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FIXED POINT THEOREMS FOR MEASURABLE SEMIGROUPS OF OPERATIONS

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ABSTRACT. Let S be a topological semigroup, K a compact convex subset of a separated convex space E and $T: S \times K \longrightarrow K$ an affine action (denoted by $(s, x) \longrightarrow T_s(x)$, $s \in S, x \in K$) of S as continuous affine maps on K. It is shown in A. Lau and J. Wong [22] that the weakly left uniformly measurable functions WLUM(S) on S has a left invariant mean iff S has the fixed point property for weakly measurable affine actions, i.e. affine actions such that the scalar function $s \longrightarrow x^*T_s(x)$ is measurable for each $x \in K$ and $x^* \in E^*$ (the dual of E) with respect to the Borel sets in S. It is natural to ask for a "strongly" measurable analogue of this result. There are a number of ways to define such actions and the corresponding functions on S. In this paper, we obtained a neat analogue of this fixed point theorem by a suitable choice of strong measurability which naturally leads to another new fixed point theorem for separable actions. Also, we shall unify these and many known fixed point theorems and extend and generalise them to anti-actions of S as bounded linear operators on Banach spaces

1. **Introduction.** Let S be a topological semigroup with separately continuous multiplication, K a compact convex subset of a separated convex space E. An affine action of S on K is a map $T: S \times K \longrightarrow K$ (denoted by $(s, x) \longrightarrow T_s(x)$, $s \in S, x \in K$) such that (1) $T_{st} = T_s \circ T_t$ for all $s, t \in S$ and (2) $T_s: K \longrightarrow K$ is affine continuous for each $s \in S$. T is called *weakly measurable* if the scalar function $s \longrightarrow x^*T_s(x)$ is measurable with respect to the Borel sets in S for each $x \in K$ and $x^* \in E^*$, the continuous dual of E.

Let $m(S)$ be the Banach algebra (pointwise operations, sup norm) of all bounded functions on S and BM(S) the closed subalgebra of all Borel measurable bounded functions. As in Lau and Wong [22], let WLUM(S) be the space of all weakly left uniformly measurable functions in BM(S), i.e. WLUM(S) consists of all functions in BM(S) such that the scalar function $s \longrightarrow m(\ell_s f)$ is Borel measurable for each $m \in \mathrm{BM}(S)^*$, where $\ell_s f(t) = f(st)$, $s, t \in S$. It is shown in Lau and Wong [22, Theorem 2.1] that WLUM(S) has a left invariant mean iff each weakly measurable affine action T of S has a fixed point ($x_0 \in K$ such that $T_s(x_0) = x_0$ for all $s \in S$). We shall obtain a separable analogue and a strongly measurable analogue of this result, and unify these two theorems with many other known fixed point theorems, using an alternative approach. Extensions and generalizations to anti-actions on Banach spaces are also obtained.

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2. **Definitions.** For notations and terminologies not explained here, see Lau and Wong [22] and also BJM [2]. A function $f \in m(S)$ is called *left separable* if its left orbit $O_L(f) = \{ \ell_s f : s \in S\}$ is norm separable in $m(S)$. (Here $\ell_s f(t) = f(st)$, $s, t \in S$). Let $\mathrm{LS}(S)$ denote the space of all left separable functions on S and define $\mathrm{SLUM}(S) = \mathrm{WLUM}(S) \cap \mathrm{LS}(S)$. Functions in $\mathrm{SLUM}(S)$ are called *strongly left uniformly measurable*. Moreover, if $f \in \mathrm{BM}(S)$, then f is strongly left uniformly measurable iff the vector-valued function $s \longrightarrow \ell_s f$ from S into the Banach space $\mathrm{BM}(S)$ is weakly measurable and separably valued. (For measurability of Banach space-valued functions, see Hille and Phillips [16] and Dunford and Schwartz [7]). A linear supspace X in $m(S)$ is left introverted if $m_\ell(f) \in X$ for all $f \in X$ and $m \in M(S)^*$. Here $m_\ell(f)(s) = m(\ell_s f)$, $s \in S$, $m \in m(S)^*, f \in m(S)$.

Let B be a Banach space and $\tau: S \times B \longrightarrow B$ an action or anti-action (denoted by $(s, b) \longrightarrow \tau_s(b)$, $s \in S$, $b \in B$) of S, as bounded linear operators in B (For an anti-action, $\tau_{st} = \tau_t \circ \tau_s$, $s, t \in S$). τ is said to be separable if the orbit $\{\tau_s(b) : s \in S\}$ is norm separable in B for each $b \in B$. It is weakly measurable if the scalar function $s \longrightarrow m\big(\tau_s(b)\big)$ is measurable with respect to the Borel sets in S for each $b \in B$ and $m \in B^*$. It is strongly measurable if it is both weakly measurable and separable.

If $T: S \times K \longrightarrow K$ is an affine action on the compact convex set K, then T induces an anti-action $\tau: S \times A(K) \longrightarrow A(K)$ of S on the Banach space $A(K)$ of all continuous affine functions on K with sup norm $\|\cdot\|_K$, defined by $\tau_s(h) = h \circ T_s$, $s \in S$ and $h \in A(K)$. In fact $\|\tau_s(h)\|_K \leq \|h\|_K$ and $\|\tau_s\| \leq 1$, $\tau_s(1) = 1$ for all $s \in S$, $h \in A(K)$. The converse is also true. Each anti-action on $A(K)$ such that $\|\tau_s\| \leq 1$ and $\tau_s(1) = 1$ is induced by an affine action $T: S \times K \longrightarrow K$. We shall discuss this and study the relationship between these actions, and anti-actions, in §4 below when we impose more measurability, continuity (separate and joint) and equicontinuity conditions on them.

Before we present the main results, we first, gather some technical lemmas, which are of independent interest.

LEMMA 2.1. *(a)* $\mathrm{LS}(S)$ *is a norm closed left invariant left introverted subalgebra of* $m(S)$ *containing the constants.*

(b) $\mathrm{SLUM}(S)$ *is a norm closed left invariant left introverted linear subspace of* $m(S)$ *containing the constants.*

PROOF. We shall only sketch the proof.

(a) $\mathrm{LS}(S)$ is a left invariant subalgebra because subspaces of a separable metric space are also separable (see Gamelin and Greene [9, Theorem 5.7, p. 23]). $\mathrm{LS}(S)$ is left introverted because $m_\ell(\ell_s f) = \ell_s m_\ell(f)$ and the operator $m_\ell: m(S) \longrightarrow m(S)$ is bounded linear ($\|m_\ell\| \leq \|m\|$). It is also norm closed because a countable union of countable sets is countable. The rest is obvious.

(b) Since $\mathrm{SLUM}(S) = \mathrm{WLUM}(S) \cap \mathrm{LS}(S)$ and $\mathrm{WLUM}(S)$ is a norm closed left invariant left introverted linear subspace of $m(S)$ containing the constants (see Lau and Wong [22, §2 p. 549]), so is $\mathrm{SLUM}(S)$ by (a) above.

There is no need to separately define functions $f \in m(S)$ such that $\{ \ell_s f : s \in S\}$ is weakly separable because of the following

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LEMMA 2.2. *For a subset A in a Banach space B, the following statements are equivalent:*

(a) A is norm separable.

(b) A is weakly separable.

PROOF. The proof is contained in H. Dzinotyiweyi [8, Lemma 4.3, p. 57]. For completeness, we include the proof here. Let $^-$ and $^=$ denote the weak and norm closures in B respectively. Suppose D is a countable weakly dense subset of A. Consider $\langle D\rangle^=$ where $\langle\ \rangle$ denotes the linear span. We have $\langle D\rangle^= = \{\sum_{i=1}^n \lambda_i d_i : d_i \in D, \lambda_i \text{ rational}\}^=$. Therefore $\langle D\rangle^=$ is separable in norm since the set $\{\sum_{i=1}^n \lambda_i d_i : d_i \in D, \lambda_i \text{ rational}\}$ is countable. But $\langle D\rangle^= = \langle D\rangle^- \supset D^- \supset A$. Hence A is also norm separable.

REMARK. The arguments used in the proof actually show that both (a) and (b) are equivalent to the statement (c) A_1 is norm or weakly separable. Here $A \subset A_1 \subset \langle A\rangle^=$. For example $A_1 = \mathrm{CO}(A)^-$ the closed (norm or weak) convex hull of A.

Let AP(S) and WAP(S) denote the almost periodic and weakly almost periodic (continuous bounded) functions on S respectively and let S_d denote the semigroup S with the discrete topology (see BJM [2] for definitions).

LEMMA 2.3. *(a)* $\mathrm{AP}(S) \subset \mathrm{AP}(S_d) \subset \mathrm{LS}(S)$.

(b) If S is separable, then $\mathrm{WAP}(S) \subset \mathrm{LS}(S)$.

PROOF. (a) This follows from the fact that compact metric spaces are separable (Gamelin and Greene [9, Theorem 5.12, p. 25]).

(b) By BJM [2, III. 14.3 pp. 124–125], $\mathrm{WAP}(S) \subset \mathrm{WLUC}(S)$, the space of all $f \in \mathrm{CB}(S)$ (continuous bounded functions) such that $s \longrightarrow \ell_s f$ is weakly continuous. If S is separable, then $\{\ell_s f : s \in S\}$ is weakly separable. Such an f must be in LS(S), by Lemma 2.1.

3. **Fixed point theorems.**

THEOREM 3.1. *The following statements are equivalent:*

(1) LS(S) *has a left invariant mean.*

(2) For any affine action $T: S \times K \longrightarrow K$ *such that the induced anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *is separable, there is some* $x_0 \in K$ *such that* $T_s(x_0) = x_0$ *for all* $s \in S$.

PROOF. (1) implies (2): Assume that LS(S) has a left invariant mean and let $T: S \times K \longrightarrow K$ be any affine action such that the induced anti-action $\tau: S \times A(K) \longrightarrow A(K)$ (where $\tau_s(h) = h \circ T_s$, $s \in S$, $h \in A(K)$) is separable. For $x \in K$ define $T_x: A(K) \longrightarrow m(S)$ by $T_x h(s) = h\big(T_s(x)\big) = \tau_s h(x)$, $h \in A(K)$, $s \in S$. Then $\ell_s(T_x h) = T_x(\tau_s h)$. Since T_x is bounded linear ($\|T_x h\|_S \leqq \|h\|_K$), it follows that $T_x h \in \mathrm{LS}(S)$ by separability of τ. Hence by Argabright's ([1, Theorem 1, p. 128]) fixed point theorem, there is some $x_0 \in K$ such that $T_s(x_0) = x_0\ \forall s \in S$.

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(2) implies (1): As usual, take K to be the set of all means in $E = \mathrm{LS}(S)^*$ with weak* topology and define $T: S \times K \longrightarrow K$ by $T_s(m) = \ell_s^* m$, $s \in S$, $m \in K$. The induced anti-action $\tau: S \times A(K) \longrightarrow A(K)$ of S on $A(K)$ where $\tau_s(h) = h \circ T_s$ is separable. For if $h \in E^*|K$, then $h = \hat{f}$ ($f \in \mathrm{LS}(S)$), considered as an affine function on K by $\hat{f}(m) = m(f)$, $m \in K$. Then $\tau_s(h) = \ell_s f$ and $\{\tau_s(h) : s \in S\}$ is norm separable since $\|\hat{f}\|_K = \sup\{|\hat{f}(m)| : m \in K\} = \|f\|_S$. The same holds for general $h \in A(K)$ since $E^*|K + \mathbb{R}$ is norm dense in $A(K)$. Any fixed point of T is a left invariant mean on $\mathrm{LS}(S)$.

THEOREM 3.2. *The following statements are equivalent:*

(1) $\mathrm{SLUM}(S)$ *has a left invariant mean.*

(2) For any affine action $T: S \times K \longrightarrow K$ *such that the induced anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *is strongly measurable there is some* $x_0 \in K$ *such that* $T_s(x_0) = x_0\ \forall s \in K$.

PROOF. (1) implies (2): Define $T_x: A(K) \longrightarrow m(S)$ by $T_x h(s) = h\big(T_s(x)\big) = \tau_x h(x)$, $h \in A(K)$ and $s \in S, x \in K$. Weak measurability of τ is equivalent to weak measurability of T. Hence $T_x h \in \mathrm{WLUM}(S)$ for $h \in A(K)$ and $x \in K$ (see Lau and Wong [22, Theorem 2.1, proof of (1) implies (2), p. 550]). Separability of τ implies $T_x h \in \mathrm{LS}(S)$. Hence $T_x h \in \mathrm{SLUM}(S)$ and the rest proceeds as before using Argabright [1, Theorem 1, p. 128] again.

(2) implies (1): Similar to Theorem 3.1; we omit the details.

REMARK. One can also define an anti-action $\tau: S \times A(K) \longrightarrow A(K)$ to be "strongly" measurable if the map $s \longrightarrow \tau_s h$ is norm Borel measurable for each $h \in A(K)$. That is, inverse images of norm Borel sets in $A(K)$ are Borel measurable in S. Also, we may define $f \in \mathrm{BM}(S)$ to be "strongly" left uniformly measurable if the map $s \longrightarrow \ell_s f$ is norm Borel measurable in the same sense. However, there is one (and only one) obstruction to a complete analogue of Theorem 3.2. Namely, the set of all such f need not form a linear space because the sum of two Banach space valued measurable functions need not be measurable unless the Banach space is separable (see J. Nedoma [26]).

Since $\mathrm{LS}(S)$ is also an algebra, one can consider multiplicative left invariant means on $\mathrm{LS}(S)$ and ask for an analogue of Theorem 3.1. But first a definition. Let Y be a compact Hausdorff space and $T: S \times Y \longrightarrow Y$ an action (not necessarily affine) of S on Y as continuous mappings from Y into Y. T induces an anti-action $\tau: S \times C(Y) \longrightarrow C(Y)$ of S on the Banach space $C(Y)$ (sup norm) such that $\tau_s h = h \circ T_s$, $s \in S$, $h \in C(Y)$. Clearly $\|\tau_s\| \leq 1, \tau_s(1) = 1$ and $\tau_s(hk) = \tau_s h \cdot \tau_s k\ \forall s \in S; h, k \in C(Y)$.

THEOREM 3.3. *The following statements are equivalent:*

(1) $\mathrm{LS}(S)$ *has a multiplicative left invariant mean.*

(2) For any action $T: S \times Y \longrightarrow Y$ *of* S *on a compact Hausdorff space* Y *such that the induced anti-action* $\tau: S \times C(Y) \longrightarrow C(Y)$ *is separable, there is some* $y_0 \in Y$ *such that* $T_s(y_0) = y_0$ *for all* $s \in S$.

PROOF. (1) implies (2): This is similar to Theorem 3.1, using $C(Y)$, ($A(Y)$ is not available) and Mitchell's fixed point theorem [23, Theorem 2, p. 121] instead of Argabright's. We omit the details.

(2) implies (1): Again, similar to Theorem 3.1. Take Y to be the set of all multiplicative means in $E = \mathrm{LS}(S)^*$ with the weak* topology and define $T_s(m) = \ell_s^* m$, $s \in S$, $m \in Y$. The induced anti-action $\tau : S \times C(Y) \longrightarrow C(Y)$ is separable because $\tau_s(h) = \ell_s f$ if $h = \hat{f}$, $f \in \mathrm{LS}(S)$ where $\hat{\ }: \mathrm{LS}(S) \longrightarrow C(Y)$ is the Gelfand isometric isomorphism.

4. **An alternative approach and connection with other fixed point theorems.** Let $T : S \times K \longrightarrow K$ be an affine action of S on a compact convex set K and $\tau : S \times A(K) \longrightarrow A(K)$ its induced anti-action on the Banach space $A(K)$ such that $\tau_s h = h \circ T_s$, $h \in A(K)$, $s \in S$. Then $\|\tau_s\| \leq 1$ and $\tau_s(1) = 1$ $\forall s \in S$. Conversely, if $\tau : S \times A(K) \longrightarrow A(K)$ is any anti-action of S on the Banach space $A(K)$ as bounded linear operators in $A(K)$ with $\|\tau_s\| \leq 1$ (hence uniformly bounded) and $\tau_s(1) = 1$ $\forall s \in S$, then τ is induced by a unique affine action $T : S \times K \longrightarrow K$ of S on K. In fact $T_s(x)$ is uniquely defined by the formula $h\big(T_s(x)\big) = \tau_s h(x)$, $x \in K$, $s \in S$, $h \in A(K)$, using Argabright [1, Lemma 2, p. 127] and the fact that $A(K)$ separates points of K. If $x_0 \in K$ is a fixed point of T, then $h\big(T_s(x_0)\big) = h(x_0)$ $\forall s \in S$, $h \in A(K)$. This is equivalent to the statement that the (affine) functions $\{\tau_s - h : s \in S, h \in A(K)\}$ generate a proper ideal in the Banach algebra $C(K)$ because any proper ideal in $C(K)$ is contained in a maximal one and each maximal ideal in $C(K)$ is fixed, i.e. of the form. $M_{x_0} = \{f \in C(K) : f(x_0) = 0\}$ (see Hewitt and Ross [15, § C. 31, p. 438]). Therefore Theorems 3.1 and 3.2 can be reformulated algebraically, in terms of the Banach space action τ as

THEOREM 4.1. *The following statements are equivalent:*

(1) LS(S) [SLUM(S)] *has a left invariant mean.*

(2) For each separable [strongly measurable] anti-action $\tau : S \times A(K) \longrightarrow A(K)$ *of* S *on the Banach space* $A(K)$ *such that* $\|\tau_s\| \leq 1$ *and* $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s h - h : s \in S \text{ and } h \in A(K)\}$ *generate a proper ideal in* $C(K)$.

There is also an analogous reformulation of Theorems 3.3 as well. See § 5 below.

The following three questions naturally arise:

(a) Can other known fixed point theorems for affine actions on compact convex sets K (e.g. those in Mitchell [24] and Lau [20]) be reformulated for anti-actions on the Banach spaces $A(K)$?

(b) Are there analogues of Theorem 4.1 for anti-actions τ on the Banach space $A(K)$ with other measurability, continuity or compactness conditions on τ?

(c) What about anti-actions on general Banach spaces?

We answer the first two questions by giving a complete picture of the relationship between an affine action T and its induced anti-action τ with respect to measurability, continuity, and compactness conditions on them. The last problem is discussed in § 5 below as generalizations and extensions of fixed point theorems.

Let $T : S \times K \longrightarrow K$ be an affine action on the compact convex set K and $\tau : S \times A(K) \longrightarrow A(K)$ its induced anti-action of S on the Banach space $A(K)$ (Recall that $E^* | K + \mathbb{R}$ is norm dense in $A(K)$).

THEOREM 4.2. *(1) T is weakly measurable iff τ is weakly measurable.*

(2) T is jointly continuous [separately continuous] iff τ is separately or jointly continuous when $A(K)$ has the norm topology [weak topology].

PROOF. (1) is clear and was already used in the proof of Theorem 3.2. To prove (2), first assume that T is jointly continuous on $S \times K$ and define for each $h \in A(K)$ the function $g: K \times S \longrightarrow \mathbb{R}$ by $g(x,s) = h\big(T_s(x)\big) = \tau_s h(x)$, $x \in K$, $s \in S$. Then g is bounded (by $\|h\|_K$). g is jointly continuous since T is. By BJM [2, Lemma I. 1.8 p. 4], the map $s \longrightarrow g(\cdot, s)$ is continuous from S into $\big(C(K), \|\cdot\|_K\big)$. i.e. $s \longrightarrow \tau_s h$ is $\|\cdot\|_K$-continuous. So τ is separately continuous, hence also jointly continuous since $\|\tau_s\| \leq 1$ $\forall s \in S$ (i.e. uniformly bounded).

Conversely, if $s \longrightarrow \tau_s h$ is $\|\cdot\|_K$-continuous $\forall h \in A(K)$. The same BJM lemma implies that g is jointly continuous for each $h \in A(K)$ where $g(x,s) = h\big(T_s(x)\big) = \tau_s h(x)$. So is T because the topology of K is determined by $A(K)$.

Next, T is separately continuous iff τ is separately continuous when $A(K)$ has the weak topology because the weak topology on $A(K)$ is the same as the pointwise topology in $A(K)$ (see Lau and Wong [22, Theorem 2.1 proof of (1) implies (2), p. 550]), and because a linear operator in a Banach space is norm-norm continuous iff it is weak-weak continuous.

It remains to show that when $A(K)$ has the weak (or pointwise) topology, separate continuity of τ implies also joint continuity of τ.

To prove this, suppose τ is separately continuous when $A(K)$ has the weak topology. Since $\|\tau_s\| \leq 1$ $\forall s \in S$, the family $F = \{\tau_s : s \in S\}$ is equicontinuous from $A(K) \longrightarrow A(K)$ as linear operators in the Banach space $A(K)$ (Robertson and Robertson [29, Theorem 3, p. 69]). It follows that F is also equicontinuous when $A(K)$ has the weak topology. Now let F have the relative product topology of $A(K)^{A(K)}$ where each factor $A(K)$ has the weak topology. Define the maps $\phi: S \times A(K) \longrightarrow F \times A(K)$ and $\psi: F \times A(K) \longrightarrow A(K)$ by $\phi(s,h) = (\tau_s, h)$ and $\psi(\theta, h) = \theta(h)$; $s \in S$, $h \in A(K)$, $\theta \in F$. ϕ is continuous on $S \times A(K)$ by separate continuity of τ. ψ is continuous by Kelley [18, Theorem 15, p. 232] and weak-weak equicontinuity of F. Hence $\psi \circ \phi$ is continuous on $S \times A(K)$. But $\psi \circ \phi(s,h) = \tau_s h$. This completes the proof.

For definitions of equicontinuous and quasi-equicontinuous affine actions $T: S \times K \longrightarrow K$, the reader is referred to BJM [2, p. 154].

THEOREM 4.3. *The following statements are equivalent:*

(1) $\{T_s : s \in S\}$ is quasi-equicontinuous in K^K.

(2) For each $h \in A(K)$, the map $x \longrightarrow T_x h$ is continuous from K into (CB(S), weak).

(3) $\{\tau_s h : s \in S\}$ is quasi-equicontinuous in $\mathbb{R}^K$ for each $h \in A(K)$.

(4) The family $\{\tau_s : s \in S\}$ is a weakly almost periodic semigroup of bounded linear operators in the Banach space $A(K)$ i.e. the weak closure of $\{\tau_s h : s \in S\}$ is weakly compact in $A(K)$ for each $h \in A(K)$.

PROOF. (1) and (2) are equivalent by BJM [2, Theorem IV. 1.9 (d) p. 155].

(1) iff (3): Assume that $\{T_s : s \in S\}$ is quasi-equicontinuous in K^K and let $F = p\,\mathrm{CL}\{T_s : s \in S\}$, the pointwise closure of $\{T_s : s \in S\}$ in the pointwise topology (= product topology) p in K^K. Then $F \subset C(K,K)$ by definition of quasi-equicontinuity. Now for each $x \in K$, $\{f(x) : f \in F\} \subset K$, which is compact. By Kelley [18, Theorem 1, p. 218], F is pointwise compact in K^K, hence in $C(K,K)$. Suppose now $h \in A(K)$ and $\tau_{s_\alpha} h \xrightarrow{p_1} k \in \mathbb{R}^K$ where p_1 is the pointwise topology of $\mathbb{R}^K$. There is some subnet s_β such that $T_{s_\beta} \xrightarrow{p} T \in C(K,K)$. Hence $\forall x \in K$, $k(x) = \lim_\beta \tau_{s_\beta} h(x) = \lim_\beta h\big(T_{s_\beta}(x)\big) = h\big(T(x)\big)$. So $k = h \circ T$ is also continuous and (1) implies (3). Conversely assume (3) that $\{\tau_s h : s \in S\}$ is quasi-equicontinuous in $\mathbb{R}^K$ for each $h \in A(K)$ and suppose $T_{s_\alpha} \xrightarrow{p} T \in K^K$. Then for each $h \in A(K)$, $\tau_{s_\alpha} h \xrightarrow{p_1} h \circ T \in \mathbb{R}^K$ and $h \circ T \in C(K)$ by quasi-equicontinuity of $\{\tau_s h : s \in S\}$ in $\mathbb{R}^K$. Hence T is continuous since $A(K)$ determines the topology of K. Consequently (3) implies (1).

(3) iff (4): Finally, assume that $\{\tau_s h : s \in S\}$ is quasi-equicontinuous in $\mathbb{R}^K$ for each $h \in A(K)$. Let $F = p_1\,\mathrm{CL}\{\tau_s h : s \in S\}$ be the pointwise closure of $\{\tau_s h : s \in S\}$ in the pointwise topology p_1 in $\mathbb{R}^K$. Clearly $F \subset A(K)$ (by quasi-equicontinuity) and $|k(x)| \le \|h\|_K$ $\forall x \in K$ and $k \in F$. Hence by Kelley [18, Theorem 1, p. 218], F is pointwise compact in $\mathbb{R}^K$, hence in $A(K)$. Since the weak topology in $A(K)$ coincides with p_1 in $A(K)$. F is weakly compact in $A(K)$. So is weak $\mathrm{CL}\{\tau_s h : s \in S\} \subset F$. Hence (3) implies (4). Obviously (4) implies (3). This completes the proof.

THEOREM 4.4. *The following statements are equivalent:*

(1) $\{T_s : s \in S\}$ *is equicontinuous on* K *(into* K*).*

(2) For each $h \in A(K)$*, the map* $x \longrightarrow T_x h$ *is continuous from* K *into* $(\mathrm{CB}(S), \|\cdot\|_S)$.

(3) For each $h \in A(K)$, $\{\tau_s h : s \in S\}$ *is equicontinuous on* K *(into* $\mathbb{R}$*).*

(4) The family $\{\tau_s : s \in S\}$ *is an almost periodic semigroup of bounded linear operators in the Banach space* $A(K)$*, i.e., the norm closure of* $\{\tau_s h : s \in S\}$ *is norm compact in* $A(K)$ *for each* $h \in A(K)$.

PROOF. The equivalence of (1) and (2) is found in BJM [2, proof of Theorem IV. 1.9 (e) p. 156] (see also Lau [20, p. 71]).

(3) is just another way of saying that the map $x \longrightarrow T_x h$ is $\|\cdot\|_S$-continuous. Hence (3) iff (2).

(3) iff (4): Assume (3) that $\{\tau_s h : s \in S\}$ is equicontinuous on K (into $\mathbb{R}$). Again let $F = p_1\,\mathrm{CL}\{\tau_s : s \in S\}$ where p_1 is the pointwise topology in $\mathbb{R}^K$. By Kelley [18, Theorem 14, p. 232], F is also equicontinuous. Hence $F \subset A(K)$. By Kelley [18, Theorem 15, p. 232], the topology p_1 coincides with the topology τ_c of uniform convergence on compacta of $C(K)$ on F. Since K is compact, the τ_c topology of $C(K)$ is the sup norm topology $\|\cdot\|_K$. Now F is pointwise compact in $\mathbb{R}^K$ by Kelley [18, Theorem 1, p. 218]. (Again, note that $|f(x)| \le \|h\|_K$ for all $x \in K, f \in F$). Therefore F is norm compact in $A(K)$. So is norm-$\mathrm{CL}\{\tau_s h : s \in S\} \subset F$. Hence (3) implies (4). Conversely, suppose (4) holds then $\{\tau_s h : s \in S\}$ is relatively compact in the norm topology of $A(K)$, hence of $C(K)$. By the Ascoli's Theorem (Kelley [18, Theorem 17, p. 233]), the family $\{\tau_s h : s \in S\}$ is equicontinuous on K (into $\mathbb{R}$). This completes the proof.

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It follows that Mitchell's fixed point theorems, Junghenn and Lau's fixed point theorem and Lau's fixed point theorem in BJM [2, Theorem IV. 1.9 (a), (b), (d) and (e), p. 154] can all be reformulated in the theorem below (parts (2) to (5) respectively).

THEOREM 4.5 (Reformulations). *(1)* WLUM(S) *has a left invariant mean iff for any weakly measurable anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *on the Banach space* $A(K)$ *such that* $\|\tau_s\| \leq 1$ *and* $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s h - h : s \in S, h \in A(K)\}$ *generate a proper ideal in* $C(K)$. *(Lau and Wong [22]).*

(2) WLUC(S) *has a left invariant mean iff for any separately (or jointly) continuous anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *where* $A(K)$ *has the weak topology and* $\|\tau_s\| \leq 1$, $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s(h) - h : s \in S, h \in A(K)\}$ *generate a proper ideal in* $C(K)$. *(Mitchell [24]).*

(3) LUC(S) *has a left invariant mean iff for any separately (or jointly) continuous anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *where* $A(K)$ *has the norm topology and* $\|\tau_s\| \leq 1$, $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s h - h : s \in S, h \in A(K)\}$ *generate a proper ideal in* $C(K)$. *(Mitchell [24]).*

(4) WAP(S) *has a left invariant mean iff for any weakly almost periodic anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *such that* $\|\tau_s\| \leq 1$ *and* $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s h - h : s \in S, h \in A(K)\}$ *generate a proper ideal in* $C(K)$. *(Junghenn [17] and Lau [21]).*

(5) AP(S) *has a left invariant mean iff for any almost periodic anti-action* $\tau: S \times A(K) \longrightarrow A(K)$ *such that* $\|\tau_s\| \leq 1$ *and* $\tau_s(1) = 1$ $\forall s \in S$, *the functions* $\{\tau_s h - h : s \in S, h \in A(K)\}$ *generate a proper ideal in* $C(K)$. *(Lau [20]).*

If S is a uniform semigroup, i.e. a semigroup with a uniform structure $\mathcal{U}$ such that the maps $s \longrightarrow sa$ and $s \longrightarrow as$ are uniformly continuous for each $a \in S$ (e.g. when S is a topological group with left (or right) uniform structure), an affine action, $T: S \times K \longrightarrow K$ is equi-uniformly continuous if for any neighbourhood N (of 0) in E, there is some $U \in \mathcal{U}$ such that $(s, t) \in U$ implies $T_s(x) - T_t(x) \in N$ $\forall x \in K$. In Wong [31, Theorem 2.1, p. 229], it is shown that $\mathcal{LUC}(S)$ has a left invariant mean iff any equi-uniformly continuous affine action $T: S \times K \longrightarrow K$ has a fixed point. Here $\mathcal{LUC}(S)$ is the space of all $f \in \mathrm{CB}(S)$ (S with the uniform topology) such that the map $s \longrightarrow \ell_s f$ is uniformly continuous with respect to the uniformity $\mathcal{U}$ on S and the uniformity on CB(S) induced by the sup norm. Now if $\tau: S \times A(K) \longrightarrow A(K)$ is the induced anti-action. Equi-uniform continuity of T is equivalent to the condition that τ is uniformly continuous when $A(K)$ has the norm topology, i.e. the map $s \longrightarrow \tau_s(h)$ is uniformly continuous from $(S, \mathcal{U})$ into $\big(A(K), \|\cdot\|_K\big)$ for each $h \in A(K)$. Hence Wong's fixed point theorem in [31, Theorem 2.1, p. 229] also has a similar reformulation.

From the above discussions, we see that for many fixed point theorems, we lose nothing by considering (anti) actions on the Banach space $A(K)$ instead of affine actions on compact convex sets K in locally convex spaces. Indeed, more is known about Banach space actions. Presumably, this makes them easier to handle and enables us to unify many diverse fixed point theorems as well as to make extensions and generalizations to actions on general Banach spaces in the next section.

5. **Extensions and generalizations to actions on general Banach spaces.** The preceding fixed point theorems and their reformulations can all be generalised to actions on general Banach spaces. We shall dicuss only the cases of LS(S) and separable anti-actions and respectively SLUM(S) and strongly measurable anti-actions.

Note that LS(S) is a norm closed subalgebra of BM(S), hence also a sublattice, (R. G. Douglas [6]) while SLUM(S) in general is only a norm closed linear subspace. However, they are typical of the general situation. In fact, the results of this section have their analogues for any norm closed left introverted, left invariant subalgebra and (respectively) linear subspace of BM(S) containing the constants.

Let B be any Banach space and $\tau: S \times B \longrightarrow B$ an anti-action of S on B such that $\|\tau_s\| \leq 1 \; \forall s \in S$. Define for $b \in B$ and $b^* \in B^*$ a function $\tau(b, b^*) \in m(S)$ by

$$\tau(b, b^*)(s) = b^* \tau_s(b), \quad s \in S.$$

(a representation function). Since $\|\tau(b, b^*)\|_S \leq \|b^*\| \, \|b\|$, $\tau(b, b^*)$ is bounded linear in b for fixed $b^* \in B^*$ (and vice versa). Also

$$\ell_s\big(\tau(b, b^*)\big) = \tau\big(\tau_s(b), b^*\big), \quad s \in S.$$

(This is because τ is an *anti-action*).

THEOREM 5.1. *The following statements are equivalent:*

(1) LS(S) *has a* LIM.

(2) For any separable anti-action $\tau: S \times B \longrightarrow B$ *of* S *on a Banach space such that (a)* $\|\tau_s\| \leq 1 \; \forall s \in S$ *and (b) there is some* $e \neq 0$ *in* B *such that* $\tau_s(e) = e \; \forall s \in S$, *the linear span* $\langle \tau_s b - b : s \in S, b \in B \rangle$ *is not norm dense in* B.

PROOF. (1) implies (2): Suppose LS(S) has a LIM and let $\tau: S \times B \longrightarrow B$ be such an anti-action on the Banach space B with $\{\tau_s b : s \in S\}$ separable for each $b \in B$. Then for each $b \in B$, $b^* \in B^*$ the representation function $\tau(b, b^*) \in$ LS(S). Now choose $b^* \in B^*$ such that $\|b^*\| = 1$ and $b^*(e) = 1$. (We may assume $\|e\| = 1$ and use the Hahn-Banach Extension Theorem). Let m be a LIM on LS(S) and define $n(b) = m\big(\tau(b, b^*)\big)$, $b \in B$. Then $n \in B^*$ and $|n(b)| \leq \|b\|$ and $n(e) = m\big(\tau(e, b^*)\big) = m(1) = 1$. In particular, $n \neq 0$. Now left invariance of m implies that $n(\tau_s b - b) = m\Big(\tau\big(\tau_s(b), b^*\big)\Big) - m\big(\tau(b, b^*)\big) = m\big(\ell_s \tau(b, b^*)\big) - m\big(\tau(b, b^*)\big) = 0 \; \forall s \in S$, $b \in B$. So $\langle \tau_s b - b : s \in S, b \in B \rangle^- \subset \ker(n) \underset{\neq}{\subset} B$.

(2) implies (1): Take $B =$ LS(S) and define $\tau: S \times B \longrightarrow B$ by $\tau_s(f) = \ell_s f$, $s \in S$, $f \in$ LS(S). Then τ is a separable anti-action such that $\|\tau_s\| \leq 1$ and $\tau_s(1) = 1 \; \forall s \in S$. By assumption, $H = \{\sum_{i=1}^n \ell_{s_i} f_i - f_i : s_i \in S, f_i \in \text{LS}(S)\}$ is not norm dense in LS(S). Hence by the Hahn-Banach Separation Theorem, there is some $m \in \text{LS}(S)^*$, $m \neq 0$ such that $m(\ell_s f) = m(f) \; \forall f \in$ LS(S), $s \in S$. Now LS(S) is a closed subalgebra of $m(S)$ containing the constants. Hence LS(S) must be a lattice (as a sublattice of the lattice $m(S)$, pointwise lattice operations) by R. G. Douglas [6]. Therefore LS(S) is an M space with unit (Kelley and Namioka [19, § 24]). Using the invariance arguments in Namioka

[25, Proposition 3.2], we conclude that both m^+ and m^- are left invariant $\big(m^+(\ell_s f) = m^+(f)\ \forall s \in S, f \in \mathrm{LS}(S)\big)$ and one of them say m^+ is non-zero. Then $m_1 = \frac{1}{m^+(1)}\, m$ is a LIM on LS(S).

REMARKS. (a) Namioka [25] studied anti-actions (or right actions) of semigroups (on special Banach spaces) $\tau: S \times M \longrightarrow M$ where M is an M-space with unit e such that $\tau_s \geq 0$ and $\tau_s(e) = e\ \forall s \in S$. ($\tau_s$ is necessarily continuous). However, he also required τ to be separately continuous because he was interested in left invariant means on the algebra (hence a lattice) LUC(S) of left uniformly continuous bounded functions on S and the anti-action $\tau_s(f) = \ell_s f$ of S on the M-space LUC(S). Our setting is different since we assume τ to be separable but no separate continuity and is more general since our semigroups act on Banach spaces. Theorem 5.1 has an analogue for other algebras $X \subset m(S)$ containing the constants and anti-actions $\tau: S \times B \longrightarrow B$ such that the representation functions $\tau(b, b^*) \in X\ \forall b \in B,\ b^* \in B^*$. For example, when X is the algebra LUC(S), WAP(S) and AP(S) and the corresponding anti-action is repsectively separately continuous, weakly almost periodic and almost periodic. We omit the details. When X is not an algebra, say when X = SLUM(S) (or WLUM(S)), Namioka's invariance arguments in [25, Proposition 3.2] cannot carry over, since such an X is not a lattice (R. G. Douglas [6]). However, we still have an analogue. See Theorem 5.3 below.

(b) The conclusion in (2) that the linear span $\langle \tau_s b - b : s \in S, b \in B\rangle$ is not norm dense in B is clearly equivalent to the existence of some non-zero $n \in B^*$ invariant under the adjoint action i.e. $\tau_s^* n = n\ \forall s \in S$ (Hahn-Banach separation Theorem). That is, the adjoint action has a non-zero fixed point.

(c) For similar but independent results on left invariant means on the full algebra $m(S)$ and anti-actions of S on Banach spaces, the reader is referred to Granirer [13] and Glicksberg [10].

For multiplicative left invariant means on LS(S), we have

THEOREM 5.2. *The following statements are equivalent:*

(1) LS(S) has a multiplicative left invariant mean.

(2) For any separable anti-action $\tau: S \times B \longrightarrow B$ of S on a Banach algebra B with identity e such that

(a) $\|\tau_s\| \leq 1, \quad \tau_s(ab) = \tau_s(a)\tau_s(b) \quad \forall a, b \in B,\ s \in S$ and

(b) $\tau_s(e) = e\ \forall s \in S$, $\{\tau_s b - b : s \in S, b \in B\}$ generates a proper ideal in B or equivalently, the ideal $\langle\langle \tau_s b - b : s \in S, b \in B\rangle\rangle$ generated is not norm dense in B.

PROOF. (1) implies (2): As in Theorem 5.1, we pick $b^* = \varphi$, a non-zero multiplicative linear functional on B (assumed to exist). Then $\|\varphi\| = 1$ and $\varphi(e) = 1$, $\tau(ab, \varphi) = \tau(a, \varphi)\tau(b, \varphi)\ \forall a, b \in B$. The functional n defined by $n(b) = m\big(\tau(b, \varphi)\big)$, $b \in B$ is then multiplicative (since both m and τ are) and $n \neq 0$ since $n(e) = 1$. So by left invariance of m, $\{\tau_s b - b : s \in S, b \in B\} \subset \mathrm{Ker}(n)$ which is a maximal (hence closed) ideal in B. So the ideal generated by $\{\tau_s b - b : s \in S, b \in B\}$ is proper. Now an ideal in B is proper iff it is not norm dense because every proper ideal is contained in a maximal one and each maximal one is closed. Hence (1) implies (2).

(2) implies (1): Take $B = \mathrm{LS}(S)$ and define $\tau: S \times B \longrightarrow B$ by $\tau_s f = \ell_s f$ as before. By assumption the ideal $K = \{\sum_{i=1}^{n} g_i(\ell_{s_i} f_i - f_i) : s_i \in S, f_i, g_i \in \mathrm{LS}(S)\}$ is not norm dense in $\mathrm{LS}(S)$. Hence $\mathrm{LS}(S)$ has a MLIM by Granirer [12, Theorem 2, p. 101]. (Note that for any subalgebra $X \subset m(S)$ with $1 \in X$, each evaluation functional is a non-zero multiplicative linear functional).

REMARKS. (a) With notations as in the proof of (1) implies (2), we have $1 = |n(e)| = \left|n\left(e - \sum_{i=1}^{n} a_i(\tau_{s_i} b_i - b_i)\right)\right| \leq \|e - \sum_{i=1}^{n} a_i(\tau_{s_i} b_i - b_i)\|$ $\forall s_i \in S$; $a_i, b_i \in B$. Hence $\inf\{\|e - k\| : k \in K\} \geq 1$. But then $\inf\{\|e-k\| : k \in K\} = 1 = \|e\|$. (See Granirer [13, p. 59] for related results). Here $K = \{\sum_{i=1}^{n} a_i(\tau_{s_i} b_i - b_i) : s_i \in S, a_i, b_i \in B\}$.

(b) The conclusion in (2) of Theorem 5.2 that $\{\tau_s b - b : s \in S, b \in B\}$ generates a proper ideal in the Banach algebra B is clearly implied by the existence of a non-zero multiplicative linear functional in B^* invariant under the adjoint action. If B is also commutative, the converse is also true because for such B, every maximal ideal is the kernel of a non-zero multiplicative linear functional.

For the space SLUM(S) (or WLUM(S)) which is not necessarily an algebra, the situation is different but more interesting:

THEOREM 5.3. *The following statements are equivalent*

(1) SLUM(S) *has a left invariant mean.*

(2) For any strongly measurable (i.e. weakly measurable and separable) anti-action $\tau: S \times B \longrightarrow B$ *on a Banach space* B *such that (a)* $\|\tau_s\| \leq 1$ $\forall s \in S$ *and (b) there is some* $e \neq 0$ *in* B *such that* $\tau_s(e) = e$ $\forall s \in S$, *there exists* $n \in B^*$ *with* $\|n\| = n(e) = 1$ *and* $\tau_s^* n = n$ $\forall s \in S$.

PROOF. (1) implies (2) by the same arguments as in Theorem 5.1. Conversely, define the anti-action $\tau: S \times B \longrightarrow B$, where $B = \mathrm{SLUM}(S)$ by $\tau_s f = \ell_s f$. τ is strongly measurable. Take $e = 1$. If $n \in \mathrm{SLUM}(S)^*$ satisfies $\|n\| = n(1)$ and $\tau_s^* n = n$ $\forall s \in S$, then n is necessarily a left invariant mean on SLUM(S).

REMARKS. (a) The existence of $\phi \in B^*$ with $\tau_s^* \phi = \phi$ $\forall s \in S$ and $\|\phi\| = \phi(e) = 1$ for some $e \in B$ implies that the linear span M of $\{\tau_s b - b : s \in S, b \in B\}$ is not norm dense in B (see Remark (b) after Theorem 5.1). The converse may not hold.

(b) If $B = A(K)$ with K compact convex then the existence of $\phi \in A(K)^*$ such that $\tau_s^* \phi = \phi$ $\forall s \in S$ and $\|\phi\| = \phi(1) = 1$ is equivalent to $\{\tau_s h - h : s \in S, h \in A(K)\}$ generating a proper ideal in $C(K)$ because each maximal ideal in $C(K)$ is fixed and each mean on $A(K)$ is evaluation at some point of K.

(c) If $B = \mathrm{SLUM}(S)$ and $\tau_s f = \ell_s f s \in S, f \in \mathrm{SLUM}(S)$, then the existence of $0 \neq \Psi \in \mathrm{SLUM}(S)^*$ with $\tau_s^* \Psi = \Psi$ $\forall s \in S$ may not imply that SLUM(S) has a left invariant mean because SLUM(S) is not necessarily an algebra and Namioka's invariance arguments cannot carry over, unlike the situation of the algebra LS(S) in Theorem 5.1.

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