STATIONARY MARKOVIAN ARRIVAL PROCESSES: RESULTS AND OPEN PROBLEMS

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Abstract

We consider two classes of irreducible Markovian arrival processes specified by the matrices C and D: the Markov-modulated Poisson process (MMPP) and the Markov-switched Poisson process (MSPP). The former exhibits a diagonal matrix D while the latter exhibits a diagonal matrix C. For these two classes we consider the following four statements: (I) the counting process is overdispersed; (II) the hazard rate of the event-stationary interarrival time is nonincreasing; (III) the squared coefficient of variation of the event-stationary process is greater than or equal to one; (IV) there is a stochastic order showing that the time-stationary interarrival time dominates the event-stationary interarrival time. For general MSPPs and order two MMPPs, we show that (I)–(IV) hold. Then for general MMPPs, it is easy to establish (I), while (II) is shown to be false by a counter-example. For general simple point processes, (III) follows from (IV). For MMPPs, we conjecture that (IV) and thus (III) hold. We also carry out some numerical experiments that fail to disprove this conjecture. Importantly, modelling folklore has often treated MMPPs as "bursty", and implicitly assumed that (III) holds. However, to the best of our knowledge, proving this is still an open problem.

1. Introduction

Point processes on the line, generated by transitions of continuous-time Markov chains (CTMCs), have been studied intensely by the applied probability community over the past few decades under the umbrella of matrix analytic methods (MAM) [19].

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These have been applied to teletraffic [1], business networks [17], social operations research [29] and biological systems [25]. The typical model referred to as the Markovian arrival process (MAP) is comprised of a finite-state irreducible CTMC, which generates events at selected instances of state change or according to Poisson processes modulated by the CTMC. MAPs are dense in the class of point processes so that they can essentially approximate any point process [5]. Yet, at the same time, they are analytically tractable and may often be incorporated effectively within more complex stochastic models [24].

In general, treating point processes as *stationary* often yields a useful mathematical perspective that matches scenarios when there is no known dependence on time. In describing a point process, we use $N(t)$ to denote the number of events during the interval $[0, t]$, and further use the sequence $\{T_n\}$ to denote the sequence of interarrival times. Two notions of stationarity are useful in this respect. A point process is *time-stationary* if the distribution of the number of events within a given set is invariant to time translations. That is, for a given subset of the timeline, $\mathcal{T} \subset \mathbb{R}$, the distribution of the number of events during $\mathcal{T}$ is the same as the distribution of the number of events during the shifted set $\{u + t \mid u \in \mathcal{T}\}$ for any time shift t.

A point process is *event-stationary* if the joint distribution of $T_{k_1}, \ldots, T_{k_n}$ is the same as that of $T_{k_1+\ell}, \ldots, T_{k_n+\ell}$ for any integer sequence of indices $k_1, \ldots, k_n$ and any integer shift ℓ. For a given model of a point process, one may often consider either the event-stationary or the time-stationary case. The probability laws of both cases agree in the case of the Poisson process; however, this is not true in general. For MAPs, time-stationarity and event-stationarity are easily characterized by the initial distribution of the background CTMC. Starting it at its stationary distribution yields time-stationarity, and starting at the stationary distribution of an associated discrete-time Markov chain yields event-stationarity. This associated discrete-time Markov chain, with transition probability matrix P, defined later (see Section 2), records the state at every time point T_n and is indexed by the (discrete) time index, n.

A common way to parametrise MAPs is by considering the generator Q of a finite-state irreducible CTMC and setting $Q = C + D$. The off-diagonal elements of the matrix C determine state transitions without event counts, while the diagonal elements of C are negative. The matrix D determines state transitions associated with event counts. Such parametrisation hints at considering two special cases: Markov-modulated Poisson processes (MMPPs) arising from a diagonal matrix D, and Markov-switched Poisson processes (MSPPs) arising from a diagonal matrix C. We note that a single probability law of a MAP can be represented in multiple ways, and the diagonal D or the diagonal C representations for MMPP or respectively MSPP are only one of many forms.

MMPPs are a widely used class of processes in modelling and are a typical example of a Cox process, also known as a doubly stochastic Poisson process [11, 28]. For a detailed outline of a variety of classic MMPP results, see the work of Fischer and Meier-Hellstern [10] and the references therein. MSPPs were introduced by Liu and Neuts [20] and to date, have not been very popular for modelling. However, the duality

of diagonal D versus diagonal C motivates us to consider and contrast both these processes. We also note that hyper-exponential renewal processes are special cases of MSPPs as well as Markovian transition counting processes (MTCPs). The latter is an alternative model for the MMPPs [3].

There are two key random variables associated with the given MAP, which we denote as T_1^π and T_1^α. These are distributed as the first interarrival time, T_1, in the time-stationary and event-stationary versions, respectively. We note that π and α notionally indicate the nature of the initial distribution of the MAP, and make sense when considering Proposition 2.2 later.

Our focus in this paper is on second-order properties of MMPPs and MSPPs and related traits. Consider the squared coefficient of variation and the limiting index of dispersion of counts given by

$$c^2 = \frac{\mathrm{Var}(T_1^\alpha)}{\mathbb{E}^2[T_1^\alpha]} \quad \text{and} \quad d^2 = \lim_{t\to\infty} \frac{\mathrm{Var}(N(t))}{\mathbb{E}[N(t)]}, \tag{1.1}$$

respectively.

From a modelling perspective, MMPPs are often used for bursty traffic. In fact, in certain cases, modelling folklore has assumed that for MMPPs, $c^2 \geq 1$ (see for example, [16]). The $c^2 \geq 1$ assumption is perhaps used without direct proof due to the fact that $d^2 \geq 1$ is straightforward to verify, and there is a similarity between these measures (for example, for a renewal process, $c^2 = d^2$). However, as we highlight in this paper, determining such "burstiness" properties is not straightforward.

A related property is that T_1^α exhibits a decreasing hazard rate (DHR), where for a random variable with probability distribution function (PDF) $f(t)$ and cumulative distribution function (CDF) $F(t)$, the hazard rate is

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

A further related property is the stochastic order (see Ch. VII of [4]) $T_1^\pi \geq_{\mathrm{st}} T_1^\alpha$. We denote these properties as follows:

$$d^2 \geq 1, \tag{I}$$

$$T_1^\alpha \text{ exhibits DHR,} \tag{II}$$

$$c^2 \geq 1, \tag{III}$$

$$T_1^\pi \geq_{\mathrm{st}} T_1^\alpha. \tag{IV}$$

All these properties are related and in this paper, we highlight the relationships between (I), (II), (III) and (IV) and establish the following. For MSPPs and MMPPs of order 2, properties (I)–(IV) hold. For general MMPPs, it is known that (I) holds; however, a counter-example shows that (II) does not hold and we conjecture that (III) and (IV) hold. We note that (I) differs from (II)–(IV) as d^2 is not directly related to the random variable T_1.

Our interest in this class of problems stemmed from relationships between different types of MAPs; as discussed by Nazarathy and Weiss [23], and Asanjarani and Nazarathy [3]. Once it became evident that $c^2 \geq 1$ for MMPPs is an open problem even though it is acknowledged as a modelling fact in folklore, we searched for alternative proof avenues. This led to the stochastic order in (IV) as well as to considering DHR properties.

The remainder of the paper is structured as follows. In Section 2, we present preliminaries, focusing on the relationships between (I)–(IV) as well as defining MMPPs and MSPPs. In Section 3, we present our main results and some conjectures about MMPPs. We conclude the paper in Section 4.

2. Preliminaries

First, consider (I)–(IV) and their relationships. To establish (III), there are several possible avenues based on (I), (II) and (IV). We now explain these relationships.

- Using (I): First, from the theory of simple point processes on the line, note the relationship between d^2 and c^2:

$$d^2 = c^2 \Big(1 + 2 \sum_{j=1}^{\infty} \frac{\mathrm{Cov}(T_0^\alpha, T_j^\alpha)}{\mathrm{Var}(T_0^\alpha)} \Big),$$

  (for more details, see [12]). However, the autocorrelation structure is typically intractable, and hence does not yield results. If we were focusing on a renewal process where T_i and T_j are independent for $i \neq j$, then this immediately shows that $d^2 = c^2$. However, our focus is broader, and hence (I) (indicating that $d^2 \geq 1$) does not appear to yield a path for (III).
- Using (II): An alternative way is to consider (II) and use the fact that for any DHR random variable, we have $c^2 \geq 1$ (see [27]). Hence, if (II) holds, then (III) holds.
- Using (IV): We have the following lemma, implying that (III) is a consequence of the stochastic order (IV).

LEMMA 2.1. *Consider a simple nontransient point process on the line, and let T_1^π and T_1^α represent the first interarrival time in the time-stationary case and the event-stationary case, respectively. Then, $c^2 \geq 1$ if and only if $\mathbb{E}[T_1^\pi] \geq \mathbb{E}[T_1^\alpha]$.*

PROOF. From point process theory (see for example, (3.4.17) of [9]),

$$\mathbb{E}[T_1^\pi] = \tfrac{1}{2} \lambda^* \mathbb{E}[(T_1^\alpha)^2],$$

where

$$\lambda^* = \lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{\mathbb{E}[T_1^\alpha]}.$$

Now,

$$c^2 = \frac{\mathbb{E}[(T_1^\alpha)^2] - (\mathbb{E}[T_1^\alpha])^2}{(\mathbb{E}[T_1^\alpha])^2} = 2\frac{\mathbb{E}[T_1^\pi]}{\mathbb{E}[T_1^\alpha]} - 1,$$

and we obtain the result.   □

MAPs: We now describe the MAP. A MAP of order p (MAP$_p$) is generated by a two-dimensional Markov process $\{(N(t), X(t)); t \geq 0\}$ on state space $\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, p\}$. The counting process $N(\cdot)$ counts the number of "events" in $[0, t]$ with $\mathbb{P}(N(0) = 0) = 1$. The phase process $X(\cdot)$ is an irreducible CTMC with state space $\{1, \ldots, p\}$, initial distribution $\boldsymbol{\eta}$ and generator matrix Q. A MAP is characterized by parameters $(\boldsymbol{\eta}, C, D)$, where the matrix C has negative diagonal elements and nonnegative off-diagonal elements, and records the rates of phase transitions which are not associated with an event. The matrix D has nonnegative elements and describes the rates of phase transitions which are accompanied with an event (an increase of $N(t)$ by 1). Moreover, we have $Q = C + D$. More details are in the books by Asmussen [4, Ch. XI] and He [13, Ch. 2].

MAPs are attractive due to the tractability of many of their properties, including distribution functions, generating functions, and moments of both counting process $N(t)$ and the sequence of interarrival times $\{T_n\}$.

Since Q is assumed irreducible and finite, it has a unique stationary distribution $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi}Q = \boldsymbol{0}'$ and $\boldsymbol{\pi}\boldsymbol{1} = 1$, where $\boldsymbol{0}'$ is a row vector of 0's and $\boldsymbol{1}$ is a column vector of 1's. Note that from $Q\boldsymbol{1} = \boldsymbol{0}'$, we have $-C\boldsymbol{1} = D\boldsymbol{1}$. Of further interest is the discrete-time Markov chain with irreducible stochastic matrix $P = (-C)^{-1}D$ and stationary distribution $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}P = \boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\boldsymbol{1} = 1$.

Observe the relation between the stationary distributions $\boldsymbol{\pi}$ and $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\pi}D}{\boldsymbol{\pi}D\boldsymbol{1}} \quad \text{and} \quad \boldsymbol{\pi} = \frac{\boldsymbol{\alpha}(-C)^{-1}}{\boldsymbol{\alpha}(-C)^{-1}\boldsymbol{1}} = \lambda^*\boldsymbol{\alpha}(-C)^{-1}, \tag{2.1}$$

where $\lambda^* = \boldsymbol{\pi}D\boldsymbol{1} = -\boldsymbol{\pi}C\boldsymbol{1}$.

The following known proposition, as distilled from the literature [4, Ch. XI], provides the key results on MAPs that we use in this paper. It shows that T_1 is a phase-type (PH) random variable with parameters $(\boldsymbol{\eta}, C)$, for the initial distribution of the phase and the sub-generator matrix, respectively. It further shows that the initial distribution of the phase process may render the MAP as time-stationary or event-stationary. This proposition also motivates the notation T_1^π and T_1^α.

PROPOSITION 2.2. *Consider a MAP with parameters $(\boldsymbol{\eta}, C, D)$, then for $t \geq 0$,*

$$\mathbb{P}(T_1 > t) = \boldsymbol{\eta}e^{Ct}\boldsymbol{1}.$$

Further, if $\boldsymbol{\eta} = \boldsymbol{\pi}$, then the MAP is time-stationary, and if $\boldsymbol{\eta} = \boldsymbol{\alpha}$, it is event-stationary, where $\boldsymbol{\pi}$ and $\boldsymbol{\alpha}$ are the stationary distributions of Q and P, respectively.

Note that for such a $PH(\boldsymbol{\eta}, C)$ random variable, the density $f(t)$ and the hazard rate $h(t)$ are respectively

$$f(t) = \boldsymbol{\eta} e^{Ct} D\mathbf{1} \quad \text{and} \quad h(t) = \frac{\boldsymbol{\eta} e^{Ct} D\mathbf{1}}{\boldsymbol{\eta} e^{Ct}\mathbf{1}}.$$

Further, to show DHR, the derivative of the hazard rate

$$h'(t) = \frac{\boldsymbol{\eta} C e^{Ct}(-C)\mathbf{1}\,\boldsymbol{\eta} e^{Ct}\mathbf{1} - \boldsymbol{\eta} C e^{Ct}\mathbf{1}\,\boldsymbol{\eta} e^{Ct}(-C)\mathbf{1}}{(\boldsymbol{\eta} e^{Ct}\mathbf{1})^2}.$$

We now describe the second-order properties associated with each case.

Event-stationary case: For an event-stationary MAP (sometimes an event-stationary MAP is referred to as an interval-stationary MAP [10]), where $\boldsymbol{\eta} = \boldsymbol{\alpha}$, the (generic) interarrival time is phase-type distributed, $PH(\boldsymbol{\alpha}, C)$, and thus the kth moment is

$$M_k = \mathbb{E}[T_n^k] = k!\,\boldsymbol{\alpha}(-C)^{-k}\mathbf{1} = (-1)^{k+1}\,k!\,\frac{1}{\lambda^*}\boldsymbol{\pi}(C^{-1})^{k-1}\mathbf{1},$$

with the first and second moments (here represented in terms of $\boldsymbol{\pi}$ and C)

$$M_1 = \frac{1}{\lambda^*}\boldsymbol{\pi}\mathbf{1} = \frac{1}{\lambda^*} \quad \text{and} \quad M_2 = 2\frac{1}{\lambda^*}\boldsymbol{\pi}(-C)^{-1}\mathbf{1}.$$

Hence, the squared coefficient of variation (SCV) of events (intervals) has the following simple formula:

$$c^2 + 1 = \frac{M_2}{M_1^2} = \frac{-2\,(1/\lambda^*)\,\boldsymbol{\pi}\,C^{-1}\mathbf{1}}{(1/\lambda^*)^2} = 2\boldsymbol{\pi}C\mathbf{1}\boldsymbol{\pi}C^{-1}\mathbf{1}. \tag{2.2}$$

Time-stationary case: For the time-stationary MAP with parameters $(\boldsymbol{\eta}, C, D)$ (where $\boldsymbol{\eta} = \boldsymbol{\pi}$) (see [4]),

$$\mathbb{E}[N(t)] = \boldsymbol{\pi}D\mathbf{1}\,t, \tag{2.3}$$

$$\text{Var}(N(t)) = (\boldsymbol{\pi}D\mathbf{1} + 2\,\boldsymbol{\pi}DD_Q^\sharp D\mathbf{1})\,t + o(t), \tag{2.4}$$

where $o(t)/t$ vanishes as $t \to \infty$, and $D_Q^\sharp$ is the *deviation matrix* associated with Q defined by the formula

$$D_Q^\sharp = \lim_{t\to\infty} D_Q^\sharp(t) = \int_0^\infty (e^{Qu} - \mathbf{1}\boldsymbol{\pi})\,du.$$

Note that in some sources [4, 21], the variance formula (2.4) is presented in terms of the matrix $Q^- := (\mathbf{1}\boldsymbol{\pi} - Q)^{-1}$. The relation between these two matrices is $Q^- = D_Q^\sharp + \mathbf{1}\boldsymbol{\pi}$ (see [8]).

Applying (2.3) and (2.4), we can write d^2 in terms of MAP parameters as

$$d^2 = 1 + \frac{2}{\lambda^*}\,\boldsymbol{\pi}DD_Q^\sharp D\mathbf{1}. \tag{2.5}$$

MMPP: A MAP with a diagonal matrix D is an MMPP. MMPPs correspond to doubly stochastic Poisson processes (also known as Cox processes), where the modulating process is driven by a CTMC. MMPPs have been used extensively in stochastic modelling and analysis [10]. The parameters of an MMPP$_p$ are $D = \text{diag}(\lambda_i)$, where $\lambda_i \geq 0$ for $i = 1, \ldots, p$, and $C = Q - D$. For MMPPs, (2.3) and (2.4) can be simplified by using the relations:

$$\lambda^* = \boldsymbol{\pi}D\mathbf{1} = \sum_{i=1}^{p} \pi_i \lambda_i, \quad D\mathbf{1} = (\lambda_1, \ldots, \lambda_p)', \quad \alpha_i = \frac{\lambda_i}{\lambda^*}\pi_i,$$

where the last equation comes from (2.1).

MSPP: A MAP with a diagonal matrix C is an MSPP [2, 13]. For MSPP$_p$, events switch between p Poisson processes, each with rate $D_{11}, \ldots, D_{pp}$, where each phase transition also incurs an event. This is in contrast to MMPPs where the transition of the phase process never generates events. We denote the diagonal elements of C with $-\gamma_i$ and hence,

$$\lambda^* = \boldsymbol{\pi}(-C)\mathbf{1} = \sum_{i=1}^{p} \pi_i \gamma_i, \quad (-C)\mathbf{1} = (\gamma_1, \ldots, \gamma_p)' \quad \text{and} \quad \alpha_i = \frac{\gamma_i}{\lambda^*}\pi_i.$$

Note that analysing MSPPs is considerably easier than MMPPs, because a diagonal C is much easier to handle than a nondiagonal C, and in an (irreducible) MMPP, C must be nondiagonal.

Properties (I)–(IV) *for MAPs:* Using the results above, for any irreducible MAP with matrices C and D, the main properties (I)–(IV) of this paper can be formulated as

$$\text{(I)} \qquad \boldsymbol{\pi}DD_Q^{\sharp}D\mathbf{1} \geq 0, \tag{2.6}$$

$$\text{(II)} \qquad \boldsymbol{\alpha}Ce^{Ct}(-C)\mathbf{1} \; \boldsymbol{\alpha}e^{Ct}\mathbf{1} + (\boldsymbol{\alpha}Ce^{Ct}\mathbf{1})^2 \leq 0 \quad \text{for all } t \geq 0, \tag{2.7}$$

$$\text{(III)} \qquad \boldsymbol{\pi}C\mathbf{1}\boldsymbol{\pi}C^{-1}\mathbf{1} \geq 1, \tag{2.8}$$

$$\text{(IV)} \qquad \boldsymbol{\pi}e^{Ct}\mathbf{1} \geq \boldsymbol{\alpha}e^{Ct}\mathbf{1} \quad \text{for all } t \geq 0, \tag{2.9}$$

respectively, where $D_Q^{\sharp}$ is the *deviation matrix* defined by the formula

$$D_Q^{\sharp} = \int_0^{\infty} (e^{Qu} - \mathbf{1}\boldsymbol{\pi}) \, du.$$

3. Main results

We now show that an MSPP and an MMPP$_2$ satisfy (I)–(IV). Establishing (I), $d^2 \geq 1$, is not a difficult task for both MMPPs and MSPPs.

PROPOSITION 3.1. *An MMPP and an MSPP exhibit $d^2 \geq 1$.*

PROOF. This is a well-known result that for all doubly stochastic Poisson processes (Cox processes), $d^2 \geq 1$ [18, Ch. 6]. So, we have the proof for MMPPs.

The fact that for a given MMPP, we have $d^2 \geq 1$ and using (2.5) results in

$$\pi D D_Q^\sharp D \mathbf{1} \geq 0 \quad \text{for any diagonal nonnegative matrix } D. \tag{3.1}$$

However, all MAPs satisfy $\pi D D_Q^\sharp D \mathbf{1} = \pi(-C) D_Q^\sharp (-C) \mathbf{1}$. Since for an MSPP, $-C$ is a diagonal nonnegative matrix, from (3.1), we have (2.6). □

It is not difficult to show that for (II), DHR holds for MSPP.

PROPOSITION 3.2. *For an MSPP, the hazard rate of the stationary interarrival time is nonincreasing.*

PROOF. By denoting $C = \text{diag}(-\gamma_i)$ and $\mathbf{u} = e^{Ct}\mathbf{1}$, the left-hand side of (2.7) can be written as

$$-\left(\sum_{i=1}^{p} \alpha_i \gamma_i^2 u_i \right)\left(\sum_{i=1}^{p} \alpha_i u_i \right) + \left(\sum_{i=1}^{p} \alpha_i \gamma_i u_i \right)^2.$$

Denoting $v_i = \alpha_i u_i$, setting $q_i = v_i / (\sum_{i=1}^{p} v_i)$ and dividing by $(\sum_{i=1}^{p} v_i)^2$ results in

$$-\sum_{i=1}^{p} \gamma_i^2 q_i + \left(\sum_{i=1}^{p} \gamma_i q_i \right)^2.$$

The above expression can be viewed as the negative of the variance of a random variable that takes values γ_i with probability q_i. Therefore, we have (2.7). □

However, somewhat surprisingly, MMPPs do not necessarily possess DHR. An exception is MMPP$_2$, as shown in Proposition 3.7; however, DHR for higher-order MMPPs does not always hold. The gist of the following example was communicated to us by Milkós Telek and Illés Horváth.

EXAMPLE 3.3. Set

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

As shown in Figure 1, the hazard rate function for an MMPP with the above matrices is not monotone. Hence, at least for general MMPPs, trying to show (III), $c^2 \geq 1$, via hazard rates is not a viable avenue.

Since hazard rates do not appear to be a viable path for establishing (III) for MMPPs, an alternative may be to consider the stochastic order (IV). Starting with MSPPs, we see that this property holds.

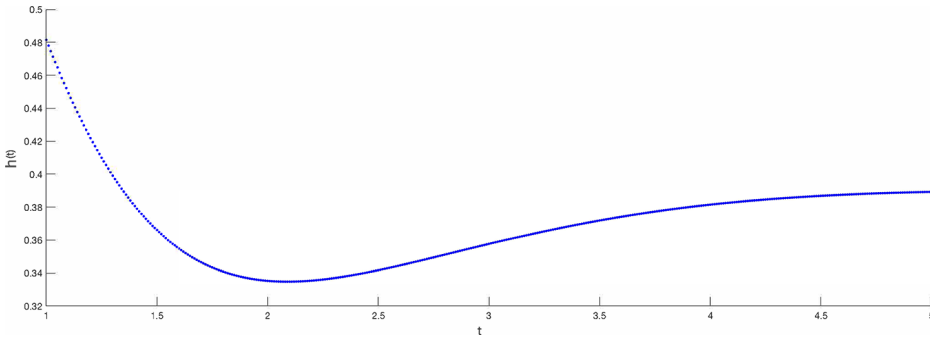PROPOSITION 3.4. *For an MSPP, $T_1^\pi \geq_{st} T_1^\alpha$.*

FIGURE 1. Nonmonotone hazard rate for an MMPP$_4$.

PROOF. Using inequality (2.9), we claim

$$(\boldsymbol{\pi} - \boldsymbol{\alpha})\, e^{Ct}\mathbf{1} \geq 0 \quad \text{for all } t \geq 0.$$

Without loss of generality, we assume that there is an order $0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_p$ (with $\gamma_i \neq \gamma_j$ for some i, j) for diagonal elements of the matrix $(-C)$. There is a possibility that for $1 < p' < p$, $0 = \gamma_1 = \gamma_2 = \cdots = \gamma_{p'-1}$ and $0 < \gamma_{p'}$; however, in the rest of the proof, we assume that $p' = 1$, meaning that all γ_i are strictly positive. Adapting to the case of $p' > 1$ is straightforward.

Now, $\{\lambda^* - \gamma_i\}_{i=1,\dots,p}$ is a nonincreasing sequence and therefore in the sequence $\{\pi_i - \alpha_i\} = \{\pi_i(\lambda^* - \gamma_i)/\lambda^*\}$, when an element $\pi_k - \alpha_k$ is negative, all the elements $\pi_i - \alpha_i$ for $i \geq k$ are also negative. Moreover, both $\boldsymbol{\pi}$ and $\boldsymbol{\alpha}$ are probability vectors, so $(\boldsymbol{\pi} - \boldsymbol{\alpha})\mathbf{1} = \sum_i(\pi_i - \alpha_i) = 0$. Therefore, at least the first element in the sequence $\{\pi_i - \alpha_i\} = \{\pi_i(\lambda^* - \gamma_i)/\lambda^*\}$ is positive. Hence, there exists an index $1 < k \leq p$ such that $\pi_i - \alpha_i$ for $i = 1, \dots, k-1$ is nonnegative and for $i = k, \dots, p$ is negative. Therefore,

$$(\boldsymbol{\pi} - \boldsymbol{\alpha})\, e^{Ct}\mathbf{1} = \underbrace{\sum_{i=1}^{k-1}(\pi_i - \alpha_i)\, e^{-\gamma_i t}}_{\text{nonnegative}} + \underbrace{\sum_{i=k}^{p}(\pi_i - \alpha_i)e^{-\gamma_i t}}_{\text{negative}}$$

$$\geq e^{-\gamma_{k-1}t}\sum_{i=1}^{k-1}(\pi_i - \alpha_i) + e^{-\gamma_k t}\sum_{i=k}^{p}(\pi_i - \alpha_i)$$

$$= L(e^{-\gamma_{k-1}t} - e^{-\gamma_k t}) \geq 0,$$

where $L = \sum_{i=1}^{k-1}(\pi_i - \alpha_i) = \sum_{i=k}^{p}(\alpha_i - \pi_i)$. Here the inequalities are a consequence of the ordering of γ_i. $\qquad\square$

Hence, via Lemma 2.1 or alternatively via the DHR property in Proposition 3.2, we have the following result.

COROLLARY 3.5. *For an MSPP, $c^2 \geq 1$.*

In fact, for MSPPs, this is an easy result and it can also be proved independently by using the Cauchy–Schwarz inequality. Further, we can find an upper bound for c^2 as follows.

PROPOSITION 3.6. *An MSPP with diagonal matrix $C = -\mathrm{diag}(\gamma_i)$ satisfies*

$$1 \le c^2 \le 2\frac{\kappa^2}{\gamma^2} - 1,$$

where $\kappa = (\min \gamma_i + \max \gamma_i)/2$ and $\gamma = \sqrt{(\min \gamma_i)(\max \gamma_i)}$.

PROOF. From (2.2), we have $c^2 + 1 = 2\pi C \mathbf{1} \pi C^{-1} \mathbf{1}$. For $C = -\mathrm{diag}(\gamma_i)$, we have $C^{-1} = -\mathrm{diag}(1/\gamma_i)$ and so,

$$\frac{c^2 + 1}{2} = \pi C \mathbf{1} \pi C^{-1} \mathbf{1} = \Big(\sum_{i=1}^{p} \pi_i \gamma_i \Big)\Big(\sum_{i=1}^{p} \pi_i \frac{1}{\gamma_i} \Big). \tag{3.3}$$

From the Cauchy–Schwarz inequality and the Kantorovich inequality [26],

$$1 = \Big(\sum_{i=1}^{p} \pi_i \gamma_i^{1/2} \frac{1}{\gamma_i^{1/2}} \Big)^2 \le \Big(\sum_{i=1}^{p} \pi_i \gamma_i \Big)\Big(\sum_{i=1}^{p} \pi_i \frac{1}{\gamma_i} \Big) \le \frac{\kappa^2}{\gamma^2}. \tag{3.4}$$

Combination of (3.3) and (3.4) results in

$$1 \le \frac{c^2 + 1}{2} \le \frac{\kappa^2}{\gamma^2},$$

which completes the proof.   □

For MMPPs, we do not have general proofs for properties (III) and (IV). Still, for MMPPs of order two (MMPP$_2$), things are easier and we are able to show that all properties (I)–(IV) hold.

PROPOSITION 3.7. *For an MMPP$_2$, $c^2 \ge 1$ and $d^2 \ge 1$, $h(t)$ is DHR and the stochastic order $T_1^\pi \ge_{st} T_1^\alpha$ holds. Further, $c^2 = 1$ and $d^2 = 1$ if and only if $\lambda_1 = \lambda_2$.*

PROOF. Consider an MMPP$_2$ with parameters

$$C = \begin{pmatrix} -\sigma_1 - \lambda_1 & \sigma_1 \\ \sigma_2 & -\sigma_2 - \lambda_2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.5}$$

Then, $\pi = (\sigma_2, \sigma_1)/(\sigma_1 + \sigma_2)$. As in [15], evaluation of the transient deviation matrix through (for example) Laplace transform inversion yields

$$\begin{aligned}
\frac{\mathrm{Var}(N(t))}{\mathbb{E}[N(t)]} = 1 &+ \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_1\sigma_2 + \lambda_2\sigma_1)} \\
&- \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^3(\lambda_1\sigma_2 + \lambda_2\sigma_1)t}(1 - e^{-(\sigma_1 + \sigma_2)t}).
\end{aligned}$$

Therefore, from (1.1),

$$d^2 = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_1\sigma_2 + \lambda_2\sigma_1)} \geq 1.$$

Further, explicit computation yields

$$c^2 = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_2\sigma_1 + \lambda_1(\lambda_2 + \sigma_2))} \geq 1.$$

Thus, it is evident that the MMPP$_2$ has $d^2 > 1$ and $c^2 > 1$ as long as $\lambda_1 \neq \lambda_2$, and $d^2 = c^2 = 1$ when $\lambda_1 = \lambda_2$.

For DHR and the stochastic order, first we note that for an MMPP$_2$ with the above parameters, $\alpha = (\sigma_2\lambda_1, \sigma_1\lambda_2)/(\sigma_1\lambda_2 + \sigma_2\lambda_1)$. By setting $a = \sigma_2\lambda_1 + \lambda_2(\sigma_1 + \lambda_1) > 0$ and $b = \sigma_1 + \sigma_2 + \lambda_1 + \lambda_2 > 0$, after some simplification, the left-hand side of (2.7) is given by

$$-\frac{ae^{-bt}\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_2\lambda_1 + \sigma_1\lambda_2)^2},$$

which is strictly negative for $\lambda_1 \neq \lambda_2$ and is zero for $\lambda_1 = \lambda_2$. For the stochastic order, by evaluating the matrix exponential and simplifying,

$$(\boldsymbol{\pi} - \boldsymbol{\alpha})e^{Ct}\mathbf{1} = \frac{e^{-(b+\sqrt{b^2-4a})/2}(e^{t\sqrt{b^2-4a}} - 1)\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)(\sigma_1\lambda_2 + \sigma_2\lambda_1)\sqrt{b^2 - 4a}},$$

which is strictly positive for $\lambda_1 \neq \lambda_2$ and is zero for $\lambda_1 = \lambda_2$. □

We note that $c^2 > 1$ in Proposition 3.7 may also be derived by considering a canonical form representation of second-order MAPs as presented by Bodrog et al. [7]. In that paper, they suggested two canonical form representations for MAPs of order 2. MMPPs fit the first canonical form which is represented in Definition 1 of [7] via,

$$\tilde{C} = \begin{bmatrix} -\tilde{\lambda}_1 & (1-\tilde{a})\tilde{\lambda}_1 \\ 0 & -\tilde{\lambda}_2 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} \tilde{a}\tilde{\lambda}_1 & 0 \\ (1-\tilde{b})\tilde{\lambda}_2 & \tilde{b}\tilde{\lambda}_2 \end{bmatrix}. \tag{3.6}$$

We use tildes so as not to confuse the notation of Bodrog et al. [7] with the notation of our current paper. Given a representation of an MMPP$_2$ as in (3.5), one may use a similarity transform jointly for C and D to find an equivalent canonical form representation. In this case, a similarity transformation matrix which satisfies $BC = \tilde{C}B$ and $BD = \tilde{D}B$ can be represented as

$$B = \begin{bmatrix} b_1 & 1 - b_1 \\ 1 - b_2 & b_2 \end{bmatrix}.$$

Solving a system of equations for b_1, b_2, and the parameters of the canonical form, we find four different equivalent solutions in the sense that the resulting canonical form agrees with the original MMPP$_2$. One possible solution is

$$b_1 = 0 \quad \text{and} \quad b_2 = -\frac{\mathrm{r} - \lambda_1 + \lambda_2 + \sigma_1 + \sigma_2}{2\lambda_1 - 2\lambda_2}.$$

Here, $r = \sqrt{b^2 - 4a}$, where a and b are as in the proof of Proposition 3.7. With this similarity transform, the parameters of the canonical form (3.6) yielding MMPP$_2$ are

$$\tilde{\lambda}_1 = \frac{b - r}{2}, \quad \tilde{\lambda}_2 = \frac{b + r}{2}, \quad \tilde{a} = \frac{\lambda_2(b + r)}{2a}, \quad \tilde{b} = \frac{\lambda_1(b - r)}{2a}.$$

Having this representation in hand, from the work of Bodrog et al. [7, Lemma 1], we observe that $c^2 > 1$.

Note that one can also represent MSPPs of order 2 in terms of their canonical representations [7]. However, for MSPP, the correlation coefficient can be both positive and negative, and thus one needs to choose between two alternative canonical representations.

3.1. Conjectures for MMPP   We embarked on this research due to the folklore assumption that for MMPP, $c^2 \geq 1$ (III) (see for example, [16]). To date, there is no known proof for an arbitrary irreducible MMPP. Still, we conjecture that both (III) and (IV) hold for MMPPs.

CONJECTURE 3.8. *For an irreducible MMPP, $c^2 \geq 1$.*

CONJECTURE 3.9. *For an irreducible MMPP, $T_1^\pi \geq_{\mathrm{st}} T_1^\alpha$.*

In an attempt to disprove these conjectures, we carried out a numerical experiment by generating random instances of MMPPs. Each instance is created by first generating a matrix Q where off-diagonal entries are generated from a uniform $(0, 1)$ distribution and diagonal entries that ensure row sums are 0. We then generate a matrix D with diagonal elements that are exponentially distributed with rate 1. Such a (Q, D) pair then implies π and α using their definitions in Section 2. For each such MMPP, we calculate $\pi C \mathbf{1} \pi C^{-1} \mathbf{1} - 1$ as in (2.8) (for Conjecture 3.8) and $(\pi - \alpha)e^{Ct}\mathbf{1}$ as in (2.9) (for Conjecture 3.9), where we take $t \in \{0, 0.2, 0.4, \ldots, 9.8, 10.0\}$. Getting a strictly negative value on any of these would imply a counterexample for the associated conjecture.

We repeated this experiment for random MMPP instances of orders $3, 4, 5$ and 6 with 10^6 instances for each order. In all the cases, the calculated quantities were greater than -10^{-15}. Note that in certain cases, the quantity associated with (IV) was negative and lying in the range $(-10^{-15}, -10^{-16}]$. We attribute these negative values to numerical error stemming from the calculation of the matrix exponential e^{Ct}.

We ran our experiments with the Julia programming language, V1.0 [22]. The calculation time was approximately 1.5 hours on a desktop computer.

The purpose of carrying out such experiments is to try to disprove the conjectures; however, we failed to do so. Also note that when considering uniformly generated instances, it is possible that we miss corner cases that may disprove the conjectures. In trying to find some classes of such corner cases, we considered random cyclic Q matrices with nonzero entries similar to (3.2). We generated 10^6 such (order 4) examples, and all agreed with (III) and (IV) as well.

4. Conclusion

We have highlighted various related properties for point processes on the line and MAPs exhibiting diagonal matrices (C or D), in particular. Bodrog et al. [7] showed $c^2 \geq 1$ for MMPPs of order 2, and here we have derived it alongside $T_1^\pi \geq_{st} T_1^\alpha$ using alternative means. Further, using a similar technique to our MSPP proof (proof of Proposition 3.2), we can also show properties for MMPPs with symmetric C matrices. However, for general MMPPs of order greater than 2, proving these properties remains an open problem.

We note that stepping outside of the matrix analytic paradigm and considering general Cox processes is also an option. In fact, since any Cox process can be approximated by an MMPP, one may formulate versions of conjectures 1 and 2 for Cox processes under suitable regularity conditions.

There is also a related branch of questions dealing with characterizing the Poisson process via $c^2 = 1$ and considering when an MMPP is Poisson. For example, for the general class of MAPs, Bean and Green [6] provide a condition for determining if a given MAP is Poisson. It is not hard to construct a MAP with $c^2 = 1$ that is not Poisson. However, we conjecture that all MMPPs with $c^2 = 1$ are Poisson. Yet, we do not have a proof. Further, we conjecture that for an MMPP, if $c^2 = 1$, then all λ_i are equal (the converse is trivially true). We do not have a proof for orders beyond two. Related questions also hold for the more general Cox processes.

We also note that the MSPP class of processes generalizes hyper-exponential renewal processes, as well as a class of processes called a Markovian transition counting process (MTCP), as in the authors' previous work [3]. Further, since the stationary interval interarrival times of MMPP processes are PH distributed, it is also of interest to consider related problems dealing with moments of phase-type distributions (see for example, [14] and the references therein).

Acknowledgements

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