 1, and $\mathbb{1}$ denotes an indicator function. It is clear that the two expressions of X_{t+1} in (1.2) and (1.3) are noticeably different; they are identical if and only if $b_i^*(d_1, d_2) \leq d_i$ for $i = 1, 2$, but we have shown in (1.1) that the opposite inequality holds, namely, $b_i^*(d_1, d_2) > d_i$, when rate class 1 is preferred. As such, ignoring the impact of underreporting will lead to a sub-optimal contract for the insured, justifying the need to study the problem posed in Question 2.

Solving Question 2 turns out to be highly nontrivial; the main challenge is that the equilibrium reporting strategy $b_i^*(d_1, d_2)$ is obtained in a semi-closed form, subject to the unique root of the nonlinear function $\hbar$ (defined in (3.11)), and it further feeds into the wealth dynamics in (1.3) in a nonsmooth way (see the term with the indicator function in (1.3)). Because of these technical challenges, finding an *analytical* solution to Question 2 seems unlikely even under additional assumptions on the loss distribution. As such, we resort to the numerical approach to identify the optimal (equilibrium) deductible pair $\left(d_1^*, d_2^*\right)$ under strategic underreporting. When the positive loss follows a Gamma distribution (or Pareto distribution), we numerically obtain the equilibrium deductible pair $\left(d_1^*, d_2^*\right)$ and the corresponding equilibrium strategy $\left(b_1^*, b_2^*\right) := \left(b_1^*\left(d_1^*, d_2^*\right), b_2^*\left(d_1^*, d_2^*\right)\right)$. The first main result is that both equilibrium deductibles are strictly positive (i.e., $d_1^* > 0$ and $d_2^* > 0$); therefore, full insurance under strategic underreporting is *not* optimal to the insured, which is consistent with the literature on moral hazard (see Winter, 2013). Second, we obtain $d_2^* > d_1^*$, suggesting that insureds in rate class 2 (bad class) purchase less insurance than those in rate class 1 (good class), which is due to the higher premium loading associated with rate class 2. This finding implies a negative relation between insurance demand and riskiness, which is empirically confirmed for auto insurance (see Li *et al*., 2007) and for flood insurance (see Dombrowski *et al*., 2020). Third, in our numerical examples, we confirm that the amount of hidden losses is strictly positive, which means that the insured economically prefers rate class 1, as one would expect. Last, we conduct extensive sensitivity analysis to investigate how various model parameters affect the insured's equilibrium insurance and reporting decisions. An important finding is that the higher the risk aversion,

[9] Holtan (2001) studies optimal insurance under bonus-malus contracts, which is related to, but significantly different from, our Question 2. The model in Holtan (2001) builds upon the classical one of Arrow (1963) but changes the insurance indemnity I to an adjusted one $\tilde{I}$, defined by $\tilde{I}(z) = (I(z) - \psi)_+$, in which $\psi > 0$ is the present value of all future premium savings if the insured hides a loss and is assumed to be *independent* of the insurance contract (i.e., ψ does *not* depend on I). In contrast, the insured's reporting is *endogenously* determined in equilibrium in our work.

the more the insurance coverage, as implied by a lower equilibrium deductible d_i^*, which is also found when moral hazard is not considered in the study of optimal insurance (see Arrow, 1974). However, the impact of risk aversion on the amount of hidden losses is complex and not captured by any monotone relation. On the positive side, we observe that the amount of hidden losses is positively correlated with the difference in the premium loadings between the two rate classes, $\theta_2 - \theta_1$.

We remark that underreporting losses is a type of *ex post* moral hazard, which can be generally defined as insureds' actions of manipulating the distribution of losses reported to the insurer following an accident; see, for example, Chiappori (2000). In that regard, this paper contributes to the literature of optimal insurance under *ex post* moral hazard. We refer the readers to Winter (2013) for a survey on this topic, including both *ex ante* and *ex post* moral hazard. However, the existing literature of optimal insurance under *ex post* moral hazard is largely motivated by the inefficiency of medical expenditures *ex post* in health insurance; see, for instance, Zeckhauser (1970) and Ma and McGuire (1997). Another type of *ex post* moral hazard is insurance fraud (i.e., intentionally increasing the number or amount of claims) and can be theoretically ruled out by imposing the so-called incentive compatibility (IC) condition, also termed no-sabotage condition, on the indemnity function of an insurance contract. There is a growing stream of literature on optimal insurance or reinsurance under the IC condition; see Xu *et al*. (2019), Chi and Tan (2021), Boonen and Jiang (2022), and Jin *et al*. (2023) for recent contributions to this field. Note that experience rating is *not* considered in those works, so their claim that the IC constraint rules out *ex post* moral hazard is only true under their setup but fails in a realistic model where submitting a claim to the insurer often increases the insured's future premium.

The remainder of the paper is organized as follows. In Section 2, we present the insurance model allowing loss underreporting. In Section 3, for a given pair of deductibles, we formulate the insured's optimal reporting problem as a noncooperative Nash game and obtain her equilibrium barrier reporting strategies in semi-explicit form. In Section 4, we study optimal insurance under strategic underreporting, and in Section 5, we determine how underreporting affects the insurer's expected profit. Section 6 concludes the paper. Additional numerical results on the equilibrium deductibles when the positive loss follows a Pareto distribution are collected in an online appendix.

2. Model

In this section, we follow Cao *et al*. (2023b) to define a discrete-time insurance market to study the loss reporting problem of an insured (she/her). The key difference from Cao *et al*. (2023b) is that we now allow the insured to buy deductible insurance, instead of restricting her coverage to full insurance.

2.1. Insurance market

We model the insured's risk by a series of independent and identically distributed (i.i.d.) nonnegative random variables $\{Z_t\}_{t=1,2,\ldots}$, in which $Z_t \stackrel{d}{=} Z$ denotes her loss over the t^{th} period $[t-1, t)$. As suggested by claim data from short-term insurance, we assume that the distribution of Z is a mixture of a point mass at 0, with probability $\mathbb{P}(Z=0) = p_0 \in (0, 1)$, and a continuous, positive random variable, with full support over $(0, \infty)$ and a probability density function f. Let F (resp. $S = 1 - F$) denote the cumulative distribution function (resp. survival function) of Z, that is, $F(x) = p_0 + \int_0^x f(t)\mathrm{d}t$ for $x \geq 0$. See, for example, Zacks and Levikson (2004) for the same setup.

To mitigate her risk exposure, the insured buys deductible insurance from a representative seller/insurer (it), who adopts a two-class BMS in ratemaking. Under our BMS, if the insured files a claim in the previous period, she moves to or stays in class 2 for the next period and pays premium π_2; otherwise, she moves to or stays in class 1 and pays premium π_1.[10]

[10]Here, we do not assume a specific premium principle for computing π_i because the results in Section 3 do not rely on the premium principle. However, the analysis in Section 4 requires knowing how the deductible choice affects the contract premium; in that section, we will assume that the premium is given by the expected-value principle so that we can determine the equilibrium deductibles.

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As supported by common practice, by following related research (i.e., Zacks and Levikson, 2004; Charpentier *et al*., 2017, and Cao *et al*., 2023b), we assume that the insured applies a *barrier* strategy to decide whether she should report a loss. Such a reporting strategy is fully captured by a pair $(b_1, b_2) \in \mathbb{R}_+^2$; when the insured is in rate class $i = 1, 2$, she reports a loss if and only if it is greater than the barrier b_i. Because the payoff of deductible insurance is zero when the loss size is less than the deductible, we assume, without loss of generality, that $b_i \geq d_i$ for $i = 1, 2$. Let $\{X_t\}_{t=0,1,\dots}$ denote the insured's wealth process under a chosen barrier strategy (with dependence on (b_1, b_2) suppressed for notational simplicity); here, X_t is the insured's wealth at time t *before* receiving income and paying premium to the insurer. Then, the dynamics of her wealth when she is in rate class i over $[t, t+1)$ follows:

$$X_{t+1} = X_t + c - \pi_i - Z_{t+1} + (Z_{t+1} - d_i)_+ \cdot \mathbb{1}_{\{Z_{t+1} > b_i\}}, \quad i = 1, 2, \tag{2.1}$$

in which $c > 0$ is the insured's per-period income, π_i is the premium payable when she is in rate class i, Z_{t+1} is the loss over $[t, t+1)$, and $\mathbb{1}_{\{Z_{t+1} > b_i\}}$ equals 1 if and only if $Z_{t+1} > b_i$ and equals 0 otherwise. We impose the following assumption on the income c:

$$c > \max\{\pi_1 + d_1,\ \pi_2 + d_2\}, \tag{2.2}$$

which implies that the insured is able to pay the premium and deductible regardless of her rate class. This assumption is, though technical, easily satisfied in real life.[11]

2.2. Insured's preferences

We assume the insured is risk-averse, and her preferences are characterized by the standard expected utility theory with an exponential utility $U(x) = -\mathrm{e}^{-\gamma x}$, in which $\gamma > 0$ is the insured's (constant) absolute risk aversion (see, e.g., Ghossoub *et al*., 2023 and Meng *et al*., 2022 for the same preference assumption). The goal of the insured is to choose equilibrium barrier and deductible strategies to maximize the expected utility of her "terminal" wealth (see, e.g., Merton, 1969 for a standard reference on this criterion).

However, instead of a finite or an infinite planning horizon, we assume that the insured's planning horizon τ is *random* and exogenously given. Several recent papers (see Cao *et al*., 2022 and follow-ups) on optimal (re)insurance problems also consider a random horizon; in particular, footnote 18 in Cao *et al*. (2023a) discusses in detail the mathematical and economic justifications for the choice of a random horizon.[12] Because we assumed that the insured adopts a *homogeneous* barrier strategy in reporting, it is natural to assume that the random horizon τ follows a geometric distribution, which is the only *memoryless* discrete distribution. Under this assumption, we have

$$\mathbb{P}(\tau = t) = (1 - p)\, p^{t-1}, \quad t = 1, 2, \dots,$$

in which $p = \mathbb{P}(\tau > 1) \in (0, 1)$.

Based on the above setup, the insured's objective function when she is currently in rate class $i = 1, 2$ is given by:

$$J_i(x) = \mathbb{E}\left(-\mathrm{e}^{-\gamma X_\tau} \,\middle|\, X_0 = x, I_0 = i\right), \tag{2.3}$$

in which $X_0 = x \in \mathbb{R}$ is the insured's initial wealth, and $I_0 = i \in \{1, 2\}$ is her initial rate class. For notational simplicity, we often suppress the dependence of J_i upon (b_1, b_2, d_1, d_2). Several remarks are due regarding the objective function J_i in (2.3). First, J_i is *time-independent* because we only consider

[11] For example, take auto insurance and income in the USA: Forbes's analysis in 2023 finds that the national average cost for car insurance (full coverage) is \$2, 150 per year; the average deductible amount is \$500 estimated by American Family Insurance; and the latest (year 2021) official median household income is \$70, 784.

[12] In addition to their arguments, such a modeling choice removes the strong horizon effect on the insured's decision-making. Indeed, suppose that the insured has a finite-time horizon T, then her optimal strategy for the last period is to report all losses over the deductible, and this strategy is unlikely to be optimal for other periods.

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time-homogeneous deductible and barrier strategies, losses are i.i.d., and τ follows a geometric distribution. Therefore, the condition in (2.3), though written at time 0, can be equivalently understood as at any time. Second, J_i is a function of *both* (b_1, d_1) and (b_2, d_2) for $i = 1, 2$, and this reflects the *game* nature embedded in the BMS. To be more concrete, the joint effect of (b_1, d_1) and (b_2, d_2) on J_i is due to the fact that there is a strictly positive probability that the insured will move from one rate class to another. Consequently, the insured plays against herself in a *noncooperative Nash game*; please see Remark 3.1 in Cao *et al*. (2023a) for a detailed discussion on this game feature and Björk *et al.* (2014) and (2017) for standard references on such a game-theoretical approach.

We end this section with a useful lemma that reveals the relation between J_1 and J_2.

Lemma 2.1. *The objectives J_1 and J_2 defined in (2.3) satisfy*

$$J_1(x) = -(1-p)e^{-\gamma(x+c-\pi_1)}\,\mathbb{E}\left(e^{\gamma(Z-(Z-d_1)_+\cdot\mathbb{1}_{\{Z>b_1\}})}\right) + pF(b_1)\mathbb{E}\big(J_1(x+c-\pi_1-Z)\big|Z\le b_1\big)$$
$$+\, pS(b_1)\mathbb{E}\big(J_2(x+c-\pi_1-(Z\wedge d_1))\big|Z>b_1\big), \tag{2.4}$$

$$J_2(x) = -(1-p)e^{-\gamma(x+c-\pi_2)}\,\mathbb{E}\left(e^{\gamma(Z-(Z-d_2)_+\cdot\mathbb{1}_{\{Z>b_2\}})}\right) + pF(b_2)\mathbb{E}\big(J_1(x+c-\pi_2-Z)\big|Z\le b_2\big)$$
$$+\, pS(b_2)\mathbb{E}\big(J_2(x+c-\pi_2-(Z\wedge d_2))\big|Z>b_2\big), \tag{2.5}$$

in which

$$\mathbb{E}\left(e^{\gamma\left(Z-(Z-d_i)_+\cdot\mathbb{1}_{\{Z>b_i\}}\right)}\right) = \int_0^{b_i} e^{\gamma z}dF(z) + S(b_i)e^{\gamma d_i}, \quad i=1,2. \tag{2.6}$$

Proof. By a standard argument in renewal theory, the objective function J_1 satisfies the following recursion:

$$J_1(x) = -\mathbb{E}_x\big(e^{-\gamma X_\tau}\,\big|\tau=1\big)\mathbb{P}(\tau=1)$$
$$+\big\{\mathbb{E}_x(J_1(X_1)|\tau>1, Z\le b_1)\mathbb{P}(Z\le b_1) + \mathbb{E}_x(J_2(X_1)|\tau>1, Z>b_1)\mathbb{P}(Z>b_1)\big\}\mathbb{P}(\tau>1),$$

in which $\mathbb{E}_x$ denotes conditioning on $X_0 = x$. The above recursion, along with (2.1), implies (2.4); by a similar argument, we obtain (2.5). The result in (2.6) is straightforward by splitting the integration region into $[0, b_i]$ and (b_i, ∞). Note that the Riemann integral in (2.6) includes the jump at 0, that is, $\int_0^{b_i} e^{\gamma z}dF(z) = p_0 + \int_{(0,b_i]} e^{\gamma z}f(z)dz$; we follow this convention throughout this paper when writing integrals. □

3. Equilibrium reporting strategy under deductible insurance

This section solves Question 1 from the Introduction to obtain the insured's equilibrium reporting strategy for a fixed deductible insurance contract. To this end, we first formally define the insured's equilibrium reporting strategy as follows.

Definition 3.1. *Assume the deductibles $(d_1, d_2) \in \mathbb{R}_+^2$ are given, along with the premiums $(\pi_1, \pi_2) \in \mathbb{R}_+^2$. For a fixed $b_2 \ge d_2$, let $\bar{b}_1(b_2)$ denote* $\arg\sup_{b_1\ge d_1} J_1(x; b_1, b_2)$*; for a fixed $b_1 \ge d_1$, let $\bar{b}_2(b_1)$ denote* $\arg\sup_{b_2\ge d_2} J_2(x; b_1, b_2)$*. A strategy $\left(b_1^*, b_2^*\right)$ is called an* equilibrium barrier strategy *if it is a fixed point of the mapping $(b_1, b_2) \mapsto (\bar{b}_1(b_2), \bar{b}_2(b_1))$*[13]

Because the insured's objectives in (2.3) involve exponential utility and because the wealth dynamics in (2.1) is linear, we conjecture that the solution of (2.4)–(2.5) is of the form:

$$J_i(x) = -A_i\, e^{-\gamma x}, \quad i=1,2, \tag{3.1}$$

[13]In this definition, we suppress the dependence of the functions upon (d_1, d_2) for notational simplicity.

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in which $A_i > 0$ is yet to be determined. Note that A_i depends on the insured's barrier strategy (b_1, b_2) and on the deductibles (d_1, d_2), but we suppress this dependence frequently to simplify notation. By using the ansatz in (3.1), the recursions in (2.4)–(2.5) become

$$e^{\gamma(c-\pi_1)}A_1 = (1-p)\left\{\int_0^{b_1} e^{\gamma z}dF(z) + S(b_1)e^{\gamma d_1}\right\} + pA_1\int_0^{b_1} e^{\gamma z}dF(z) + pS(b_1)e^{\gamma d_1}A_2,$$

and

$$e^{\gamma(c-\pi_2)}A_2 = (1-p)\left\{\int_0^{b_2} e^{\gamma z}dF(z) + S(b_2)e^{\gamma d_2}\right\} + pA_1\int_0^{b_2} e^{\gamma z}dF(z) + pS(b_2)e^{\gamma d_2}A_2.$$

Solving the above system yields

$$A_1(b_1; b_2) = (1-p)\,\frac{e^{\gamma(c-\pi_2)}\left\{\int_0^{b_1} e^{\gamma z}dF(z) + e^{\gamma d_1}S(b_1)\right\} + N\big(b_1, b_2\big)}{D(b_1, b_2)}, \tag{3.2}$$

and

$$A_2(b_2; b_1) = (1-p)\,\frac{e^{\gamma(c-\pi_1)}\left\{\int_0^{b_2} e^{\gamma z}dF(z) + e^{\gamma d_2}S(b_2)\right\} + N(b_1, b_2)}{D(b_1, b_2)}, \tag{3.3}$$

in which N and D equal, respectively,

$$N(b_1, b_2) = p\left\{e^{\gamma d_1}S(b_1)\int_0^{b_2} e^{\gamma z}dF(z) - e^{\gamma d_2}S(b_2)\int_0^{b_1} e^{\gamma z}dF(z)\right\},$$

and

$$D(b_1, b_2) = \left\{e^{\gamma(c-\pi_1)} - p\int_0^{b_1} e^{\gamma z}dF(z)\right\}\left(e^{\gamma(c-\pi_2)} - pe^{\gamma d_2}S(b_2)\right) - p^2 e^{\gamma d_1}S(b_1)\int_0^{b_2} e^{\gamma z}dF(z). \tag{3.4}$$

Because the insured is a utility maximizer, her objective $J_i(x)$ in (2.3) must be strictly increasing with respect to the initial wealth x, which is why we require A_i in (3.1) to be strictly positive. However, the expressions in (3.2) and (3.3) cannot guarantee positiveness of A_1 and A_2; thus, we derive conditions under which both are positive.

Lemma 3.1. *A_1 in (3.2) is positive if and only if*

$$\left\{e^{\gamma(c-\pi_1)} - p\int_0^{b_1} e^{\gamma z}dF(z)\right\}\left(e^{\gamma(c-\pi_2)} - pe^{\gamma d_2}S(b_2)\right) > p^2 e^{\gamma d_1}S(b_1)\int_0^{b_2} e^{\gamma z}dF(z). \tag{3.5}$$

If inequality (3.5) holds, then A_2 in (3.3) is positive.

Proof. For ease of notation in this proof, let

$$\alpha_i = e^{\gamma(c-\pi_i)}, \qquad \beta_i = e^{\gamma d_i}S(b_i), \qquad \delta_i = \int_0^{b_i} e^{\gamma z}dF(z), \quad i = 1, 2,$$

and

$$\Delta_1 = \alpha_1 - p\delta_1, \qquad \Delta_2 = \alpha_2 - p\beta_2.$$

Note that $\Delta_2 > 0$ because $e^{\gamma(c-(\pi_2+d_2))} > 1 > pS(b_2)$ by the assumption in (2.2).

First, rewrite A_1 and A_2 as follows (by factoring out the positive constant $1-p$):

$$A_1 \propto \frac{N_1}{D}, \quad A_2 \propto \frac{N_2}{D},$$

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in which

$$N_1 = \Delta_2\delta_1 + (\alpha_2 + p\delta_2)\beta_1 = \Delta_2\delta_1 + (\Delta_2 + p\beta_2 + p\delta_2)\beta_1 = \Delta_2(\delta_1 + \beta_1) + (\delta_2 + \beta_2)p\beta_1,$$

$$N_2 = \Delta_1\beta_2 + (\alpha_1 + p\beta_1)\delta_2 = \Delta_1\beta_2 + (\Delta_1 + p\delta_1 + p\beta_1)\delta_2 = \Delta_1(\delta_2 + \beta_2) + (\delta_1 + \beta_1)p\delta_2,$$

and

$$D = \Delta_1\Delta_2 - p^2\beta_1\delta_2.$$

Because $\Delta_2 > 0$, we see that $N_1 > 0$. A_1 is positive if and only if D is positive, which is equivalent to inequality (3.5). Finally, if $D > 0$, then

$$\Delta_1 = \mathrm{e}^{\gamma(c-\pi_1)} - p\int_0^{b_1} \mathrm{e}^{\gamma z}\mathrm{d}F(z) > 0,$$

which implies $N_2 > 0$ and eventually $A_2 > 0$. □

Because the condition in (3.5) is sufficient and necessary in order for both A_1 and A_2 to be positive, as required by the ansatz in (3.1), we assume in the subsequent analysis that this condition holds, which we formalize next.

Assumption 3.1. *Henceforth, assume the deductibles d_1 and d_2 are such that (3.5) holds when $(b_1, b_2) = (d_1, d_2)$, or equivalently, $D(d_1, d_2) > 0$, in which D is given in (3.4).*

Remark 3.1. The insured's objective J_i, defined as a conditional expectation in (2.3), is well defined and unique almost surely (although 2.3 does not guarantee that J_i is finite). Under the ansatz for J_i in (3.1), the recursion system satisfied by J_1 and J_2 (2.4)–(2.5) leads to a unique pair of solutions (A_1, A_2). When Assumption 3.1 is imposed, both A_1 and A_2 are positive, and thus the ansatz in (3.1) with A_1 and A_2 given by (3.2) and (3.3) yields the unique, finite solution of J_i in (2.3).

For a deductible pair (d_1, d_2) satisfying Assumption 3.1, define the set of admissible barrier strategies by:

$$\mathcal{D} := \left\{(b_1, b_2) \in [d_1, \infty) \times [d_2, \infty) \,\middle|\, D(b_1, b_2) > 0\right\},$$

in which D is defined in (3.4). As an aside, because the function D is continuous, Assumption 3.1 implies $\mathcal{D}$ contains a neighborhood of (d_1, d_2). In what follows, when we consider optimization problems over b_1 or b_2, we always implicitly assume the optimization region is a projection of $\mathcal{D}$ onto the corresponding argument space. After we obtain a candidate for the equilibrium barrier strategy $\left(b_1^*, b_2^*\right)$, we will verify that it is, indeed, in the set $\mathcal{D}$.

We first analyze the case when the insured is in rate class 1 at time 0, and we solve the problem:

$$\bar{b}_1(b_2) := \underset{b_1 \ge d_1}{\arg\sup}\, J_1(x) = \underset{b_1 \ge d_1}{\arg\inf}\, A_1(b_1; b_2),$$

for a fixed $b_2 \ge d_2$. The solution is presented in the next proposition.

Proposition 3.1. *For a fixed $b_2 \ge d_2$, the function $b_1 \mapsto A_1(b_1; b_2)$ has a unique minimizer $\bar{b}_1(b_2)$ on $[d_1, \infty)$. If*

$$\mathrm{e}^{\gamma\pi_1}\left\{\int_0^{d_1} \mathrm{e}^{\gamma z}\mathrm{d}F(z) + \mathrm{e}^{\gamma d_1}S(d_1)\right\} \ge \mathrm{e}^{\gamma\pi_2}\left\{\int_0^{b_2} \mathrm{e}^{\gamma z}\mathrm{d}F(z) + \mathrm{e}^{\gamma d_2}S(b_2)\right\},$$

then $\bar{b}_1(b_2) = d_1$. Otherwise, $\bar{b}_1(b_2)$ is the unique zero of $g_1(\,\cdot\,; b_2)$ in (d_1, ∞), in which

$$g_1(b_1; b_2) = \left(\mathrm{e}^{\gamma(b_1-d_1)} - 1\right) + p\mathrm{e}^{-\gamma(c-\pi_1)}\left\{\int_0^{b_1} \mathrm{e}^{\gamma z}\mathrm{d}F(z) + e^{\gamma b_1}S(b_1)\right\}$$
$$- p\mathrm{e}^{-\gamma(c-\pi_2)}\left\{\int_0^{b_2} \mathrm{e}^{\gamma z}\mathrm{d}F(z) + \mathrm{e}^{\gamma(b_1-d_1)}\mathrm{e}^{\gamma d_2}S(b_2)\right\}. \quad (3.6)$$

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Proof. After factoring $\frac{1-p}{D^2}\,\mathrm{e}^{\gamma(c-\pi_1)}\mathrm{e}^{\gamma(c-\pi_2)}\,\mathrm{e}^{\gamma d_1}f(b_1)\big(\mathrm{e}^{\gamma(c-\pi_2)}-p\mathrm{e}^{\gamma d_2}S(b_2)\big)>0$ from the derivative of A_1 with respect to b_1, we obtain that this derivative is positively proportional to g_1 in (3.6). The derivative of g_1 with respect to b_1 equals

$$\frac{\partial g_1}{\partial b_1}=\gamma\,\mathrm{e}^{\gamma(b_1-d_1)}\big(1+p\mathrm{e}^{-\gamma(c-(\pi_1+d_1))}S(b_1)-p\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}S(b_2)\big)>0,$$

which is positive because $\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}<1$. Next,

$$\begin{aligned}&\lim_{b_1\to\infty} g_1(b_1;b_2)\\&=\lim_{b_1\to\infty}\mathrm{e}^{\gamma(b_1-d_1)}\left(1+p\mathrm{e}^{-\gamma(c-(\pi_1+d_1))}\left\{\int_0^{b_1}\mathrm{e}^{-\gamma(b_1-z)}\mathrm{d}F(z)+S(b_1)\right\}-p\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}S(b_2)\right)\\&=\infty,\end{aligned}$$

in which the last line follows because $0<\int_0^{b_1}\mathrm{e}^{-\gamma(b_1-z)}\mathrm{d}F(z)+S(b_1)<1$ for all b_1 and because $\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}<1$.

Finally,

$$g_1(d_1;b_2)\propto \mathrm{e}^{\gamma\pi_1}\left\{\int_0^{d_1}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma d_1}S(d_1)\right\}-\mathrm{e}^{\gamma\pi_2}\left\{\int_0^{b_2}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma d_2}S(b_2)\right\}.$$

If this expression is nonnegative, then the minimizer $\bar{b}_1(b_2)=d_1$ because g_1 increases to ∞; otherwise, if this expression is negative, then $\bar{b}_1(b_2)$ equals the unique zero of $g_1(\,\cdot\,;b_2)$ in (d_1,∞). □

In the next proposition, we solve the problem:

$$\arg\sup_{b_2\ge d_2} J_2(x)=\arg\inf_{b_2\ge d_2} A_2(b_2;b_1),$$

which corresponds to the case when the insured is in rate class 2.

Proposition 3.2. *For a fixed $b_1\ge d_1$, the function $b_2\mapsto A_2(b_2;b_1)$ has a unique minimizer $\bar{b}_2(b_1)$ on $[d_2,\infty)$. If*

$$\mathrm{e}^{\gamma\pi_1}\left\{\int_0^{b_1}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma d_1}S(b_1)\right\}\ge \mathrm{e}^{\gamma\pi_2}\left\{\int_0^{d_2}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma d_2}S(d_2)\right\},$$

then $\bar{b}_2(b_1)=d_2$. Otherwise, $\bar{b}_2(b_1)$ is the unique zero of $g_2(\,\cdot\,;b_1)$ in (d_2,∞), in which

$$\begin{aligned}g_2(b_2;b_1)=\big(\mathrm{e}^{\gamma(b_2-d_2)}-1\big)+p\mathrm{e}^{-\gamma(c-\pi_1)}\left\{\int_0^{b_1}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma(b_2-d_2)}\mathrm{e}^{\gamma d_1}S(b_1)\right\}\\-p\mathrm{e}^{-\gamma(c-\pi_2)}\left\{\int_0^{b_2}\mathrm{e}^{\gamma z}\mathrm{d}F(z)+\mathrm{e}^{\gamma b_2}S(b_2)\right\}.\end{aligned}\tag{3.7}$$

Proof. After factoring $\frac{1-p}{D^2}\,\mathrm{e}^{\gamma(c-\pi_1)}\mathrm{e}^{\gamma(c-\pi_2)}\,\mathrm{e}^{\gamma d_2}f(b_2)\big(\mathrm{e}^{\gamma(c-\pi_1)}-p\int_0^{b_1}\mathrm{e}^{\gamma z}\mathrm{d}F(z)\big)>0$ from the derivative of A_2 with respect to b_2, we obtain that this derivative is positively proportional to g_2 in (3.7). The derivative of g_2 with respect to b_2 equals

$$\frac{\partial g_2}{\partial b_2}=\gamma\,\mathrm{e}^{\gamma(b_2-d_2)}\big(1+p\mathrm{e}^{-\gamma(c-(\pi_1+d_1))}S(b_1)-p\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}S(b_2)\big),$$

which is positive because $\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}<1$. We obtain the claimed result by following the same argument as in the proof of Proposition 3.1. □

Recall from Definition 3.1 that the insured's equilibrium barrier strategy, if exists, is a fixed point of the mapping $(b_1,b_2)\mapsto(\bar{b}_1(b_2),\bar{b}_2(b_1))$, in which $\bar{b}_1(b_2)\ge d_1$ and $\bar{b}_2(b_1)\ge d_2$ are obtained in the previous two propositions, respectively. We naturally ask whether the fixed point will be achieved at the boundary

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corner (d_1, d_2); in that case, the insured reports all losses above the deductible, and there is no *ex post* moral hazard in reporting. The next theorem provides an answer to this question.

Theorem 3.1. *If d_1 and d_2 satisfy*

$$e^{\gamma\pi_1}\left\{\int_0^{d_1} e^{\gamma z}dF(z)+e^{\gamma d_1}S(d_1)\right\}\geq e^{\gamma\pi_2}\left\{\int_0^{d_2} e^{\gamma z}dF(z)+e^{\gamma d_2}S(d_2)\right\}, \tag{3.8}$$

then the equilibrium strategy $\left(b_1^*, b_2^*\right)$ is (d_1, d_2).

Proof. Note that

$$\frac{\partial}{\partial b}\left[e^{\gamma\pi_1}\left\{\int_0^{b} e^{\gamma z}dF(z)+e^{\gamma d}S(b)\right\}\right]=\left(e^{\gamma b}-e^{\gamma d}\right)f(b)\geq 0, \tag{3.9}$$

for all $b \geq d$. Thus, if (3.8) is satisfied, then, by using (3.9) and $b_1 \geq d_1$, we get

$$e^{\gamma\pi_1}\left\{\int_0^{b_1} e^{\gamma z}dF(z)+e^{\gamma d_1}S(b_1)\right\}\geq e^{\gamma\pi_2}\left\{\int_0^{d_2} e^{\gamma z}dF(z)+e^{\gamma d_2}S(d_2)\right\}.$$

Under the above condition, Propositions 3.1 and 3.2, along with Assumption 3.1, imply that (d_1, d_2) is the unique equilibrium barrier strategy. □

One can interpret inequality (3.8) as follows: Consider the expected utility of the per-period outgo of the insured when she is in rate class i and when she files claims for all losses in excess of the deductible (or equivalently, $b_i = d_i$), that is,

$$\mathbb{E}\left(-e^{-\gamma(-\pi_i-(Z\wedge d_i))}\right)=-e^{\gamma\pi_i}\left\{\int_0^{d_i} e^{\gamma z}dF(z)+e^{\gamma d_i}S(d_i)\right\}.$$

Inequality (3.8) means that the expected utility of this outgo is (weakly) less when the insured is in rate class 1 than when she is in rate class 2. Therefore, the insured (weakly) prefers to be in rate class 2 and has no incentive to hide her losses in excess of the deductible. This possible preference for rate class 2 defeats its purpose as a less desirable rate class, or "punishment" for filing claims, so for the remainder of this section, we assume that d_1 and d_2 satisfy

$$e^{\gamma\pi_1}\left\{\int_0^{d_1} e^{\gamma z}dF(z)+e^{\gamma d_1}S(d_1)\right\}< e^{\gamma\pi_2}\left\{\int_0^{d_2} e^{\gamma z}dF(z)+e^{\gamma d_2}S(d_2)\right\}. \tag{3.10}$$

Given (3.10), we show in the next theorem that the insured employs a nontrivial barrier strategy in equilibrium and has incentives to hide certain losses above the deductible, which is in stark contrast to the finding of Theorem 3.1.

Theorem 3.2. *Assume d_1 and d_2 satisfy (3.10). The insured's equilibrium barrier strategy is uniquely given by $\left(b_1^*, b_2^*\right)=(b_1^*, b_1^*-d_1+d_2)\in\mathcal{D}$, in which b_1^* equals the unique zero of $\hbar$ on (d_1,∞), with $\hbar$ defined by:*

$$\begin{aligned}\hbar(b)=&\left(e^{\gamma(b-d_1)}-1\right)+pe^{-\gamma(c-\pi_1)}\left\{\int_0^{b} e^{\gamma z}dF(z)+e^{\gamma b}S(b)\right\}\\ &-pe^{-\gamma(c-\pi_2)}\left\{\int_0^{b-d_1+d_2} e^{\gamma z}dF(z)+e^{\gamma(b-d_1+d_2)}S(b-d_1+d_2)\right\}.\end{aligned} \tag{3.11}$$

Proof. We start by showing that $b_i^* \neq d_i$ for $i = 1, 2$, if the inequality in (3.10) holds. First, (3.9) and (3.10) imply, for $b_2 \geq d_2$,

$$e^{\gamma\pi_1}\left\{\int_0^{d_1} e^{\gamma z}dF(z)+e^{\gamma d_1}S(d_1)\right\}< e^{\gamma\pi_2}\left\{\int_0^{b_2} e^{\gamma z}dF(z)+e^{\gamma d_2}S(b_2)\right\}.$$

It follows from Proposition 3.1 that $\bar{b}_1(b_2) > d_1$ for all $b_2 \geq d_2$.

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Next, we show that $(\bar{b}_1(d_2), d_2)$ is not an equilibrium. Assuming the contrary holds, we have $g_1(\bar{b}_1(d_2); d_2) = 0$, which, by (3.6), is equivalent to

$$0 = \left(\mathrm{e}^{\gamma(\bar{b}_1(d_2)-d_1)} - 1\right) + p\mathrm{e}^{-\gamma(c-\pi_1)} \left\{ \int_0^{\bar{b}_1(d_2)} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + e^{\gamma \bar{b}_1(d_2)} S(\bar{b}_1(d_2)) \right\}$$
$$- p\mathrm{e}^{-\gamma(c-\pi_2)} \left\{ \int_0^{d_2} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma(\bar{b}_1(d_2)-d_1)} \mathrm{e}^{\gamma d_2} S(d_2) \right\}.$$

Using $\bar{b}_1(b_2) > d_1$, (3.9), and the above equality, we obtain

$$\begin{aligned}
&\mathrm{e}^{\gamma \pi_1} \left\{ \int_0^{\bar{b}_1(b_2)} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma d_1} S(\bar{b}_1(b_2)) \right\} \\
&\leq \mathrm{e}^{\gamma \pi_1} \left\{ \int_0^{\bar{b}_1(b_2)} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma \bar{b}_1(b_2)} S(\bar{b}_1(b_2)) \right\} \\
&= -\frac{\mathrm{e}^{\gamma c}}{p} \left(\mathrm{e}^{\gamma(\bar{b}_1(b_2)-d_1)} - 1\right) + \mathrm{e}^{\gamma \pi_2} \left\{ \int_0^{d_2} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma(\bar{b}_1(b_2)-d_1)} \mathrm{e}^{\gamma d_2} S(d_2) \right\} \\
&= \mathrm{e}^{\gamma \pi_2} \left\{ \int_0^{d_2} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma d_2} S(d_2) \right\} - \left(\mathrm{e}^{\gamma(\bar{b}_1(b_2)-d_1)} - 1\right) \left(\frac{\mathrm{e}^{\gamma c}}{p} - \mathrm{e}^{\gamma(\pi_2+d_2)} S(d_2) \right) \\
&< \mathrm{e}^{\gamma \pi_2} \left\{ \int_0^{d_2} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma d_2} S(d_2) \right\},
\end{aligned}$$

in which the last inequality follows from $c > \pi_2 + d_2$ in (2.2). Thus, Proposition 3.2 implies $\bar{b}_2\left(\bar{b}_1(d_2)\right) > d_2$, a contradiction! Therefore, $(\bar{b}_1(d_2), d_2)$ is not an equilibrium.

It follows that an equilibrium $\left(b_1^*, b_2^*\right)$, if it exists, simultaneously solves the two equations $g_1(b_1; b_2) = 0$ and $g_2(b_2; b_1) = 0$, in which g_1 and g_2 are given in (3.6) and (3.7), respectively. By subtracting those two equations, we obtain

$$0 = \left(\mathrm{e}^{\gamma(b_1-d_1)} - \mathrm{e}^{\gamma(b_2-d_2)}\right) \left\{1 + p\left(\mathrm{e}^{-\gamma(c-(\pi_1+d_1))} S(b_1) - \mathrm{e}^{-\gamma(c-(\pi_2+d_2))} S(b_2)\right)\right\},$$

which implies $b_1^* - d_1 = b_2^* - d_2$ because the factor in curly brackets is positive.

By substituting $b_2^* = b_1^* - d_1 + b_2$ into $g_1(b_1; b_2) = 0$, we get $h(b_1^*) = 0$, in which h is given by (3.11). Next, we show that h has a unique positive zero in (d_1, ∞). To that end, note

$$h(d_1) = p\mathrm{e}^{-\gamma(c-\pi_1)} \left\{ \int_0^{d_1} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + e^{\gamma d_1} S(d_1) \right\} - p\mathrm{e}^{-\gamma(c-\pi_2)} \left\{ \int_0^{d_2} \mathrm{e}^{\gamma z} \mathrm{d}F(z) + \mathrm{e}^{\gamma(d_2)} S(d_2) \right\} < 0,$$

in which the inequality follows from (3.10). Also,

$$\begin{aligned}
\lim_{b\to\infty} h(b) = \lim_{b\to\infty} \mathrm{e}^{\gamma(b-d_1)} \Bigg[1 + p\mathrm{e}^{-\gamma(c-(\pi_1+d_1))} \left\{ \int_0^{b} \mathrm{e}^{-\gamma(b-z)} \mathrm{d}F(z) + S(b) \right\} \\
- p\mathrm{e}^{-\gamma(c-(\pi_2+d_2))} \left\{ \int_0^{b-d_1+d_2} \mathrm{e}^{-\gamma(b-d_1+d_2-z)} \mathrm{d}F(z) + S(b-d_1+d_2) \right\} \Bigg] \\
= \infty,
\end{aligned}$$

in which the limit follows from $c > \max\left(\pi_1 + d_1, \pi_2 + d_2\right)$ and $\int_0^b \mathrm{e}^{-\gamma(b-z)} \mathrm{d}F(z) + S(b) \in (0, 1)$. Finally,

$$h'(b) = \gamma \mathrm{e}^{\gamma(b-d_1)} \left\{1 + p\left(\mathrm{e}^{-\gamma(c-(\pi_1+d_1))} S(b) - \mathrm{e}^{-\gamma(c-(\pi_2+d_2))} S(b - d_1 + d_2)\right)\right\} > 0,$$

proving the claim.

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It remains to show that $\left(b_1^*, b_2^*\right) \in \mathcal{D}$, that is, $D\left(b_1^*, b_2^*\right) > 0$, which, by Lemma 3.1, is equivalent to (3.5). Setting

$$\mathcal{L}(b) = \int_0^b \mathrm{e}^{-\gamma(b-z)} \mathrm{d}F(z) + S(b),$$

which lies in (0, 1), we obtain the following equivalence relations:

$$\begin{aligned}
&D\left(b_1^*, b_2^*\right) > 0 \\
&\iff \left\{\mathrm{e}^{\gamma(c-\pi_1)} - p\int_0^{b_1^*} \mathrm{e}^{\gamma z}\mathrm{d}F(z)\right\}\left(\mathrm{e}^{\gamma(c-\pi_2)} - p\mathrm{e}^{\gamma d_2}S(b_2^*)\right) > p^2\mathrm{e}^{\gamma d_1}S(b_1^*)\int_0^{b_2^*}\mathrm{e}^{\gamma z}\mathrm{d}F(z) \\
&\iff \mathrm{e}^{\gamma(b_1^*+d_2)}\left[\left(\mathrm{e}^{\gamma(c-\pi_1-b_1^*)} - p\mathcal{L}(b_1^*)\right) + pS(b_1^*)\right]\left\{\left(\mathrm{e}^{\gamma(c-(\pi_2+d_2))} - p\mathcal{L}(b_2^*)\right) + p\int_0^{b_2^*}\mathrm{e}^{-\gamma(b_2^*-z)}\mathrm{d}F(z)\right\} \\
&\qquad > p^2\mathrm{e}^{\gamma(d_1+b_2^*)}S(b_1^*)\int_0^{b_2^*}\mathrm{e}^{-\gamma(b_2^*-z)}\mathrm{d}F(z) \\
&\iff \left(\mathrm{e}^{\gamma(c-\pi_1-b_1^*)} - p\mathcal{L}(b_1^*)\right)\left\{\left(\mathrm{e}^{\gamma(c-(\pi_2+d_2))} - p\mathcal{L}(b_2^*)\right) + p\int_0^{b_2^*}\mathrm{e}^{-\gamma(b_2^*-z)}\mathrm{d}F(z)\right\} \\
&\qquad + pS(b_1^*)\left(\mathrm{e}^{\gamma(c-(\pi_2+d_2))} - p\mathcal{L}(b_2^*)\right) > 0,
\end{aligned}$$

which holds if $p\mathrm{e}^{\gamma b_1^*}\mathcal{L}(b_1^*) < \mathrm{e}^{\gamma(c-\pi_1)}$. Note that $b \mapsto \mathrm{e}^{\gamma b}\mathcal{L}(b)$ strictly increases. If $p\mathrm{e}^{\gamma b}\mathcal{L}(b) < \mathrm{e}^{\gamma(c-\pi_1)}$ for all $b \geq d_1$, then we are done. Thus, without loss of generality, let $\tilde{b} \geq d_1$ be the unique value that solves

$$p\,\mathrm{e}^{\gamma\tilde{b}}\mathcal{L}(\tilde{b}) = \mathrm{e}^{\gamma(c-\pi_1)}. \tag{3.12}$$

As an aside, note that $\tilde{b} > d_1$; indeed, (3.12) implies $p\mathrm{e}^{\gamma(\tilde{b}-d_1)}\mathcal{L}(\tilde{b}) = \mathrm{e}^{\gamma(c-(\pi_1+d_1))}$, which is greater than 1 because $c > \pi_1 + d_1$. Thus, because $p\mathcal{L}(\tilde{b}) \in (0, 1)$, we must have $\mathrm{e}^{\gamma(\tilde{b}-d_1)} > 1$, which implies $\tilde{b} > d_1$.

Now, $p\mathrm{e}^{\gamma b_1^*}\mathcal{L}(b_1^*) < \mathrm{e}^{\gamma(c-\pi_1)}$ if and only if $b_1^* < \tilde{b}$, which is equivalent to $\hbar(\tilde{b}) > 0$ because $\hbar$ is strictly increasing. We compute

$$\begin{aligned}
\hbar(\tilde{b}) &= \left(\mathrm{e}^{\gamma(\tilde{b}-d_1)} - 1\right) + p\mathrm{e}^{-\gamma(c-\pi_1)}\mathrm{e}^{\gamma\tilde{b}}\mathcal{L}(\tilde{b}) - p\mathrm{e}^{-\gamma(c-\pi_2)}\mathrm{e}^{\gamma(\tilde{b}-d_1+d_2)}\mathcal{L}(\tilde{b}-d_1+d_2) \\
&= \mathrm{e}^{\gamma(\tilde{b}-d_1)}\left(1 - p\mathrm{e}^{-\gamma(c-(\pi_2+d_2))}\mathcal{L}(\tilde{b}-d_1+d_2)\right),
\end{aligned}$$

which is positive because $c > \pi_2 + d_2$ and $p\mathcal{L}(\tilde{b}-d_1+d_2) \in (0, 1)$. It follows that $b_1^* < \tilde{b}$, which implies $p\mathrm{e}^{\gamma b_1^*}\mathcal{L}(b_1^*) < \mathrm{e}^{\gamma(c-\pi_1)}$. Therefore, $D\left(b_1^*, b_2^*\right) > 0$ holds, and $\left(b_1^*, b_2^*\right) \in \mathcal{D}$ is verified. □

Remark 3.2. We can use Theorem 3.2 to strengthen Theorem 3.1 by asserting that $\left(b_1^*, b_2^*\right) = (d_1, d_2)$ if and only if inequality (3.8) holds.

Remark 3.3. From the proof of Theorem 3.2, we see that $b_1^* - d_1 = b_2^* - d_2$ holds for any fixed contract (d_1, d_2). This result relies heavily on the "symmetry" of the 2-class BMS with Markovian transition. Once this symmetry breaks, we expect $b_1^* - d_1 \neq b_2^* - d_2$ in general. Indeed, Section 5.2 in Cao et al. (2023b) considers a 3-class BMS under full insurance ($d_1 = d_2 = 0$) and shows that the equilibrium barriers are not equal. Note that if we were to allow the transition to depend on claim frequency or severity, the underlying BMS would naturally have more than two classes, and our results in Cao et al. (2023b) would then imply that $b_i^* - d_i \neq b_j^* - d_j$ for $i \neq j$. To see this, extend our model in Section 2 to allow the insurer to use the last two periods of claim history in ratemaking and still assume that in each period, the claim frequency is either 0 or 1; then, this model has four classes, (0, 0), (0, 1), (1, 0), and (1, 1), in which the first component indicates the claim frequency two periods ago and the second, the most recent period.

Remark 3.4. We return to an earlier comment regarding the choice of formulating the insured's loss reporting problem as a noncooperative Nash game, as in Definition 3.1. The same approach was first adopted by Thomas Björk and his coauthors (see Björk et al., 2014, 2017) to handle time-inconsistent stochastic control problems, in which the term "time inconsistency" roughly means that an optimal strategy derived by an agent at time s might not be optimal for the same agent at a later time $t > s$. One possible, certainly not the only, solution is to "partition" the single agent into many selfs (players) indexed by time and formulate a noncooperative Nash game, in which each player only controls during her own time period. If an equilibrium for such a game can be found, no player has an incentive to deviate from it, and thus such a solution can be adopted by the agent without imposing further commitment. With that in mind, the insured in our context faces the same challenge; that is, the best decision based on her current rate class might not be the best when she moves to a different rate class. In other words, $\arg\max_{(b_1,b_2)} J_1 \neq \arg\max_{(b_1,b_2)} J_2$ is possible in general. As such, following the game-theoretical approach, we interpret the insured as two players: player i represents the insured in rate class i and chooses her own action b_i, $i = 1, 2$.

As already hinted above, there are alternative formulations of the insured's loss reporting problem. For example, one could directly maximize J_1 and J_2 separably, over both b_1 and b_2. Recall from (3.1) that $J_i = -A_i\, \mathrm{e}^{-\gamma x}$, with A_1 and A_2 given by (3.2) and (3.3), respectively; thus, we would solve

$$\inf_{(b_1,b_2)\in[d_1,\infty)\times[d_2,\infty)} A_i(b_1, b_2), \quad i = 1, 2.$$

When maximizing $J_1(b_1, b_2)$ and $J_2(b_1, b_2)$ over both b_1 and b_2, we obtain the identical pair of first-order necessary conditions for the two maximization problems, which necessarily equal those in Propositions 3.1 and 3.2. Moreover, numerical work shows that the maximizers of J_1 and J_2 are equal to each other and are equal to the equilibrium $\left(b_1^*, b_2^*\right)$ as stated in Theorems 3.1 and 3.2. This nice coincidence, though not expected in general, indicates that maximizing the bivariate objectives J_1 and J_2 yields the same solution as our Nash equilibrium.

4. Equilibrium insurance under strategic underreporting

In this section, we study an optimal insurance problem for an insured who follows the equilibrium barrier strategy obtained in Theorem 3.1 or 3.2 to strategically underreport her losses. We pose this problem in Question 2 in the Introduction.

We start with a formal definition of the insured's equilibrium deductible insurance under strategic underreporting as follows. Recall that $(b_1^*(d_1, d_2), b_2^*(d_1, d_2))$ denotes the insured's equilibrium barrier strategy obtained in Theorem 3.1 or 3.2, for a fixed deductible pair $(d_1, d_2) \in \mathbb{R}_+^2$.

Definition 4.1. *For a fixed $d_2 \geq 0$, let $\bar{d}_1(d_2)$ denote* $\arg\sup_{d_1\geq 0} J_1(x; b_1^*(d_1, d_2), b_2^*(d_1, d_2), d_1, d_2)$*; for a fixed $d_1 \geq 0$, let $\bar{d}_2(d_1)$ denote* $\arg\sup_{d_2\geq 0} J_2(x; b_1^*(d_1, d_2), b_2^*(d_1, d_2), d_1, d_2)$. *A deductible pair $\left(d_1^*, d_2^*\right)$ yields the* equilibrium deductible insurance under strategic underreporting *if it is a fixed point of the mapping $(d_1, d_2) \mapsto (\bar{d}_1(d_2), \bar{d}_2(d_1))$.*

The premium will depend on the deductible chosen, and for concreteness, we assume that the insurer applies the expected-value principle to calculate premiums, but the premium loading varies by rate class. To be precise, if the insured is currently in rate class i and chooses a deductible $d_i \geq 0$, her (per-period) premium is given by:

$$\pi_i = (1 + \theta_i)\, \mathbb{E}(Z - d_i)_+, \quad i = 1, 2,$$

in which $\theta_i > 0$ is the premium loading for rate class i, and $x_+ := \max\{x, 0\}$. Because class 2 is considered as "punishment" and rate class 1 as "reward," we assume $\theta_1 < \theta_2$ throughout the paper.

To obtain the equilibrium insurance contract as stated in Definition 4.1, we solve the following problem for each rate class:

$$\sup_{d_i\geq 0} \mathcal{J}_i(x; d_1, d_2), \qquad i = 1, 2, \tag{4.1}$$

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Table 2. *Parameter values in the base case.*

Parameter	Symbol	Value
Insured's per-period income	c	35
Insured's risk aversion	γ	0.1
Risk loading for rate class 1	θ_1	20%
Risk loading for rate class 2	θ_2	50%
Gamma distribution	(κ, λ)	(2,5)
$\mathbb{P}(Z=0)$	p_0	0.1
$\mathbb{P}(\tau > 1)$	p	0.8

in which $\mathcal{J}_i$ equals

$$\mathcal{J}_i(x; d_1, d_2) = J_i\big(x; b_1^*(d_1, d_2), b_2^*(d_1, d_2), d_1, d_2\big).$$

Recall J_i is defined in (2.3), and $\big(b_1^*(d_1, d_2), b_2^*(d_1, d_2)\big)$ is obtained as in Theorem 3.1 or 3.2. If an equilibrium $\big(d_1^*, d_2^*\big)$ exists, we will write b_1^* (resp. b_2^*) to denote $b_1^*\ \big(d_1^*, d_2^*\big)\ \big($resp. $b_2^*\ \big(d_1^*, d_2^*\big)\ \big)$.

Although Theorems 3.1 and 3.2 provide a complete characterization of $\big(b_1^*(d_1, d_2), b_2^*(d_1, d_2)\big)$, the results are semi-explicit, and the related inequalities in (3.8) and (3.10) strongly depend on the loss distribution F. In consequence, finding an analytical solution to the optimization problems in (4.1) is likely impossible. So, we resort to a numerical approach for finding the equilibrium deductible startegy. To start, we assume in the rest of this section that the insured's per-period loss Z is a mixture of a point mass at zero and a continuous random variable $\tilde{Z} \sim Gamma(\kappa, \lambda)$, with weights $p_0 \in (0, 1)$ and $1 - p_0$, respectively. Recall that the $Gamma(\kappa, \lambda)$ distribution is characterized by its probability density function:

$$f(x) = \frac{x^{\kappa-1}\, \mathrm{e}^{-x/\lambda}}{\lambda^{\kappa}\, \Gamma(\kappa)}, \quad x > 0, \qquad \text{with} \quad \Gamma(\kappa) = \int_0^\infty t^{\kappa-1} \mathrm{e}^{-t} \mathrm{d}t.$$

The Gamma distribution belongs to the exponential family of distributions and can capture "fat tails" that are often encountered in insurance data; see, for instance, Section 17.3.1 in Frees (2009). In Appendix A, we consider another popular fat-tailed distribution, specifically, the Pareto distribution.

We set the parameter values for the model as listed in Table 2, which constitutes the base case of our numerical study. The key purpose of the subsequent numerical analysis is to obtain *qualitative* findings on the equilibrium deductible strategy and its implications. In addition, we conduct extensive sensitivity analysis to understand how model parameters affect the insured's equilibrium decisions on insurance and reporting. When we investigate the impact of a particular parameter on the insured's equilibrium strategy, we allow this parameter to vary over a reasonable range but keep the remaining parameters unchanged, with values as in Table 2.

Under the base case with the parameters as in Table 2, we compute

$$\big(d_1^*, d_2^*\big) = (7.9029, 12.2407) \quad \text{and} \quad \big(b_1^*, b_2^*\big) = (7.9515, 12.2893). \tag{4.2}$$

As a side note, the expected loss size is $\mathbb{E}Z = (1 - p_0)\mathbb{E}\tilde{Z} = 9$. We summarize the key common findings from the base case and the subsequent sensitivity studies as follows.

- First, the equilibrium deductibles are always strictly positive (i.e., $d_1^* > 0$ and $d_2^* > 0$). In consequence, when strategic underreporting is considered and allowed, full insurance is *not* optimal; therefore, the full insurance assumption made in Cao *et al*. (2023b) and several other papers are questionable. As underreporting is a form of *ex post* moral hazard, this result strengthens the conclusion that "full insurance is never optimal in the presence of moral hazard" (Winter, 2013, p.212).

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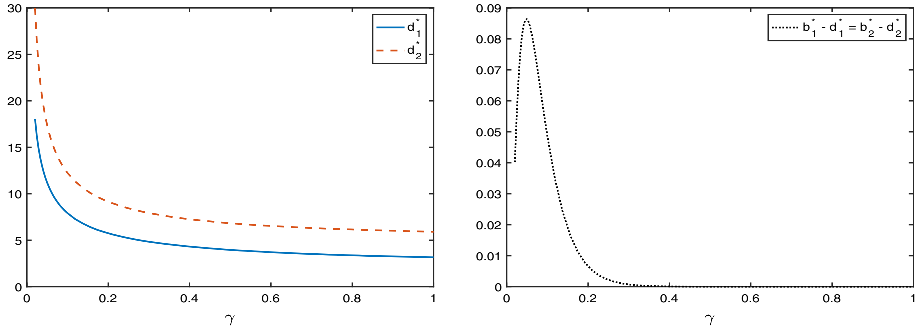

Figure 1. *Impact of insured's risk aversion γ on the equilibrium deductibles and hidden losses.*

- Second, the inequality $d_2^* > d_1^*$ always holds. This result implies that the insured's risk classification negatively affects the insurance demand; that is, the worse the rate class, the less the insurance coverage. Several empirical studies obtain the same conclusion; see, for instance, Li *et al*. (2007) for auto insurance and Dombrowski *et al*. (2020) for flood insurance.
- Third, the numerical solutions confirm

$$b_1^* - d_1^* = b_2^* - d_2^* =: b^* - d^* > 0. \tag{4.3}$$

  Since the difference is the same in both rate classes, we denote such a difference by $b^* - d^*$ and name it the *amount of hidden losses*. Recall that in the presence of a deductible d and a reporting barrier b, losses greater than d but less than b are the ones that are strategically hidden by the insured. Note that we know, from Theorem 3.1, that given an insurance contract (d_1, d_2), it is possible for the insured to report full losses, that is, $b_i^* - d_i = 0$ for $i = 1, 2$. So, the essential takeaway from (4.3) is that the amount of hidden losses is strictly positive in equilibrium, which provides theoretical explanation for the insured's underreporting and shows that inequality (3.10) holds in equilibrium. It also generalizes the result in Cao *et al*. (2023b), in which the authors show that the hidden losses are positive under full insurance; our work shows that this result still holds when the insured can choose her deductibles freely.
- Last, we can show that the "double equilibrium" contract under strategic underreporting in (4.2) yields higher welfare (as measured by expected utility) for the insured than the "single equilibrium" contract under full reporting. To arrive at this conclusion, we can force the reporting barriers to be equal to the contract deductibles, $b_i = d_i$ for $i = 1, 2$, which effectively prevents underreporting. Under this constraint, we revisit the insured's optimal deductible insurance problem in (4.1) and obtain the corresponding equilibrium deductibles $\tilde{d}_1$ and $\tilde{d}_2$:

$$\tilde{d}_1 = 7.9030 > d_1^* \quad \text{and} \quad \tilde{d}_2 = 12.2409 > d_2^*.$$

  Further, set $x = 0$ and write the insured's expected utility J_i under $(d_1, d_2, b_1, b_2) = (d_1^*, d_2^*, b_1^*, b_2^*)$ as J_i^* and under $(d_1, d_2, b_1, b_2) = (\tilde{d}_1, \tilde{d}_2, \tilde{d}_1, \tilde{d}_2)$ as $\tilde{J}_i$, $i = 1, 2$. Then, a direct computation shows that $J_i^* > \tilde{J}_i$ holds for both $i = 1, 2$, although the two quantities are close. For instance, under the scaling from J_i to $-100 \ln(-J_i)$, we obtain

$$-100 \ln(-J_1^*) = 403.6277 > 403.6268 = -100 \ln(-\tilde{J}_1),$$
$$-100 \ln(-J_2^*) = 396.5032 > 396.5025 = -100 \ln(-\tilde{J}_2).$$

In the first sensitivity study, we investigate how the insured's risk aversion γ affects her equilibrium deductibles $\left(d_1^*, d_2^*\right)$ and the amount of hidden losses $b^* - d^*$, as defined in (4.3). The left panel of Figure 1 shows that both d_1^* and d_2^* are decreasing in γ. This finding is consistent with the classic literature *without* moral hazard, namely, that a more risk-averse agent demands more insurance coverage; see Arrow (1974).

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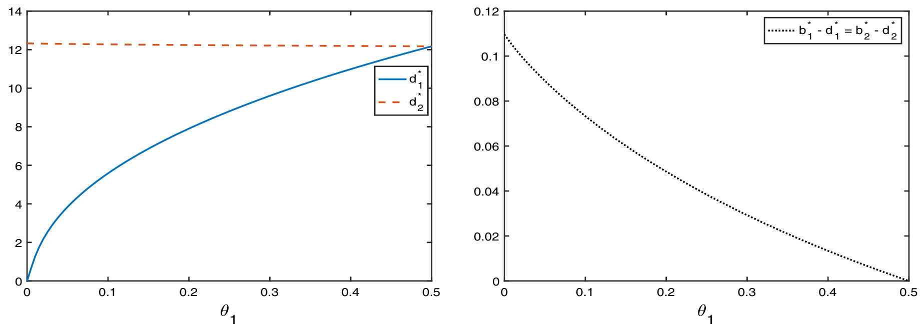

Figure 2. *Impact of premium loading θ_1 on equilibrium deductibles and hidden losses.*

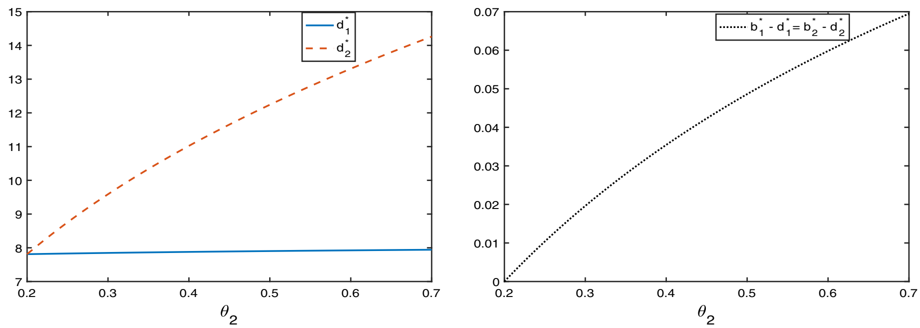

Figure 3. *Impact of premium loading θ_2 on equilibrium deductibles and hidden losses.*

Surprisingly, the amount of hidden losses $b^* - d^*$ is not monotone in γ, as shown in the right panel of Figure 1. When the risk aversion parameter γ increases from 0 to 1, $b^* - d^*$ first increases and then decreases. To understand this non-monotonicity of $b^* - d^*$ with respect to γ, one can show that for the function $\hbar$ defined in (3.11), $\lim_{\gamma \to 0^+} \frac{\hbar(b_1)}{\gamma} = b_1 - d_1 - p(\pi_2 - \pi_1)$. As $\gamma \to 0^+$, the insured becomes risk-neutral and has no demand for insurance, this implies both d_1^* and d_2^* converge to infinity, which implies $\lim_{\gamma \to 0^+} \pi_2 = \lim_{\gamma \to 0^+} \pi_1 = 0$ and $\lim_{\gamma \to 0^+} (b_1^* - d_1^*) = 0$. On the other hand, as γ gets arbitrarily large, the individual becomes very risk-averse, and her equilibrium deductibles and barriers all approach 0. Thus, applying Occam's razor, we expect $b^* - d^*$ either (1) to increase from 0 to some positive number and, then, decrease to 0 as γ increases, or (2) to stay identically 0. Our numerical work shows that the former occurs.

We next consider the impact of the premium loadings θ_1 and θ_2 on the equilibrium deductibles and hidden losses. From the two left panels in Figures 2 and 3, we see that the equilibrium deductible in each rate class increases with respect to the premium loading of that rate class and does not change much with respect to the premium loading for the other rate class. Therefore, the insured purchases less insurance as the premium loading for her current rate class increases. Also, as the penalty decreases (i.e., as $\theta_2 - \theta_1$ decreases), then $d_2^* - d_1^*$ decreases.

From the two right panels in Figures 2 and 3, we deduce that the amount of hidden losses decreases in θ_1 and increases in θ_2. The reason for this effect is that, with a smaller θ_1 or larger θ_2, the difference in premium loading between the two rate classes is larger, or in other words, the penalty for changing from rate class 1 to rate class 2 (or the reward for moving in the other direction) is larger. As a result, the insured has more incentive to hide her losses. Also, note that $d_1^* = 0$ when $\theta_1 = 0$, which means the insured purchases full insurance in rate class 1 when the premium loading is 0. However, this result also depends on other model parameters; for example, in work not shown here, if the per-period income rate is low enough, the insured will not necessarily purchase full insurance even if $\theta_1 = 0$.

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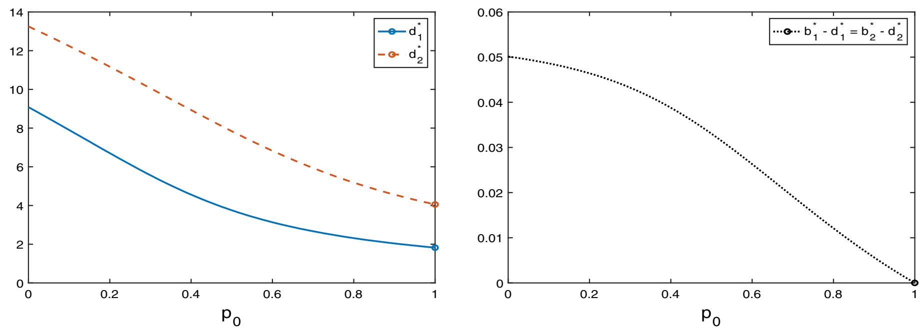

Figure 4. *Impact of probability mass at zero p_0 on equilibrium deductibles and hidden losses.*

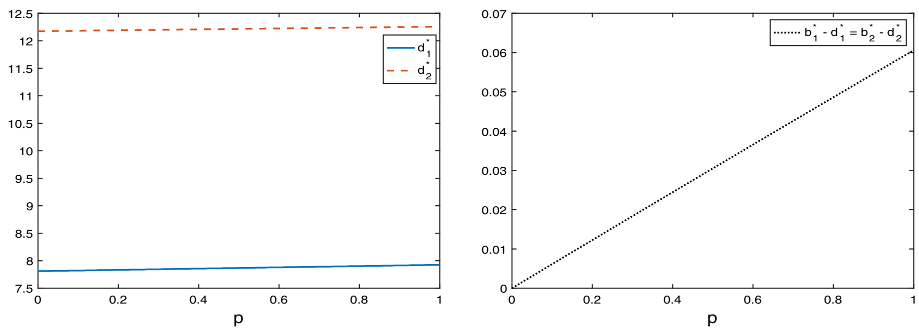

Figure 5. *Impact of renewal probability p on equilibrium deductibles and hidden losses.*

In Figure 4, we study the impact of $p_0 = \mathbb{P}(Z = 0)$, the probability of no loss per period, on the insured's equilibrium strategies. The right panel shows that the amount of hidden losses decreases in p_0. The reason lies behind the fact that the motivation for underreporting comes from the difference in the premiums between the two rate classes. When p_0 increases, a positive loss is less likely to occur and the difference in the premiums between the two rate classes shrinks. Thus, the insured will hide less of her loss. The left panel of Figure 4 shows that the two equilibrium deductibles d_1^* and d_2^* both decrease in p_0, suggesting the insured will purchase more insurance for more unlikely losses, which seems counterintuitive. However, note that with a larger value of p_0, the premium also decreases, and for the same amount spent on insurance premium, the insured could purchase more insurance via a lower deductible.

Figure 5 presents the effect of p on the equilibrium deductibles and hidden losses. Recall that $p = \mathbb{P}(\tau > 1)$, with τ being the insured's planning horizon, measures the probability of receiving the reward for underreporting (or penalty for reporting). When $p = 0$, the contract ends at the end of the current period with probability one; thus, there is no need to consider the possible penalty or reward for the next period. In this case, the insured reports her full loss, explaining why $b^* - d^* = 0$ when $p = 0$. As p increases, the probability of reward for underreporting increases, which motivates the insured to hide more losses. This argument explains the increase of $b^* - d^*$ with respect to p observed in the right panel of Figure 5. We observe from the left panel of Figure 5 that both d_1^* and d_2^* increase in p, meaning the insured will buy less insurance in both rate classes if the game is more likely to continue. The reason for this phenomenon is, as p increases, the expected time horizon $\mathbb{E}\tau$ also increases. If the insured were to purchase the same amount of insurance, the per-period premium would stay the same but the premium over the entire time horizon would increase, which would reduce the insured's terminal wealth. Thus, the insured is better-off purchasing less insurance and relying more on self-insuring (because both b_1^* and b_2^* increase) for a longer expected time horizon.

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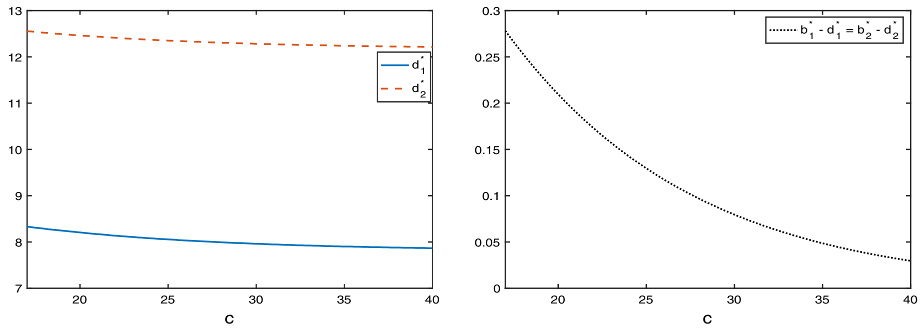

Figure 6. *Impact of income rate c on equilibrium deductibles and hidden losses.*

Finally, Figure 6 demonstrates the impact of the per-period income rate c on equilibrium strategies. The left panel shows that both equilibrium deductibles d_1^* and d_2^* decrease with respect to c. Thus, an insured with higher income will purchase more insurance, because the insurance is relatively less expensive. Similarly, with a higher income rate, the reward for underreporting is less significant, which induces the insured to hide less losses, as shown in the right panel of Figure 6.

5. Impact of underreporting on insurer's profit

We analyzed strategic loss reporting for a risk-averse insured in Section 3 and examined how underreporting affects her insurance demand in Section 4. In this section, we shift our attention to study how the insured's underreporting affects the insurer's profit.

Suppose the insured has chosen a pair of deductibles $\vec{d} := (d_1, d_2) \in \mathbb{R}_+^2$ and a pair of reporting barriers $\vec{b} := (b_1, b_2) \in [d_1, \infty) \times [d_2, \infty)$, both of which can be arbitrarily chosen from their respective domain. Let Y_t denote the insurer's *cumulative* profit over $[0, t]$ from this insurance policy for $t = 1, 2, \ldots$. Given that the insured is in rate class i, $i = 1, 2$, during $[t, t+1)$, the dynamics of the insurer's profit is governed by:

$$Y_{t+1} = Y_t + \pi_i - (Z_{t+1} - d_i) \cdot \mathbb{1}_{\{Z_{t+1} > b_i\}}, \qquad Y_0 = y \in \mathbb{R},$$

in which Z_{t+1} is the insured's loss in this period. We now define the insurer's total expected profit at the insured's exit time τ by:

$$V_i(y) = \mathbb{E}(Y_\tau | Y_0 = y, I_0 = i), \qquad i = 1, 2, \tag{5.1}$$

in which I_0 denotes the insured's initial rate class. It is clear the $V_i(y)$ depends on both $\vec{d}$ and $\vec{b}$, and if this dependence relations need to be emphasized, we will expand the arguments accordingly. For instance, $V_i(y, b_1, b_2)$ emphasizes the dependence on the barrier strategy (b_1, b_2). Similar to the treatment of the insured's objectives J_i in Section 3, we obtain an expression for $V_i(y)$, as summarized in the next lemma.

Lemma 5.1. *The insurer's expected profit V_i, defined in (5.1), equals*

$$V_i(y) = y + \frac{M_i}{G}, \qquad i = 1, 2, \tag{5.2}$$

in which

$$M_1 = pS(b_1)\left\{\pi_2 - \int_{b_2}^{\infty} (z - d_2)\mathrm{d}F(z)\right\} + (1 - pS(b_2))\left\{\pi_1 - \int_{b_1}^{\infty} (z - d_1)\mathrm{d}F(z)\right\}, \tag{5.3}$$

$$M_2 = (1 - pF(b_1))\left\{\pi_2 - \int_{b_2}^{\infty} (z - d_2)\mathrm{d}F(z)\right\} + pF(b_2)\left\{\pi_1 - \int_{b_1}^{\infty} (z - d_1)\mathrm{d}F(z)\right\}, \tag{5.4}$$

$$G = (1 - p)\big(1 + p(S(b_1) - S(b_2))\big). \tag{5.5}$$

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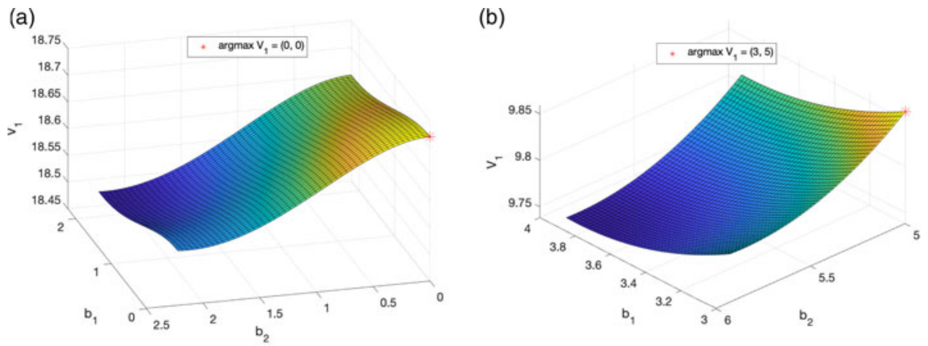

$(d_1, d_2) = (0, 0)$ $(d_1, d_2) = (3, 5)$

Figure 7. *Insurer's expected profit V_1 as a function of the insured's reporting strategy (b_1, b_2).*

Proof. By following a similar proof to Lemma 2.1, one can show that V_1 and V_2 satisfy the following system of equations:

$$\begin{aligned} V_1(y) &= (1-p)\big(y + \pi_1 - S(b_1)\,\mathbb{E}(Z - d_1|Z > b_1)\big) \\ &\quad + p\Big(F(b_1)V_1(y+\pi_1) + S(b_1)\mathbb{E}\big(V_2(y+\pi_1 - Z + d_1)\big|Z > b_1\big)\Big), \\ V_2(y) &= (1-p)\big(y + \pi_2 - S(b_2)\,\mathbb{E}(Z - d_2|Z > b_2)\big) \\ &\quad + p\Big(F(b_2)V_1(y+\pi_2) + S(b_2)\mathbb{E}\big(V_2(y+\pi_2 - Z + d_2)\big|Z > b_2\big)\Big). \end{aligned}$$

To solve the above system, we consider an ansatz of V_i of the form $V_i(y) = y + B_i$, with $B_i \in \mathbb{R}$ yet to be determined. After a lengthy, but straightforward, computation, we verify that B_i is uniquely given by $B_i = \frac{M_i}{G}$, in which M_1, M_2, and G are defined in (5.3), (5.4), and (5.5), respectively. In addition, we easily verify that $M_1, M_2, G > 0$ and thus $V_i(y) - y > 0$. □

Although Lemma 5.1 finds a fully explicit expression for the insurer's expected profit V_i from an insured in rate class i, it is still challenging to obtain *analytical* results about the sensitivity of V_i with respect to the insured's decisions $\vec{d}$ and $\vec{b}$. As such, we proceed with a numerical study to achieve this goal. Without the loss of generality, set $x = 0$ and $y = 0$ in the subsequent analysis and suppress the y-dependence in V_i. Indeed, recall from (3.1) and (5.2) that the objectives are *separable* in the initial wealth x and initial profit y, and, in particular, the insured's equilibrium reporting strategy is *independent* of x and y. In the numerical study of this section, we set the model parameters as in Table 2.

First, fix the insurance contract (i.e., fix the deductible pair $\vec{d} = (d_1, d_2)$) and study how the reporting barriers b_1 and b_2 affect the insurer's expected profit. For comparison, we consider two contracts: full insurance with $(d_1, d_2) = (0, 0)$, and a nontrivial deductible contract with $(d_1, d_2) = (3, 5)$. In Figure 7, we plot the insurer's expected profit $V_1 := V_1(b_1, b_2)$ as a function of the reporting barriers b_1 and b_2 in a neighborhood of the equilibrium point $(b_1^*(d_1, d_2), b_2^*(d_1, d_2))$. (The graphs for V_2 are similar and all the results discussed below hold for V_2 as well.) The most important finding is that, for both contracts, $V_1(d_1, d_2) > V_1(b_1^*(d_1, d_2), b_2^*(d_1, d_2))$; equivalently, the insurer's expected profit when the insured follows the equilibrium reporting strategy is strictly less than that when the insured fully reports all losses above the deductibles. This result makes intuitive sense because the savings from strategic underreporting to the insured is connected with lost profit to the insurer. In addition, we observe that the maximum value of the insurer's expected profit over the *considered region* is achieved when the insured does *not* hide any losses (above the deductible), that is, arg max $V_1(b_1, b_2) = (d_1, d_2)$.

Next, we study how the insured's *two* decisions, deductibles $\vec{d}$ and reporting barriers $\vec{b}$, affect the insurer's expected profit, along with her own expected utility. We consider three different contracts (d_1, d_2): Contract I is the insured's equilibrium contract, and Contract II (resp., Contract III)

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Table 3. *Insured's expected utility and insurer's expected profit under different strategies.*

Deductibles (d_1, d_2)	Barriers (b_1, b_2)	Insured's expected utility $(-100\ln(-J_1), -100\ln(-J_2))$	Insurer's expected profit (V_1, V_2)
Contract I:	(7.9029, 12.2407)	(403.6268, 396.5025)	(3.6475, 3.8203)
(7.9029, 12.2407)	(7.9515, 12.2893)	(403.6277, 396.5032)	(3.6460, 3.8190)
	(8, 13)	(403.6209, 396.3596)	(3.6561, 3.8349)
Contract II:	(10, 15)	(403.0839, 395.6812)	(2.5288, 2.5922)
(10, 15)	(10.0508, 15.0508)	(403.0848, 395.6818)	(2.5285, 2.5920)
	(11, 16)	(402.7856, 395.4577)	(2.6202, 2.6753)
Contract III:	(6, 10)	(403.1190, 395.7804)	(5.0431, 5.3390)
(6, 10)	(6.0504, 10.0504)	(403.1199, 395.7813)	(5.0400, 5.3360)
	(7, 11)	(402.7680, 395.4774)	(5.1171, 5.4089)

has higher (resp., lower) deductibles than those of Contract I. For each fixed contract (d_1, d_2), we further consider three different reporting strategies (b_1, b_2): the first strategy is full reporting with $(b_1, b_2) = (d_1, d_2)$, the second strategy is the corresponding equilibrium reporting strategy with $(b_1, b_2) = (b_1^*(d_1, d_2), b_2^*(d_1, d_2)) > (d_1, d_2)$, and the third strategy is the pair of integers obtained by rounding up from the equilibrium strategy, that is, $(b_1, b_2) = (\lceil b_1^*(d_1, d_2)\rceil, \lceil b_2^*(d_1, d_2)\rceil)$.[14] Under each strategy, we report the insured's scaled expected utility $(-100\ln(-J_1), -100\ln(-J_2))$ (we scale (J_1, J_2) only to make the comparison easier) and the insurer's expected profit (V_1, V_2) in Table 3, with the largest value highlighted in blue. Taking the insurer's standpoint, we observe a consistent result that $V_1(d_1, d_2) > V_1(b_1^*(d_1, d_2), b_2^*(d_1, d_2))$ for all three contracts, also confirmed by Figure 7 previously. As indicated by the third strategy under each contract, $V_1(b_1, b_2) > V_1(d_1, d_2)$ is possible when (b_1, b_2) is above the equilibrium choice, implying that, in theory, the insurer could benefit from underreporting when the insured hides large losses. However, the reporting decision is entirely the insured's, and the best outcome to the insured is always achieved by the equilibrium reporting strategy under each fixed contract. Table 3 further shows that the best among all nine combinations for the insured is the "double equilibrium," that is, taking the equilibrium contract $\left(d_1^*, d_2^*\right)$ and following the equilibrium reporting strategy $(b_1^*\left(d_1^*, d_2^*\right), b_2^*\left(d_1^*, d_2^*\right))$, as expected. Therefore, we conclude that the insured's strategic underreporting negatively impacts the insurer's profit.

6. Conclusions

Underreporting losses is well documented in the empirical insurance literature, but very few theoretical insurance models exist to explain this behavior, let alone its impact on insurance demand. In our previous work Cao *et al*. (2023b), we propose a BMS insurance model with two rate classes and consider a utility-maximizing insured with full insurance who chooses a barrier strategy for reporting losses. However, as argued in Introduction, the assumption of full insurance is questionable and, in particular, prevents us from studying the impact of strategic underreporting on insurance demand. Therefore, we extend the work of Cao *et al*. (2023b) from full insurance to deductible insurance and make two significant contributions. First, for a fixed insurance contract (characterized by a pair of deductibles (d_1, d_2)), we solve for the equilibrium barrier strategy $(b_1^*(d_1, d_2),\ b_2^*(d_1, d_2))$ that maximizes the expected exponential utility of the

[14] As an example, recall from (4.2) that the equilibrium contract under strategic loss reporting is $\left(d_1^*, d_2^*\right) = (7.9029, 12.2407)$, which is the deductible pair of Contract I and also the first reporting strategy under Contract I. The equilibrium reporting strategy of Contract I is $(b_1^*\left(d_1^*, d_2^*\right), b_2^*\left(d_1^*, d_2^*\right)) = (7.9515, 12.2893)$, constituting to the second strategy in consideration, and $(\lceil 7.9515\rceil, \lceil 12.2893\rceil) = (8, 13)$ is the third reporting strategy under Contract I.

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insured's terminal wealth. We obtain the equilibrium barrier strategy in semi-closed form, via the solution of a nonlinear equation. Second, we propose a novel optimal insurance problem in which the insured takes into account the fact that she will follow the equilibrium barrier strategy $(b_1^*(d_1, d_2),\ b_2^*(d_1, d_2))$ for reporting. Extensive numerical analysis confirms that the equilibrium contract deductibles d_1^* and d_2^* are strictly positive, justifying the extension from full insurance to deductible insurance. We also find that $b_1^*\left(d_1^*, d_2^*\right) > d_1^*$ and $b_2^*\left(d_1^*, d_2^*\right) > d_2^*$; that is, under the equilibrium insurance contract, the insured strategically underreports losses. We expect this last result to hold under premium principles other than the expected-value principle assumed in this paper.

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