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CONVERGENCE TO STABLE LAWS IN THE SPACE $\mathcal{D}$

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Abstract

We study the convergence of centered and normalized sums of independent and identically distributed random elements of the space $\mathcal{D}$ of càdlàg functions endowed with Skorokhod's J_1 topology, to stable distributions in $\mathcal{D}$. Our results are based on the concept of regular variation on metric spaces and on point process convergence. We provide some applications; in particular, to the empirical process of the renewal–reward process.

Keywords: Regular variation; stable process; functional convergence

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1. Introduction and main results

The main aim of this paper is to study the relation between regular variation in the space $\mathcal{D}_I$ and convergence to stable processes in $\mathcal{D}_I$. Let us first describe the framework of regular variation on metric spaces introduced by [8] and [9]. Let I be a nonempty closed subinterval of $\mathbb{R}$. We denote by $\mathcal{D}_I$ the set of real-valued càdlàg functions defined on I, endowed with the J_1 topology. Let $\mathcal{S}_I$ be the unit ball of $\mathcal{D}_I$ with respect to the uniform metric, i.e. the subset of $\mathcal{D}_I$ of functions x such that $\|x\|_I = \sup_{t\in I} |x(t)| = 1$. A random element X in $\mathcal{D}_I$ is said to be regularly varying if there exists $\alpha > 0$, an increasing sequence a_n, and a probability measure ν on $\mathcal{S}_I$, called the spectral measure, such that

$$\lim_{n\to\infty} n\mathbb{P}\left(\|X\|_I > a_n x,\ \frac{X}{\|X\|_I} \in A\right) = x^{-\alpha}\nu(A) \tag{1.1}$$

for any Borel set A of $\mathcal{S}_I$ such that $\nu(\partial A) = 0$ where ∂A is the topological boundary of A. Then $\|X\|_I$ has a regularly varying right-tail and the sequence a_n is regularly varying at infinity with index $1/\alpha$ and satisfies $\mathbb{P}(\|X\|_I > a_n) \sim 1/n$. [9, Theorem 10] states that (1.1) is equivalent to the regular variation of the finite dimensional marginal distributions of the process X together with a certain tightness criterion.

In the finite dimensional case, it is well known that if $\{X_n\}$ is an independent and identically distributed (i.i.d.) sequence of finite dimensional vectors whose common distribution is multivariate regularly varying, then the sum $\sum_{i=1}^n X_i$, suitably centered and normalized, converges to an α-stable distribution. In statistical applications, such sums appear to evaluate the asymptotic behavior of an empirical estimator around its mean. Therefore, we will consider centered sums and we shall always assume that $1 < \alpha < 2$. The case $\alpha \in (0, 1)$ is actually

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much simpler. Very general results in the case $\alpha \in (0, 1)$ can be found in [6]. In this case no centering is needed to ensure the absolute convergence of the series representation of the limiting process. In contrast, if $\alpha \in (1, 2)$, the centering raises additional difficulties. This can be seen in [13], where the point process of exceedances was first introduced for deriving the asymptotic behavior of the sum $S_n = \sum_{i=1}^n X_{i,n}$, for $X_{i,n} = Y_i \mathbf{1}_{[i/n,1]}$, with the Y_is i.i.d. regularly varying in a finite dimensional space. A thinning of the point process has to be introduced to deal with the centering. In this paper, we also rely on the point process of exceedances for more general random elements $X_{i,n}$ valued in $\mathcal{D}_I$. Our results include the case treated in [13, Proposition 3.4]; see Section 3.1. However, they do not require the centered sum $S_n - \mathbb{E}[S_n]$ to be a martingale and the limit process that we obtain is not a Lévy process in general, see the other two examples treated in Section 3. Hence, martingale-type arguments as in [10] cannot be used. We now present our main result.

Theorem 1.1. *Let $\{X_i\}$ be a sequence of i.i.d. random elements of $\mathcal{D}_I$ with the same distribution as X and assume that (1.1) holds with $1 < \alpha < 2$. For $x \in \mathcal{D}_I$, let the sets of discontinuity points of x be denoted by* $\mathrm{Disc}(x)$*. Assume that the following conditions hold.*

(i) *For all $t \in I$, $\nu(\{x \in \mathcal{S}_I, t \in \mathrm{Disc}(x)\}) = 0$.*

(ii) *For all $\eta > 0$, we have*

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\left\| \sum_{i=1}^n (X_i \mathbf{1}_{\{\|X_i\|_I \le a_n \varepsilon\}} - \mathbb{E}[X \mathbf{1}_{\{\|X\|_I \le a_n\varepsilon\}}]) \right\|_I > a_n \eta \right) = 0. \tag{1.2}$$

Then $a_n^{-1} \sum_{i=1}^n \{X_i - \mathbb{E}[X]\}$ converges weakly in $(\mathcal{D}_I, J_1)$ to an α-stable process $\aleph$, that admits the integral representation

$$\aleph(t) = c_\alpha \int_{\mathcal{S}_I} w(t)\,\mathrm{d}M(w), \tag{1.3}$$

where M is an α-stable independently scattered random measure on $\mathcal{S}_I$ with control measure ν and skewness intensity $\beta \equiv 1$ (totally skewed to the right) and $c_\alpha^\alpha = \Gamma(1-\alpha)\cos(\pi\alpha/2)$.

Remark 1.1. If x and y are two functions in $\mathcal{D}_I$, then, for the J_1 topology, addition may not be continuous at (x, y) if $\mathrm{Disc}(x) \cap \mathrm{Disc}(y) \neq \varnothing$. Condition (i) of Theorem 1.1 means that if W is a random element of $\mathcal{S}_I$ with distribution ν then, for any $t \in I$, $\mathbb{P}(t \in \mathrm{Disc}(W)) = 0$, i.e. W has no fixed jumps; see [11, p. 286]. Condition (i) of Theorem 1.1 also implies that $\nu \otimes \nu$-almost all $(x, y) \in \mathcal{S}_I \times \mathcal{S}_I$, i.e. x and y have no common jumps. Equivalently, if W and W' are i.i.d. random elements of $\mathcal{S}_I$ with distribution ν, then, almost surely, W and W' have no common jump. This implies that if $W_1, \ldots, W_n$ are i.i.d. with distribution ν, then, almost surely, addition is continuous at the point $(W_1, \ldots, W_n)$ in $(\mathcal{D}_I, J_1)^n$; cf. [16, Theorem 4.1].

It will be useful to slightly extend Theorem 1.1 by considering triangular arrays of independent multivariate càdlàg processes. To deal with ℓ-dimensional càdlàg functions, for some positive integer ℓ, we endow $\mathcal{D}_I^\ell$ with J_1^ℓ, the product J_1 topology, sometimes referred to as the weak product topology (see [17]). We then let $\mathcal{S}_{I,\ell}$ be the subset of $\mathcal{D}_I^\ell$ of functions $x = (x_1, \ldots, x_\ell)$ such that

$$\|x\|_{I,\ell} = \max_{i=1,\ldots,\ell} \sup_{t\in I} |x_i(t)| = 1.$$

Note that, in the multivariate setting, we have $\mathrm{Disc}(x) = \cup_{i=1,\ldots,\ell}\mathrm{Disc}(x_i)$. We will prove the following slightly more general result.

Theorem 1.2. *Let (m_n) be a nondecreasing sequence of integers tending to infinity. Let $\{X_{i,n},\ 1 \le i \le m_n\}$ be an array of independent random elements of $\mathcal{D}_I^\ell$. Assume that there exists $\alpha \in (1, 2)$ and a probability measure ν on the Borel sets of $(\mathcal{S}_{I,\ell}, J_1^\ell)$ such that ν satisfies condition (i) of Theorem 1.1 and, for all $x > 0$ and Borel sets A such that $\nu(\partial A) = 0$,*

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{P}\bigg(\|X_{i,n}\|_{I,\ell} > x,\ \frac{X_{i,n}}{\|X_{i,n}\|_{I,\ell}} \in A \bigg) = x^{-\alpha}\nu(A), \tag{1.4}$$

$$\lim_{n\to\infty} \max_{i=1,\dots,m_n} \mathbb{P}(\|X_{i,n}\|_{I,\ell} > x) = 0, \tag{1.5}$$

$$\lim_{x\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{E}[\|X_{i,n}\|_{I,\ell} \mathbf{1}_{\|X_{i,n}\|_{I,\ell} > x}] = 0. \tag{1.6}$$

Moreover, suppose that, for all $\eta > 0$, we have

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Bigg(\Bigg\| \sum_{i=1}^{m_n} (X_{i,n}\mathbf{1}_{\{\|X_{i,n}\|_{I,\ell} \le \varepsilon\}} - \mathbb{E}[X_{i,n}\mathbf{1}_{\{\|X_{i,n}\|_{I,\ell} \le \varepsilon\}}]) \Bigg\|_{I,\ell} > \eta \Bigg) = 0. \tag{1.7}$$

Then $\sum_{i=1}^{m_n}\{X_{i,n} - \mathbb{E}[X_{i,n}]\}$ converges weakly in $(\mathcal{D}_I^\ell, J_1^\ell)$ to an ℓ-dimensional α-stable process $\aleph$, that admits the integral representation given by (1.3) with $\mathcal{S}_I$ replaced by $\mathcal{S}_{I,\ell}$.

Remark 1.2. If $m_n = n$ and $\|X_{i,n}\|_{I,\ell} = Y_i/a_n$ where $\{Y_i,\ i \ge 1\}$ is an i.i.d. sequence and (1.4) holds, then the common distribution of the random variables Y_i has a regularly varying right-tail with index α. It follows that (1.5) trivially holds and (1.6) holds by Karamata's theorem. Note also that, obviously, if $X_{i,n} = X_i/a_n$ with $\{X_i,\ i \ge 1\}$ an i.i.d. sequence valued in $\mathcal{D}_I^\ell$, then (1.4) is equivalent to the regular variation of the common distribution of the X_is.

From Remark 1.2 we see that Theorem 1.1 is a special case of Theorem 1.2, which we will prove in Section 2.6. We conclude this section with some comments about the α-stable limit appearing in Theorem 1.1 (or Theorem 1.2). Its finite dimensional distributions are defined by the integral representation (1.3) and only depend on the probability measure ν. If $X/\|X\|_I$ is distributed according to ν and is independent of $\|X\|_I$, as in Section 3.2, then (1.1) holds straightforwardly and, provided that the negligibility condition (ii) of Theorem 1.1 holds, a byproduct of Theorem 1.1 is that the integral representation (1.3) admits a version in $\mathcal{D}_I$. The existence of càdlàg versions of α-stable processes is also a byproduct of the convergence in $\mathcal{D}_I$ of series representations as recently investigated in [5] and [1]. We will come back to this question in Section 3.2. For now, let us state an interesting consequence of the It–Nisio theorem proved in [1].

Lemma 1.1. *Let $\alpha \in (1, 2)$, ν be a probability measure on $\mathcal{S}_I$, and $\aleph$ be a process in $\mathcal{D}_I$ which admits the integral representation (1.3). Let $\{\Gamma_i,\ i \ge 1\}$ be the points of a unit rate homogeneous Poisson point process on $[0, \infty)$ and $\{W, W_i,\ i \ge 1\}$ be a sequence of i.i.d. random elements of $\mathcal{S}_I$ with common distribution ν, independent of $\{\Gamma_i\}$. Then $\mathbb{E}[W]$, defined by $\mathbb{E}[W](t) = \mathbb{E}[W(t)]$ for all $t \in I$, is in $\mathcal{D}_I$ and the series $\sum_{i=1}^\infty \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}]\mathbb{E}[W]\}$ converges uniformly almost surely in $\mathcal{D}_I$ to a limit having the same finite dimensional distribution as $\aleph$.*

Proof. The fact that $\mathbb{E}[W]$ is in $\mathcal{D}_I$ follows from dominated convergence and $\|w\|_I = 1$ almost surely (a.s.). The finite dimensional distributions of

$$S_n = \sum_{i=1}^n \{\Gamma_i^{-1/\alpha} W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}]\mathbb{E}[W_i]\}$$

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converge to those of $\aleph$ as a consequence of [14, Theorem 3.9]. Hence, to obtain the result, it suffices to show that the series $\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}]\mathbb{E}[W_i]\}$ converges uniformly a.s. Note that the series $\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\}$ converges a.s.; thus, writing

$$\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}]\mathbb{E}[W]\}$$
$$= \sum_{i=1}^{\infty}\Gamma_i^{-1/\alpha}\{W_i - \mathbb{E}[W]\} + \mathbb{E}[W]\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha} - \mathbb{E}[\Gamma_i^{-1/\alpha}]\},$$

we can assume without loss of generality that $\mathbb{E}[W] \equiv 0$. Define $T_n = \sum_{i=1}^{n} i^{-1/\alpha}W_i$. By Kolmogorov's three series theorem (see [11, Theorem 4.18]), since $\sum_{i=1}^{\infty} i^{-2/\alpha} < \infty$ and $\operatorname{var}(W_i(t)) \le 1$, for all $t \in I$, $T_n(t)$ converges a.s. to a limit, say $T_\infty(t)$.

Arguing as in [5], we apply [14, Lemma 1.5.1] to obtain that the series $\sum_{i=1}^{\infty}|\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|$ is summable. This implies that the series $\Delta = \sum_{i=1}^{\infty}(\Gamma_i^{-1/\alpha} - i^{-1/\alpha})W_i$ is uniformly convergent. Hence, $S_n - T_n$ converges uniformly a.s. to Δ and $\Delta \in \mathcal{D}_I$. Thus, for all $t \in I$, $S_n(t)$ converges a.s. to $\Delta(t) + T_\infty(t)$. Since the finite distributions of S_n converge weakly to those of $\aleph$ which belongs to $\mathcal{D}_I$ by assumption, we conclude that $\Delta + T_\infty$ has a version in $\mathcal{D}_I$. Hence, T_∞ also has a version in $\mathcal{D}_I$.

We can now apply [1, Theorem 2.1(ii)] and obtain that, suitably centered, T_n converges uniformly a.s. Moreover, for each t, we have $\mathbb{E}[T_n(t)] = \mathbb{E}[T(t)] = 0$ and $\mathbb{E}[|T(t)|^2] = \mathbb{E}[|W(t)|^2]\sum_{i=1}^{\infty} i^{-2/\alpha} \le \sum_{i=1}^{\infty} i^{-2/\alpha}$. Hence, $\{T(t),\ t \in I\}$ is uniformly integrable. Then [1, Theorem 2.1(iii)] shows that T_n converges uniformly a.s. without centering. Thus, S_n also converges almost surely uniformly.

Corollary 1.1. *The process $\aleph$ defined in Theorem 1.1 also admits the series representation*

$$\aleph(t) = \sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}W_i - \mathbb{E}[\Gamma_i^{-1/\alpha}]\mathbb{E}[W_1]\}, \tag{1.8}$$

where $\{\Gamma_i, W_i, i \ge 1\}$ are as in Lemma 1.1. This series is a.s. uniformly convergent.

It seems natural to conjecture that the limit process in Theorem 1.1 or the sum of the series in Lemma 1.1 is regularly varying with spectral measure ν (the distribution of the process W). However, such a result is not known to hold generally. It was proved in [4, Section 4] under the assumption that W has a.s. continuous paths. Under an additional tightness condition, we obtain the following result.

Lemma 1.2. *Let $\alpha \in (1, 2)$, ν be a probability measure on $\mathcal{S}_I$, and W be a random element of $\mathcal{S}_I$ with distribution ν. Assume that $\mathbb{E}[W]$ is continuous on I and that there exist $p \in (\alpha, 2]$, $\gamma > \frac{1}{2}$, and a continuous increasing function F such that, for all $s < t < u$,*

$$\mathbb{E}[|\bar{W}(s,t)|^p] \le \{F(t) - F(s)\}^\gamma, \tag{1.9}$$
$$\mathbb{E}[|\bar{W}(s,t)\bar{W}(t,u)|^p] \le \{F(u) - F(s)\}^{2\gamma}, \tag{1.10}$$

where $\bar{W}(s,t) = W(t) - W(s) - \mathbb{E}[W(t) - W(s)]$. Then the stable process $\aleph$ defined by the integral representation (1.3) admits a version in $\mathcal{D}_I$ which is regularly varying in the sense of (1.1), with spectral measure ν.

Remark 1.3. Our assumptions on W are similar to those of [1, Theorem 4.3] and [5, Theorem 1], with a few minor differences. For instance, (1.9) and (1.10) are expressed on a noncentered W in these references. Here, we only require $\mathbb{E}[W]$ to be continuous, which, under (1.9), is equivalent to condition (i) of Theorem 1.1. Indeed, take a random element W in $\mathcal{S}_I$. Then, by dominated convergence, $\mathbb{E}[W]$ is in $\mathcal{D}_I$. Condition (1.9) implies that $W-\mathbb{E}[W]$ has no pure jump. Thus, under (1.9), the process W has no pure jump if and only if $\mathbb{E}[W]$ is continuous on I.

Proof of Lemma 1.2. Proposition 3.1, below, implies that the stable process $\aleph$ defined by (1.3) admits a version in $\mathcal{D}_I$. Let us prove that this version is regularly varying in the sense of (1.1), with spectral measure ν. By Corollary 1.1, $\aleph$ can be represented as the a.s. uniformly convergent series

$$\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}W_i-\mathbb{E}[\Gamma_i^{-1/\alpha}W_i]\}=\sum_{i=1}^{\infty}\Gamma_i^{-1/\alpha}\bar{W}_i+\mathbb{E}[W]\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}-\mathbb{E}[\Gamma_i^{-1/\alpha}]\},$$

where $\bar{W}_i=W_i-\mathbb{E}[W]$. For $k\geq 1$, define $\Sigma_k=\sum_{i=k}^{\infty}\Gamma_i^{-1/\alpha}\bar{W}_i$. We proceed as in the proof of Corollary 2.2, below. Conditioning on the Poisson process and applying Burkholder's inequality, we have, for a constant C_p depending only on p, for all $t\in I$, since $\|W\|_I=1$,

$$\mathbb{E}[|\Sigma_4(t)|^p]\leq C_p\sum_{i=4}^{\infty}\mathbb{E}[\Gamma_i^{-p/\alpha}]<\infty. \tag{1.11}$$

Similarly, using (1.9) and (1.10), we obtain that there exist constants C and C' depending only on α and p such that, for $s<t<u\in I$,

$$\begin{aligned}
&\mathbb{E}\big[\big|\Sigma_4(t)-\Sigma_4(s)\big|^p\big|\Sigma_4(u)-\Sigma_4(t)\big|^p\big]\\
&\quad\leq C\mathbb{E}\bigg[\bigg(\sum_{i=4}^{\infty}\Gamma_i^{-2/\alpha}\bigg)^p\bigg]\mathbb{E}\big[\big|\bar{W}(s,t)\bar{W}(t,u)\big|^p\big]\\
&\qquad+C\mathbb{E}\bigg[\bigg(\sum_{i=4}^{\infty}\Gamma_i^{-p/\alpha}\bigg)^2\bigg]\mathbb{E}\big[\big|\bar{W}(s,t)\big|^p\big]\mathbb{E}\big[\big|\bar{W}(t,u)\big|^p\big]\\
&\quad\leq C'\{F(u)-F(s)\}^{2\gamma}.
\end{aligned}$$

This bound and (1.11) imply that $\mathbb{E}[\|\Sigma_4\|_I^p]<\infty$; see [2, Chapter 15]. Moreover, since $p/\alpha<2$ we have, for $i=2,3$, $\mathbb{E}[\Gamma_i^{-p/\alpha}]<\infty$. Using $\|W_i\|_I\leq 1$ for $i=2,3$, we finally obtain $\mathbb{E}[\|\Sigma_2\|_I^p]<\infty$ and Z can be represented as

$$\Gamma_1^{-1/\alpha}\bar{W}_1+\Sigma_2+\mathbb{E}[W]\sum_{i=1}^{\infty}\{\Gamma_i^{-1/\alpha}-\mathbb{E}[\Gamma_i^{-1/\alpha}]\}=\Gamma_1^{-1/\alpha}W_1+T,$$

where $T=\Sigma_2+\mathbb{E}[W]\sum_{i=2}^{\infty}\{\Gamma_i^{-1/\alpha}-\mathbb{E}[\Gamma_i^{-1/\alpha}]\}-\mathbb{E}[\Gamma_1^{-1/\alpha}W_1]$ satisfies $\mathbb{E}[\|T\|_I^p]<\infty$. Observe that $p>\alpha$. Since $\Gamma_1^{-1/\alpha}$ has a Frechet distribution with index α, it holds that $\Gamma_1^{-1/\alpha}W_1$ is regularly varying with spectral measure ν, which concludes the proof.

In the next section, we prove some intermediate results needed to prove Theorems 1.1 and 1.2. In particular, we give a condition for the convergence in $\mathcal{D}_I$ of the sequence of expectations.

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This is not obvious, since the expectation functional is not continuous in $\mathcal{D}_I$. We provide a criterion for the negligibility condition (1.7) and, for the sake of completeness, we recall the main tools of random measure theory we need. In Section 3, we give some applications of Theorem 1.1.

2. Some results on convergence in $\mathcal{D}_I$ and proof of the main results

2.1. Convergence of the expectation in $\mathcal{D}_I$

It may happen that a uniformly bounded sequence (X_n) converges weakly to X in $(\mathcal{D}_I, J_1)$ but $\mathbb{E}[X_n]$ does not converge to $\mathbb{E}[X]$ in $(\mathcal{D}_I, J_1)$. Therefore, to deal with the centering, we will need the following lemma.

Lemma 2.1. *Suppose that X_n converges weakly to X in $(\mathcal{D}_I, J_1)$. Suppose moreover that there exists $m > 0$ such that $\sup_n \|X_n\|_I \le m$ a.s. and X has no fixed jump, i.e. for all $t \in I$, $\mathbb{P}(t \in \mathrm{Disc}(X)) = 0$. Then the maps $\mathbb{E}[X_n] : t \to \mathbb{E}[X_n(t)]$ and $\mathbb{E}[X] : t \to \mathbb{E}[X(t)]$ are in $\mathcal{D}_I$, $\mathbb{E}[X]$ is continuous on I, and $\mathbb{E}[X_n]$ converges to $\mathbb{E}[X]$ in $(\mathcal{D}_I, J_1)$.*

Proof. Since we have assumed that $\sup_{n\ge 0} \|X_n\|_I \le m$, almost surely, it also holds that $\|X\|_I \le m$ almost surely. The fact that $\mathbb{E}[X_n]$ and $\mathbb{E}[X]$ are in $\mathcal{D}_I$ follows by bounded convergence. Because X has no fixed jump, we also find that $\mathbb{E}[X]$ is continuous on I.

By Skorokhod's representation theorem, we can assume that X_n converges to X almost surely in $\mathcal{D}_I$. By the definition of Skorokhod's metric (see e.g. [2]), there exists a sequence (λ_n) of random continuous strictly increasing functions mapping I onto itself such that $\|\lambda_n - \mathrm{id}_I\|_I$ and $\|X_n - X \circ \lambda_n\|_I$ converge almost surely to 0. By bounded convergence, it also holds that $\lim_{n\to\infty} \mathbb{E}[\|X_n - X \circ \lambda_n\|_I] = 0$. Now write

$$\|\mathbb{E}[X_n] - \mathbb{E}[X]\|_I \le \|\mathbb{E}[X_n - X \circ \lambda_n]\|_I + \|\mathbb{E}[X \circ \lambda_n - X]\|_I .$$

The first term on the right-hand side converges to zero so we only consider the second one. Denote the oscillation of a function x on a set A by

$$\mathrm{osc}(x; A) = \sup_{t\in A} x(t) - \inf_{t\in A} x(t). \tag{2.1}$$

Let the open ball centered at t with radius r be denoted by $B(t, r)$. Since X is continuous at t with probability one, it holds that $\lim_{r\to 0} \mathrm{osc}(X; B(r, t)) = 0$ a.s. Since $\|X\|_I \le m$ a.s. by dominated convergence, for each $t \in I$, we have $\lim_{r\to 0} \mathbb{E}[\mathrm{osc}(X; B(t, r))] = 0$. Let $\eta > 0$ be arbitrary. For each $t \in I$, there exists $r(t, \eta) \in (0, \eta) > 0$ such that $\mathbb{E}[\mathrm{osc}(X; B(t, r(t, \eta)))] \le \eta$. Since I is compact, it admits a finite covering by balls $B(t_i, \varepsilon_i)$, $i = 1, \dots, p$, with $\varepsilon_i = r(t_i, \eta)/2$. Fix some $\zeta \in (0, \min_{1\le i\le p} \varepsilon_i)$. Then, for $s \in B(t_i, \varepsilon_i)$ and by our choice of ζ, we have

$$\begin{aligned}
|\mathbb{E}[X \circ \lambda_n(s)] - \mathbb{E}[X(s)]| &\le \mathbb{E}[|X \circ \lambda_n(s) - X(s)|\mathbf{1}_{\{\|\lambda_n - \mathrm{id}_I\|_I \le \zeta\}}] + 2m\mathbb{P}(\|\lambda_n - \mathrm{id}_I\|_I > \zeta) \\
&\le \mathbb{E}[\mathrm{osc}(X; B(t_i, r(t_i, \eta)))] + 2m\mathbb{P}(\|\lambda_n - \mathrm{id}_I\|_I > \zeta) \\
&\le \eta + 2m\mathbb{P}(\|\lambda_n - \mathrm{id}_I\|_I > \zeta).
\end{aligned}$$

The last term does not depend on s; thus, $\|\mathbb{E}[X \circ \lambda_n] - \mathbb{E}[X]\|_I \le \eta + 2m\mathbb{P}(\|\lambda_n - \mathrm{id}_I\|_I > \zeta)$. Since $\|\lambda_n - \mathrm{id}_I\|_I$ converges a.s. to 0, we obtain $\limsup_{n\to\infty} \|\mathbb{E}[X \circ \lambda_n] - \mathbb{E}[X]\|_I \le \eta$. Since η is arbitrary, this concludes the proof.

Remark 2.1. Under the stronger assumption that X_n converges uniformly to X, Lemma 2.1 trivially holds since $\|\mathbb{E}[X_n - X]\|_I \leq \mathbb{E}[\|X_n - X\|_I]$, and the result then follows from dominated convergence. If X is a.s. continuous then the uniform convergence follows from the convergence in the J_1 topology. If X_n is a sum of independent variables converging weakly in the J_1 topology to X with no pure jumps then the convergence in the J_1 topology again implies the uniform convergence; see [1, Corollary 2.2]. However, under our assumptions, the uniform convergence does not always hold, as illustrated in the following example.

Example 2.1. For $I = [0, 1]$, set $X_n = \mathbf{1}_{[U(n-1)/n,1]}$ and $X = \mathbf{1}_{[U,1]}$ with U uniform on $[0, 1]$. Then the assumptions of Lemma 2.1 hold. However, X_n converges a.s. to X in the J_1 topology but not uniformly.

Let us now provide counterexamples in the case where the assumption of Lemma 2.1 on the limit X is not satisfied.

Example 2.2. Let $I = [0, 1]$, $X = \mathbf{1}_{[1/2,1]}$, and $X_n = \mathbf{1}_{[U_n,1]}$ where U_n is drawn uniformly on $[\frac{1}{2} - 1/n, \frac{1}{2}]$. Then $X_n \to X$ a.s. in $\mathcal{D}_I$ but $\mathbb{E}[X_n]$ does not converge to $\mathbb{E}[X] = X$ in the J_1 topology, though it does converge in the M_1 topology.

Example 2.3. Set $X_n = \mathbf{1}_{[u_n,1]}$ for all n with probability $\frac{1}{2}$ and $X_n = -\mathbf{1}_{[v_n,1]}$ for all n with probability $\frac{1}{2}$, where $u_n = \frac{1}{2} - 1/n$ and $v_n = \frac{1}{2} - 1/2n$. In the first case $X_n \to \mathbf{1}_{[1/2,1]}$ in $\mathcal{D}_I$ with $I = [0, 1]$ and in the second case $X_n \to -\mathbf{1}_{[1/2,1]}$ in $\mathcal{D}_I$. Hence, $X_n \to X$ a.s. in $\mathcal{D}_I$ for X well chosen. On the other hand, we have $\mathbb{E}[X_n] = \mathbf{1}_{[u_n,v_n)}$ which converges uniformly to the null function on $[0, u] \cup [\frac{1}{2}, 1]$ for all $u \in (0, \frac{1}{2})$, but whose supremum on $I = [0, 1]$ does not converge to 0; hence, $\mathbb{E}[X_n]$ cannot converge in $\mathcal{D}_I$ endowed with J_1, nor with the other usual distances on $\mathcal{D}_I$ such as the M_1 distance.

The assumption that $\sup_n \|X_n\|_I \leq m$ a.s. can be replaced by a uniform integrability assumption. Using a truncation argument, the following corollary is easily proved. The extension of the univariate case to the multivariate one is obvious in the product topology so we state the result in a multivariate setting.

Corollary 2.1. *Suppose that X_n converges weakly to X in $(\mathcal{D}_I^\ell, J_1^\ell)$. Suppose moreover that X has no fixed jump and $\{\|X_n\|_{I,\ell}\,,\ n \geq 1\}$ is uniformly integrable, that is*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \mathbb{E}[\|X_n\|_{I,\ell}\, \mathbf{1}_{\{\|X_n\|_{I,\ell} > M\}}] = 0.$$

Then the maps $\mathbb{E}[X_n] : t \to \mathbb{E}[X_n(t)]$ and $\mathbb{E}[X] : t \to \mathbb{E}[X(t)]$ are in $\mathcal{D}_I^\ell$, $\mathbb{E}[X]$ is continuous on I, and $\mathbb{E}[X_n]$ converges to $\mathbb{E}[X]$ in $(\mathcal{D}_I^\ell, J_1^\ell)$.

2.2. Weak convergence of random measures

Let $\mathcal{X}$ be a complete separable metric space (CSMS). Let $\mathcal{M}(\mathcal{X})$ denote the set of boundedly finite nonnegative Borel measures μ on $\mathcal{X}$, i.e. such that $\mu(A) < \infty$ for all bounded Borel sets A. A sequence (μ_n) of elements of $\mathcal{M}(\mathcal{X})$ is said to converge weakly to μ, denoted by $\mu_n \to_{w^\#} \mu$, if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for all continuous functions f with bounded support in $\mathcal{X}$. The weak convergence in $\mathcal{M}(\mathcal{X})$ is metrizable in such a way that $\mathcal{M}(\mathcal{X})$ is a CSMS. We denote by $\mathcal{B}(\mathcal{M}(\mathcal{X}))$ the corresponding Borel sigma-field. Let (M_n) be a sequence of random elements of $(\mathcal{M}(\mathcal{X}), \mathcal{B}(\mathcal{M}(\mathcal{X})))$. Then, by [3, Proposition 11.1.VIII], M_n converges weakly to M, denoted by $M_n \Rightarrow M$, if and only if $\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{-M_n(f)}] = \mathbb{E}[\mathrm{e}^{-M(f)}]$ for all bounded continuous functions f with bounded support. A point measure in $\mathcal{M}(\mathcal{X})$ is a

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measure which takes integer values on the bounded Borel sets of $\mathcal{X}$. A point process in $\mathcal{M}(\mathcal{X})$ is a random point measure in $\mathcal{M}(\mathcal{X})$. In particular, a Poisson point process has an intensity measure in $\mathcal{M}(\mathcal{X})$. In the following, we shall denote by $\mathcal{N}(\mathcal{X})$ the set of point measures in $\mathcal{M}(\mathcal{X})$ and by $\mathcal{M}_f(\mathcal{X})$ the set of finite measures in $\mathcal{M}(\mathcal{X})$.

Consider now the space $\mathcal{D}_I$ endowed with the J_1 topology. Let δ be a bounded metric generating the J_1 topology on $\mathcal{D}_I$ and which makes it a CSMS; see [2, Section 14]. From now on we denote by $\mathcal{X}_I = (\mathcal{D}_I, \delta \wedge 1)$ this CSMS, all the Borel sets of which are bounded, since we chose δ bounded. We further let $\mathcal{N}^*(\mathcal{X}_I)$ be the subset of point measures m such that, for all distinct x and y in $\mathcal{D}_I$ such that $\mathrm{Disc}(x) \cap \mathrm{Disc}(y) \neq \varnothing$, $m(\{x, y\}) < 2$. In other words, m is simple (the measure of all singletons is at most 1) and the elements of the (finite) support of m have disjoint sets of discontinuity points. To deal with multivariate functions, we endow $\mathcal{X}_I^\ell$ with the metric

$$\delta_\ell((x_1, \ldots, x_\ell), (x'_1, \ldots, x'_\ell)) = \sum_{i=1}^{\ell} \delta(x_i, x'_i),$$

so that the corresponding topology is the product topology denoted by J_1^ℓ. Finally, we define the space $\mathcal{Y}_{I,\ell} = (0, \infty] \times \mathcal{S}_{I,\ell}$, which is a CSMS when endowed with the metric $d((r, x), (r', x')) = |1/r - 1/r'| + \delta_\ell(x, x')$.

Proposition 2.1. *Let $\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell})$ and $\varepsilon > 0$ be such that*

(i) *for all $t \in I$, $\mu(\{(y, x) \in \mathcal{Y}_{I,\ell}, t \in \mathrm{Disc}(x)\}) = 0$,*

(ii) $\mu(\{\varepsilon, \infty\} \times \mathcal{S}_{I,\ell}) = 0$.

Let M be a Poisson point process on $\mathcal{Y}_{I,\ell}$ with control measure μ. Let $\{M_n\}$ be a sequence of point processes in $\mathcal{M}(\mathcal{Y}_{I,\ell})$ which converges weakly to M in $\mathcal{M}(\mathcal{Y}_{I,\ell})$. Then the weak convergence

$$\int_{(\varepsilon,\infty)} \int_{\mathcal{S}_{I,\ell}} yw M_n(\mathrm{d}y, \mathrm{d}w) \Rightarrow \int_{(\varepsilon,\infty)} \int_{\mathcal{S}_{I,\ell}} yw M(\mathrm{d}y, \mathrm{d}w) \tag{2.2}$$

holds in $(\mathcal{D}_I^\ell, J_1^\ell)$ and the limit has no pure jump.

Proof. Let us define the mapping $\psi : \mathcal{Y}_{I,\ell} \to \mathcal{X}_I^\ell$ by

$$\psi(y, w) = \begin{cases} yw & \text{if } y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Also, let $\Psi : \mathcal{M}(\mathcal{Y}_{I,\ell}) \to \mathcal{M}(\mathcal{X}_I^\ell)$ be the mapping defined by

$$[\Psi(m)](A) = m(\psi^{-1}(A) \cap ((\varepsilon, \infty) \times \mathcal{S}_{I,\ell})),$$

for all Borel subsets A in $\mathcal{M}(\mathcal{X}_I^\ell)$. Since $\partial(A \cap ((\varepsilon, \infty) \times \mathcal{S}_{I,\ell})) \subset \partial A \cap (\{\varepsilon, \infty\} \times \mathcal{S}_{I,\ell}))$, we have that $m \mapsto m(\cdot \cap ((\varepsilon, \infty) \times \mathcal{S}_{I,\ell}))$ is continuous from $\mathcal{M}(\mathcal{Y}_{I,\ell})$ to $\mathcal{M}(\mathcal{Y}_{I,\ell})$ on the set $\mathcal{A}_1 = \{\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell}) : \mu(\{\varepsilon, \infty\} \times \mathcal{S}_{I,\ell}) = 0\}$. Using the continuity of ψ on $(0, \infty) \times \mathcal{S}_{I,\ell}$, it is easy to show that $m \mapsto m \circ \psi^{-1}$ is continuous on $\mathcal{A}_2 = \{\mu \in \mathcal{M}(\mathcal{Y}_{I,\ell}) :$ there exists $M > 0, \mu([M, \infty] \times \mathcal{S}_{I,\ell}) = 0\}$. Hence, Ψ is continuous on $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. We note now that the map

$$m \mapsto \int_{(\varepsilon,\infty)} \int_{\mathcal{S}_{I,\ell}} ywm(\mathrm{d}y, \mathrm{d}w) = \int w\Psi(m)(\mathrm{d}w)$$

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is continuous as a mapping from $\Psi^{-1}(\mathcal{N}(\mathcal{X}_I))$ endowed with the $\to_{w^\#}$ topology to $(\mathcal{D}_I^\ell, J_1^\ell)$ on the set $\Psi^{-1}(\mathcal{N}^*(\mathcal{X}_I)) \cap \mathcal{A}$. This fact is a straightforward adaptation to the setting of finite point measures endowed with the $w^\#$ topology of [16, Theorem 4.1], which establishes the continuity of the summation on the subset of all $(x, y) \in \mathcal{D}_I \times \mathcal{D}_I$ (endowed with the product J_1 topology) such that $\mathrm{Disc}(x) \cap \mathrm{Disc}(y) = \varnothing$. Thus, the weak convergence (2.2) follows from the continuous mapping theorem, and by observing that the sequence (M_n) belongs to $\Psi^{-1}(\mathcal{N}(\mathcal{X}_I))$ for all n and that, by (i) and (ii), M belongs to $\Psi^{-1}(\mathcal{N}^*(\mathcal{X}_I)) \cap \mathcal{A}$ a.s. (see Remark 1.1). The fact that the limit has no pure jump also follows from (i).

2.3. Convergence in $\mathcal{D}_I^\ell$ based on point process convergence

The truncation approach is usual in the context of regular variation to exhibit α-stable approximations of the empirical mean of an infinite variance sequence of random variables. The proof relies on separating small jumps and big jumps and on point process convergence. In the following result, we have gathered the main steps of this approach. To our knowledge, such a result is not available in this degree of generality.

Theorem 2.1. *Let $\{N_n,\ n \geq 1\}$ be a sequence of finite point processes on $\mathcal{X}$ and N be a Poisson point process on $\mathcal{Y}_{I,\ell}$ with mean measure μ. Define, for all $n \geq 1$ and $\varepsilon > 0$,*

$$S_n = \int_{(0,\infty)} \int_{\mathcal{S}_{I,\ell}} yw N_n(\mathrm{d}y, \mathrm{d}w), \qquad S_n^{<\varepsilon} = \int_{(0,\varepsilon]} \int_{\mathcal{S}_{I,\ell}} yw N_n(\mathrm{d}y, \mathrm{d}w),$$
$$Z_\varepsilon = \int_{(\varepsilon,\infty)} \int_{\mathcal{S}_{I,\ell}} yw N(\mathrm{d}y, \mathrm{d}w),$$

which are well defined in $\mathcal{D}_I^\ell$ since N and N_n have finite supports in $(\varepsilon, \infty) \times \mathcal{S}_{I,\ell}$ and $(0, \infty) \times \mathcal{S}_{I,\ell}$, respectively. Assume that the following assertions hold.

(i) *$N_n \Rightarrow N$ in $(\mathcal{M}(\mathcal{Y}_{I,\ell}), \mathcal{B}(\mathcal{M}(\mathcal{Y}_{I,\ell})))$.*

(ii) *For all $t \in I$, $\mu(\{(y, x) \in \mathcal{Y}_{I,\ell}, t \in \mathrm{Disc}(x)\}) = 0$ and $\mu(\{\infty\} \times \mathcal{S}_{I,\ell}) = 0$.*

(iii) $\int_{(0,1]} y^2 \mu(\mathrm{d}y, \mathcal{S}_{I,\ell}) < \infty$.

(iv) *For each $\varepsilon > 0$, the sequence $\{\int_{(\varepsilon,\infty)} y N_n(\mathrm{d}y, \mathcal{S}_{I,\ell}),\ n \geq 1\}$ is uniformly integrable.*

(v) *For all $\eta > 0$, $\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \mathbb{P}(\| S_n^{<\varepsilon} - \mathbb{E}[S_n^{<\varepsilon}]\|_{I,\ell} > \eta) = 0$.*

Then the following assertions hold.

- *For each $\varepsilon > 0$, $Z_\varepsilon \in \mathcal{D}_I^\ell$, $\mathbb{E}[Z_\varepsilon] \in \mathcal{D}_I^\ell$, and $Z_\varepsilon - \mathbb{E}[Z_\varepsilon]$ converges weakly in $(\mathcal{D}_I^\ell, J_1^\ell)$ to a process $\bar{Z}$ as $\varepsilon \to 0$.*
- *$S_n - \mathbb{E}[S_n]$ converges weakly in $(\mathcal{D}_I^\ell, J_1^\ell)$ to $\bar{Z}$.*

Proof. For $\varepsilon > 0$, we define

$$S_n^{>\varepsilon} = \int_{(\varepsilon,\infty)} \int_{\mathcal{S}_{I,\ell}} yw N_n(\mathrm{d}y, \mathrm{d}w), \qquad \bar{S}_n^{>\varepsilon} = S_n^{>\varepsilon} - \mathbb{E}[S_n^{>\varepsilon}], \qquad \bar{S}_n^{<\varepsilon} = S_n^{<\varepsilon} - \mathbb{E}[S_n^{<\varepsilon}],$$

which are random elements of $\mathcal{D}_I^\ell$. By Proposition 2.1 and (i) and (ii), we have that $S_n^{>\varepsilon}$ converges weakly in $(\mathcal{D}_I^\ell, J_1^\ell)$ to Z_ε, provided that $\mu(\{\varepsilon\} \times \mathcal{S}_{I,\ell}) = 0$, and Z_ε has no pure jump. Since $\| S_n^{>\varepsilon}\|_{I,\ell} \leq \int_{(\varepsilon,\infty)} y N_n(\mathrm{d}y, \mathcal{S}_{I,\ell})$, by (iv) we find that $\{\| S_n^{>\varepsilon}\|_{I,\ell},\ n \geq 1\}$ is uniformly integrable. Applying Corollary 2.1, we find that $\mathbb{E}[S_n^{>\varepsilon}]$ converges to $\mathbb{E}[Z_\varepsilon]$ in $(\mathcal{D}_I^\ell, J_1^\ell)$ and

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that $\mathbb{E}[Z_\varepsilon]$ is continuous on I. Thus, addition is continuous at $(Z_\varepsilon, \mathbb{E}[Z_\varepsilon])$; see [17, p. 84]. We obtain, for all $\varepsilon > 0$, as $n \to \infty$,

$$\bar{S}_n^{>\varepsilon} \Rightarrow Z_\varepsilon - \mathbb{E}[Z_\varepsilon] \quad \text{in } \mathcal{D}_I. \tag{2.3}$$

Define $\bar{S}_n = S_n - \mathbb{E}[S_n]$. Then $\bar{S}_n = \bar{S}_n^{>\varepsilon} + \bar{S}_n^{<\varepsilon}$ and (v) can be rewritten as

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}(\|\bar{S}_n - \bar{S}_n^{>\varepsilon}\|_{I,\ell} > \eta) = 0. \tag{2.4}$$

By [2, Theorem 4.2], (a) and (2.4) imply (b). Hence, to conclude the proof, it remains to prove (a), that is, $\bar{Z}_\varepsilon$ converges weakly in $(\mathcal{D}_I^\ell, J_1^\ell)$ to a process $\bar{Z}$. For all $t \in I$ and $0 < \varepsilon < \varepsilon'$, we have

$$Z_\varepsilon(t) - Z_{\varepsilon'}(t) = \int_{(\varepsilon,\varepsilon']} \int_{\mathcal{S}_{I,\ell}} y w(t) N(\mathrm{d}y, \mathrm{d}w),$$

where N is a Poisson process with intensity measure μ. Thus, denoting by $|a|$ the Euclidean norm of vector a and by $\mathrm{Tr}(A)$ the trace of matrix A, we have

$$\begin{aligned}
\mathbb{E}[|\bar{Z}_\varepsilon(t) - \bar{Z}_{\varepsilon'}(t)|^2] &= \mathrm{Tr}(\mathrm{cov}(Z_\varepsilon(t) - Z_{\varepsilon'}(t))) \\
&= \int_{(\varepsilon,\varepsilon']} \int_{\mathcal{S}_{I,\ell}} y^2 |w(t)|^2 \mu(\mathrm{d}y, \mathrm{d}w) \\
&\le \ell \int_{(\varepsilon,\varepsilon']} y^2 \mu(\mathrm{d}y, \mathcal{S}_{I,\ell}).
\end{aligned}$$

We deduce from (iii) that $\bar{Z}_\varepsilon(t) - \bar{Z}_1(t)$ converges in L^2 as ε tends to 0. Thus, there exists a process $\bar{Z}$ such that $\bar{Z}_\varepsilon$ converges to $\bar{Z}$ pointwise in probability; hence, in the sense of finite dimensional distributions. To obtain the convergence in $(\mathcal{D}_I^\ell, J_1^\ell)$, since we use the product topology in $\mathcal{D}_I^\ell$, it only remains to show the tightness of each component. Thus, hereafter we assume that $\ell = 1$. Denote, for $x \in \mathcal{D}_I$ and $\delta > 0$,

$$w''(x, \delta) = \sup\{|x(t) - x(s)| \wedge |x(u) - x(t)|;\ s \le t \le u \in I, |u - s| \le \delta\}. \tag{2.5}$$

By [2, Theorem 15.3], it is sufficient to prove that, for all $\eta > 0$,

$$\lim_{A\to\infty} \sup_{0<\varepsilon\le 1} \mathbb{P}(\|\bar{Z}_\varepsilon\|_I > A) = 0, \tag{2.6}$$

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} \mathbb{P}(w''(\bar{Z}_\varepsilon, \delta) > \eta) = 0, \tag{2.7}$$

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} \mathbb{P}(\mathrm{osc}(\bar{Z}_\varepsilon; [a, a+\delta)) > \eta) = 0, \tag{2.8}$$

$$\lim_{\delta\downarrow 0} \limsup_{\varepsilon\downarrow 0} \mathbb{P}(\mathrm{osc}(\bar{Z}_\varepsilon; [b-\delta, b)) > \eta) = 0, \tag{2.9}$$

where $I = [a, b]$ and osc is defined in (2.1). We start by proving (2.6). For any $\varepsilon_0 \in (0, 1]$ and $\varepsilon \in [\varepsilon_0, 1]$, we have $\|\bar{Z}_\varepsilon\|_I \le \int_{(\varepsilon_0,\infty)} y N(\mathrm{d}y, \mathcal{S}_I)$; whence,

$$\sup_{\varepsilon_0\le\varepsilon\le 1} \mathbb{P}(\|\bar{Z}_\varepsilon\|_I > A) \le A^{-1} \mathbb{E}\biggl[\int_{(\varepsilon_0,\infty)} y N(\mathrm{d}y, \mathcal{S}_I)\biggr],$$

which is finite by (iv).

This yields that $\lim_{A\to\infty}\sup_{\varepsilon_0\le\varepsilon\le 1}\mathbb{P}(\|W_\varepsilon\|_I > A) = 0$ and to conclude the proof of (2.6), we only need to show that, for any $\eta > 0$,

$$\lim_{\varepsilon_0\downarrow 0}\sup_{0<\varepsilon<\varepsilon_0}\mathbb{P}(\|\bar{Z}_\varepsilon - \bar{Z}_{\varepsilon_0}\|_I > \eta) = 0. \tag{2.10}$$

The arguments leading to (2.3) can be used to show that, for all $0 < \varepsilon < \varepsilon_0$,

$$\bar{S}_n^{>\varepsilon_0} - \bar{S}_n^{>\varepsilon} \Rightarrow \bar{Z}_{\varepsilon_0} - \bar{Z}_\varepsilon \quad \text{in } \mathcal{D}_I \tag{2.11}$$

(although the latter is not a consequence of (2.3) because Z_{ε_0} and Z_ε have common jumps). By definition, we have $\bar{S}_n^{<\varepsilon_0} - \bar{S}_n^{<\varepsilon} = \bar{S}_n^{>\varepsilon} - \bar{S}_n^{>\varepsilon_0}$. By (2.11) and the continuous mapping theorem, we obtain $\|\bar{S}_n^{<\varepsilon_0} - \bar{S}_n^{<\varepsilon}\|_I \Rightarrow \|\bar{Z}_{\varepsilon_0} - \bar{Z}_\varepsilon\|_I$. Thus, by the portmanteau theorem, for all $\eta > 0$,

$$\begin{aligned}\mathbb{P}(\|\bar{Z}_{\varepsilon_0} - \bar{Z}_\varepsilon\|_I \ge \eta) &= \limsup_{n\to\infty}\mathbb{P}(\|\bar{S}_n^{<\varepsilon_0} - \bar{S}_n^{<\varepsilon}\|_I \ge \eta) \\ &\le \limsup_{n\to\infty}\mathbb{P}\left(\|\bar{S}_n^{<\varepsilon_0}\|_I \ge \frac{\eta}{2}\right) + \limsup_{n\to\infty}\mathbb{P}\left(\|\bar{S}_n^{<\varepsilon}\|_I \ge \frac{\eta}{2}\right).\end{aligned}$$

We conclude by applying (v) which precisely states that both terms in the right-hand side tend to zero as ε_0 tends to 0, for any $\eta > 0$. This yields (2.10) and (2.6) follows.

Now define the modulus of continuity of a function $x \in \mathcal{D}_I$ by

$$w(x,\delta) = \sup\{|x(t) - x(s)|,\ s,t \in I,\ |t-s| \le \delta\}.$$

We shall rely on the fact that, for any $x, y \in \mathcal{D}_I$, $w''(x+y,\delta) \le w''(x,\delta) + w(y,\delta)$. Note that this inequality is no longer true if $w(y,\delta)$ is replaced by $w''(y,\delta)$. We obtain, for any $0 < \varepsilon < \varepsilon_0$ and $\delta > 0$,

$$w''(\bar{Z}_\varepsilon,\delta) \le w''(\bar{Z}_{\varepsilon_0},\delta) + w(\bar{Z}_\varepsilon - \bar{Z}_{\varepsilon_0},\delta) \le w''(\bar{Z}_{\varepsilon_0},\delta) + 2\|\bar{Z}_\varepsilon - \bar{Z}_{\varepsilon_0}\|_I .$$

Since $\bar{Z}_{\varepsilon_0}$ is in $\mathcal{D}_I$, we have, for any fixed $\varepsilon_0 > 0$, $\lim_{\delta\to 0}\mathbb{P}(w''(\bar{Z}_{\varepsilon_0},\delta) > \eta) = 0$. Hence, with (2.10), we conclude that (2.7) holds. Similarly, since, for each subinterval T, we have

$$\operatorname{osc}(\bar{Z}_\varepsilon; T) \le \operatorname{osc}(\bar{Z}_{\varepsilon_0}; T) + 2\|\bar{Z}_\varepsilon - \bar{Z}_{\varepsilon_0}\|_I ,$$

so we obtain (2.8) and (2.9). This concludes the proof.

2.4. Regular variation in $\mathcal{D}$ and point process convergence

Now let $\{X_{i,n},\ 1 \le i \le m_n\}$ be an array of independent random elements in $\mathcal{D}_I$ and define the point process of exceedances N_n on $(0,\infty] \times \mathcal{S}_{I,\ell}$ by

$$N_n = \sum_{i=1}^{m_n} \boldsymbol{\delta}_{\|X_{i,n}\|_{I,\ell},\, X_{i,n}/\|X_{i,n}\|_I},$$

with the convention that $\boldsymbol{\delta}_{0,0/0}$ is the null mass. If the processes $X_{n,i}$, $1 \le i \le m_n$, are i.i.d. for each n, then it is shown in [7, Theorem 2.4] that (1.1) implies the convergence of the sequence of point processes N_n to a Poisson point process on $\mathcal{D}_I$. We slightly extend here this result to triangular arrays of vector-valued processes. Let N be a Poisson point process on $\mathcal{Y}_{I,\ell} = (0,\infty] \times \mathcal{S}_{I,\ell}$ with mean measure $\mu_\alpha(\mathrm{d}y\,\mathrm{d}w) = \alpha y^{-\alpha-1}\,\mathrm{d}y\nu(\mathrm{d}w)$.

Proposition 2.2. *Conditions (1.4) and (1.5) in Theorem 1.2 imply the weak convergence of N_n to N in $(\mathcal{M}(\mathcal{Y}_{I,\ell}), \mathcal{B}(\mathcal{M}(\mathcal{Y}_{I,\ell})))$.*

2.5. A criterion for negligibility

Condition (v) of Theorem 2.1 is a negligibility condition in the sup-norm. It can be checked separately on each component of $S_n^{<\varepsilon} - \mathbb{E}[S_n^{<\varepsilon}]$. We give here a sufficient condition based on a tightness criterion. Recall the definition of the modulus of continuity w'' in (2.5).

Lemma 2.2. *Let $\{U_{\varepsilon,n},\ \varepsilon > 0, n \geq 1\}$ be a collection of random elements in $\mathcal{D}_I$ such that, for all $t \in I$ and $\eta > 0$,*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \operatorname{var}(U_{\varepsilon,n}(t)) = 0, \tag{2.12}$$

$$\lim_{\delta\to 0} \sup_{0<\varepsilon\leq 1} \limsup_{n\to\infty} \mathbb{P}(w''(U_{\varepsilon,n}, \delta) > \eta) = 0. \tag{2.13}$$

Then, for all $\eta > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}(\left\|U_{\varepsilon,n}\right\|_I > \eta) = 0.$$

Proof. By (2.12) and the Bienaim–Chebyshev inequality, we obtain, for all $\eta > 0$ and $t \in I$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}(|U_{\varepsilon,n}(t)| > \eta) = 0.$$

It follows that, for any $p \geq 1$, $t_1 < \cdots < t_p$, and $\eta > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\ldots,p} |U_{\varepsilon,n}(t_k)| > \eta\Big) = 0. \tag{2.14}$$

Fix some $\zeta > 0$. By (2.13) we can choose $\delta > 0$ such that $\limsup_{n\to\infty} \mathbb{P}(w''(U_{\varepsilon,n}, \delta) > \eta) \leq \zeta$ for all $\varepsilon \in (0, 1]$. Now note that, for any $\delta > 0$, we may find an integer $m \geq 1$ and $t_1 < t_1 < \cdots < t_m$, such that, for all $x \in \mathcal{D}$, $\|x\|_I \leq w''(x, \delta) + \max_{k=1,\ldots,m} |x(t_k)|$; see [2, Proof of Theorem 15.7, p. 131]. This and (2.14) yield

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}(\left\|U_{\varepsilon,n}\right\|_I > \eta) \\
&\quad\leq \sup_{0<\varepsilon\leq 1} \limsup_{n\to\infty} \mathbb{P}(w''(U_{\varepsilon,n}, \delta) > \eta) + \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\ldots,p} |U_{\varepsilon,n}(t_k)| > \frac{\eta}{2}\Big) \\
&\quad\leq \zeta,
\end{aligned}$$

which concludes the proof since ζ is arbitrary.

It is well known that the stochastic equicontinuity condition (2.13) can be obtained by bounds on moments of the increments; see [2, Chapter 15]. We therefore obtain the following corollary.

Corollary 2.2. *Let $X, X_i,\ i \geq 1$, be i.i.d. random elements in $\mathcal{D}_I$ such that $\|X\|_I$ is regularly varying with index $\alpha \in (1, 2)$. Let $\{a_n\}$ be an increasing sequence such that*

$$\lim_{n\to\infty} n\mathbb{P}(\|X\|_I > a_n) = 1.$$

For $\varepsilon > 0$ define $\bar{X}_{\varepsilon,n} = X\mathbf{1}_{\|X\|_I\leq a_n\varepsilon} - \mathbb{E}[X\mathbf{1}_{\|X\|_I\leq a_n\varepsilon}]$ and for $s, t \in I$ define $\bar{X}_{\varepsilon,n}(s, t) = \bar{X}_{\varepsilon,n}(t) - \bar{X}_{\varepsilon,n}(s)$. Assume that there exist $p \in (\alpha, 2]$, $\gamma > \frac{1}{2}$, a continuous increasing

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function F on I, and a sequence of increasing functions F_n that converges pointwise (hence uniformly) to F such that, for all $s < t < u \in I$,

$$\sup_{0<\varepsilon\le 1} n a_n^{-p} \mathbb{E}[|\bar{X}_{\varepsilon,n}(s,t)|^p] \le \{F_n(t) - F_n(s)\}^\gamma, \tag{2.15}$$

$$\sup_{0<\varepsilon\le 1} n^2 a_n^{-2p} \mathbb{E}[|\bar{X}_{\varepsilon,n}(s,t)|^p |\bar{X}_{\varepsilon,n}(t,u)|^p] \le \{F_n(u) - F_n(s)\}^{2\gamma}. \tag{2.16}$$

Then condition (v) of Theorem 2.1 holds.

2.6. Proof of Theorem 1.2

We apply Theorem 2.1 to the point processes N_n and N defined in Section 2.4 and the measure μ_α in lieu of μ. By Proposition 2.2, we have that N_n converges weakly to N in $\mathcal{M}(\mathcal{Y}_{I,\ell})$, i.e. condition (i) of Theorem 2.1 holds. Condition (i) of Theorem 1.1 and the definition of μ_α imply condition (ii) of Theorem 2.1. Condition (1.7) corresponds to condition (v) of Theorem 2.1. Condition (iii) of Theorem 2.1 holds since

$$\int_{(0,1]} y^2 \mu_\alpha(\mathrm{d}y, \mathcal{S}_{I,\ell}) = \int_0^1 \alpha y^{2-\alpha-1}\,\mathrm{d}t = \frac{\alpha}{2-\alpha}.$$

For $0 < \varepsilon < x$, define $Y_n = \int_{(\varepsilon,\infty)} y N_n(\mathrm{d}y, \mathcal{S}_{I,\ell})$ and $Y = \int_{(\varepsilon,\infty)} y N(\mathrm{d}y, \mathcal{S}_{I,\ell})$. The weak convergence of N_n to N implies that of $N_n(\cdot \times \mathcal{S}_{I,\ell})$ to $N(\cdot \times \mathcal{S}_{I,\ell})$. In turn, by continuity of the map $m \mapsto \int_\varepsilon^\infty y m(\mathrm{d}y)$ on the set of point measures on $(0,\infty]$ without mass on $\{\varepsilon,\infty\}$, the weak convergence of $N_n(\cdot \times \mathcal{S}_{I,\ell})$ to $N(\cdot \times \mathcal{S}_{I,\ell})$ implies that of Y_n to Y. On the other hand, (1.6) and (1.4) imply that $\mathbb{E}[Y_n]$ converges to $\mathbb{E}[Y]$ and $\mathbb{E}[Y] < \infty$. Since Y_n and Y are nonnegative random variables, this implies the uniform integrability of $\{Y_n\}$, which is condition (iv) of Theorem 2.1. Finally, (1.3) follows from [14, Theorem 3.12.2].

3. Applications

The usual way to prove the weak convergence of a sum of independent regularly varying functions in $\mathcal{D}_I$ is to establish the convergence of finite dimensional distributions (which follows from the finite dimensional regular variation) and a tightness criterion. Here, we consider another approach, based on functional regular variation. We proved in Section 2.4 that functional regular variation implies the convergence of the point process of (functional) exceedances. Thus, in order to apply Theorem 1.1 or Theorem 1.2, an asymptotic negligibility condition (such as (1.2) or (1.7), respectively) must be proved. Since the functional regular variation condition takes care of the 'big jumps', the negligibility condition concerns only the 'small jumps', i.e. we must only prove the tightness of sum of truncated terms. This can be conveniently done by computing moments of any order $p > \alpha$, even though they are infinite for the original series. In this section, we provide some examples where this new approach can be fully carried out.

3.1. Invariance principle

We start by proving that the classical invariance principle is a particular case of Theorem 2.1. Let $\{z_i\}$ be a sequence of i.i.d. random variables in the domain of attraction of an α-stable law, with $\alpha \in (1,2)$. Let a_n be the $(1/n)$th quantile of the distribution of $|z_1|$ and define as usual the partial sum process S_n by $S_n(t) = a_n^{-1} \sum_{k=1}^{[nt]} (z_k - \mathbb{E}[z_1])$. For $u \in [0,1]$, denote by w_u the indicator function of the interval $[u,1]$, i.e. $w_u(t) = \mathbf{1}_{[u,1]}(t)$, and define $X_{k,n} = a_n^{-1} z_k w_{k/n}$. Then we can write $S_n = \sum_{k=1}^n (X_{k,n} - \mathbb{E}[X_{k,n}])$. We will apply Theorem 1.2 to prove the

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convergence of S_n to a stable process in $\mathcal{D}(I)$ with $I = [0, 1]$. Note that $\|X_{k,n}\|_I = z_k/a_n$. Thus, by Remark 1.2, we only need to prove that (1.4) holds with a measure ν that satisfies condition (i) of Theorem 1.1 and the negligibility condition (1.7). Let ν be the probability measure defined on $\mathcal{S}_I$ by $\nu(\cdot) = \int_0^1 \delta_{w_u}(\cdot)\,\mathrm{d}u$, μ_α be defined on $(0, \infty] \times \mathcal{S}_I$ by $\mu_\alpha((r, \infty] \times \cdot) = r^{-\alpha}\nu(\cdot)$, and μ_n be the measure in the left-hand side of (1.4). Since $\|X_{k,n}\|_I = z_k/a_n$, the random variables z_k are i.i.d., and $w_{k/n}$ are deterministic, we have, for all $r > 0$ and Borel subsets A of $\mathcal{S}_I$,

$$\mu_n((r, \infty] \times A) = (n\mathbb{P}(z_1 > a_n r)) \times \left(\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{w_{k/n} \in A\}}\right).$$

By the regular variation of z_1, the first term of this product converges to $r^{-\alpha}$. The second term of this product can be written as $P_n \circ \phi^{-1}(A)$, where $P_n = n^{-1}\sum_{k=1}^{n} \delta_{k/n}$ is seen as a probability measure on the Borel sets of $[0, 1]$ and $\phi : [0, 1] \to \mathcal{D}_I$ is defined by $\phi(u) = w_u$. Since ϕ is continuous (with $\mathcal{D}_I$ endowed by J_1) and P_n converges weakly to the Lebesgue measure on $[0, 1]$, denoted by Leb, by the continuous mapping theorem, we have that $P_n \circ \phi^{-1}$ converges weakly to $\mathrm{Leb} \circ \phi^{-1} = \nu$. This proves that (1.4) holds.

To prove that (1.7) holds, note that

$$\|S_n^{<\varepsilon}\|_I = a_n^{-1} \max_{1 \le k \le n} \left|\sum_{i=1}^{k} (z_k \mathbf{1}_{\{|z_k| \le a_n\varepsilon\}} - \mathbb{E}[z_k \mathbf{1}_{\{|z_k| \le a_n\varepsilon\}}])\right|,$$

where $S_n^{<\varepsilon}$ denotes the sum appearing in the left-hand side of (1.7). By Doob's inequality, we obtain

$$\mathbb{E}[\|S_n^{<\varepsilon}\|_\infty^2] \le 2\mathrm{var}\left(a_n^{-1}\sum_{i=1}^{n} z_k \mathbf{1}_{\{|z_k| \le a_n\varepsilon\}}\right) \le n a_n^{-2}\mathbb{E}[z_1^2 \mathbf{1}_{\{|z_k| \le a_n\varepsilon\}}] = O(\varepsilon^{2-\alpha}),$$

by regular variation of z_1. This bound and Markov's inequality yield (1.7).

3.2. Stable processes

Applying Corollary 2.2, we obtain a criterion for the convergence of partial sums of a sequence of i.i.d. processes that admit the representation RW, where R is a Pareto random variable and $W \in \mathcal{S}_I$.

Proposition 3.1. *Let $\{R, R_i\}$ be a sequence of i.i.d. real valued random variables in the domain of attraction of an α-stable law, with $1 < \alpha < 2$. Let $\{W, W_i,\ i \ge 1\}$ be an i.i.d. sequence in $\mathcal{S}_I$ with distribution ν satisfying the assumptions of Lemma 1.2, and independent of the sequence $\{R_i\}$. Then, defining a_n as an increasing sequence such that, by $\mathbb{P}(R > a_n) \sim 1/n$, $a_n^{-1}\sum_{i=1}^{n}\{R_i W_i - \mathbb{E}[R]\mathbb{E}[W]\}$ converges weakly in $\mathcal{D}_I$ to a stable process Z which admits the representation (1.3).*

Remark 3.1. By Lemma 1.1, the stable process Z also admits the series representation (1.8), which is a.s. convergent in $\mathcal{D}_I$ and by Lemma 1.2 it is regularly varying in the sense of (1.1), with spectral measure ν. As mentioned in the proof of Lemma 1.2, the proof we give here of the existence of a version of Z in $\mathcal{D}_I$ is different from the proof of [5] or [1].

Proof of Proposition 3.1. We apply Theorem 1.1 to $X_i = R_i W_i$. The regular variation condition (1.1) holds trivially since $\|X\|_I = R$ is independent of $X/\|X\|_I = W$. Condition (1.10) implies that W has no fixed jump, i.e. condition (i) of Theorem 1.1 holds.

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Thus, we only need to prove that the negligibility condition (ii) of Theorem 1.1 holds. Write $S_n^{<\varepsilon} = a_n^{-1} \sum_{i=1}^n \{R_i \mathbf{1}_{\{R_i \le \varepsilon a_n\}} W_i - \mathbb{E}[R\mathbf{1}_{\{R \le a_n \varepsilon\}}]\mathbb{E}[W]\}$ and $r_{n,i} = a_n^{-1} R_i \mathbf{1}_{\{R_i \le a_n \varepsilon\}}$. Then we have

$$S_n^{<\varepsilon} = \sum_{i=1}^n r_{n,i}\{W_i - \mathbb{E}[W]\} + \mathbb{E}[W] \sum_{i=1}^n \{r_{n,i} - \mathbb{E}[r_{n,i}]\}. \tag{3.1}$$

Since $\|\mathbb{E}[W]\|_I \le 1$, the second term's infinite norm on I can be bounded using the Bienaym–Chebyshev inequality and the regular variation of R which implies, for any $p > \alpha$, $\mathbb{E}[|r_{n,i}|^p] \sim (\alpha/(p-\alpha))\varepsilon^{p-\alpha} n^{-1}$. Hence, we only need to deal with the first term on the right-hand side of (3.1), which is hereafter denoted by $\tilde{S}_n^{<\varepsilon}$. Since R is independent of W, conditions (2.15) and (2.16) are straightforward consequences of (1.9) and (1.10). Thus, condition (ii) of Theorem 1.1 holds by Corollary 2.2. The last statement follows from Lemma 1.2

3.3. Renewal–reward process

Consider a renewal process N with i.i.d. interarrivals $\{Y_i,\ i \ge 1\}$ with common distribution function F, in the domain of attraction of a stable law with index $\alpha \in (1, 2)$. Let a_n be a norming sequence defined by $a_n = F^{\leftarrow}(1 - 1/n)$. Then, for all $x > 0$, $\lim_{n\to\infty} n\bar{F}(a_n x) = x^{-\alpha}$. Consider a sequence of rewards $\{W_i,\ i \ge 1\}$ with distribution function G and define the renewal–reward process R by $R(t) = W_{N(t)}$. Let ϕ be a measurable function and define $A_T(\phi)$ by

$$A_T(\phi) = \int_0^T \phi(R(s))\,\mathrm{d}s.$$

We are concerned with the functional weak convergence of A_T. We moreover assume that the sequence $\{(Y, W), (Y_i, W_i),\ i \ge 1\}$ is i.i.d. and that Y and W are asymptotically independent in the sense of [12], i.e.

$$\lim_{n\to\infty} n\mathbb{P}\left(\left(\frac{Y}{a_n}, W\right) \in \cdot\right) \xrightarrow{v} \mu_\alpha \otimes G^* \tag{3.2}$$

on $]0, \infty] \times \mathbb{R}$, where G^* is a probability measure on $\mathbb{R}$. This assumption is obviously satisfied when Y and W are independent, with $G^* = G$ in that case. When Y and W are independent and $\mathbb{E}[|\phi(W)|^\alpha] < \infty$, it has been proved by [15] that $a_T^{-1}\{A_T(\phi) - \mathbb{E}[A_T(\phi)]\}$ converges weakly to a stable law. Define $\lambda = (\mathbb{E}[Y])^{-1}$ and $F_0(w) = \lambda\mathbb{E}[Y\mathbf{1}_{\{W \le w\}}]$. Then F_0 is the steady state marginal distribution of the renewal–reward process and $\lim_{t\to\infty} \mathbb{P}(R(t) \le w) = F_0(w)$. For $w \in \mathbb{R}$, consider the function $\mathbf{1}_{\{\cdot \le w\}}$, which yields the usual one-dimensional empirical process:

$$E_T(w) = a_T^{-1} \int_0^T \{\mathbf{1}_{\{R(s) \le w\}} - F_0(w)\}\,\mathrm{d}s.$$

Theorem 3.1. *Assume that (3.2) holds with G^* continuous. The sequence of processes E_T converges weakly in $\mathcal{D}(\mathbb{R})$ endowed with the J_1 topology as T tends to infinity to the process E^* defined by $E^*(w) = \int_{-\infty}^{\infty} \{\mathbf{1}_{\{x \le w\}} - F_0(w)\} M(\mathrm{d}x)$, where M is a totally skewed to the right stable random measure with control measure G^*, i.e.*

$$\log \mathbb{E}[\mathrm{e}^{\mathrm{i}t \int_{-\infty}^{\infty} \phi(w) M(\mathrm{d}w)}] = -|t|^\alpha \lambda c_\alpha \mathbb{E}[|\phi(W^*)|^\alpha]\left\{1 + \mathrm{i}\,\mathrm{sign}(t)\beta(\phi)\tan\left(\frac{\pi\alpha}{2}\right)\right\},$$

where W^* is a random variable with distribution G^*, $c_\alpha^\alpha = \Gamma(1-\alpha)\cos(\pi\alpha/2)$, and $\beta(\phi) = \mathbb{E}[\phi_+^\alpha(W^*)]/\mathbb{E}[|\phi(W^*)|^\alpha]$.

Remark 3.2. We can also write $E^* = Z \circ G^* - F_0 \cdot Z(1)$, where Z is a totally skewed to the right Lévy α-stable process. If, moreover, Y and W are independent, then the marginal distribution of $R(0)$ is G, $G^* = G$, and the limiting distribution can be expressed as $Z \circ G - GZ(1)$; thus, the law of $\sup_{w \in \mathbb{R}} E^*(w)$ is independent of G.

Proof of Theorem 3.1. Write

$$\begin{aligned} E_T(w) &= a_T^{-1} \sum_{i=0}^{N(T)} Y_i \mathbf{1}_{\{W_i \le w\}} + a_T^{-1}\{T - S_{N(T)}\}\mathbf{1}_{\{W_{N}(T) \le w\}} - a_T^{-1} \lambda T \mathbb{E}[Y \mathbf{1}_{\{W \le w\}}] \\ &= a_T^{-1} \sum_{i=0}^{N(T)} \{Y_i \mathbf{1}_{\{W_i \le w\}} - \mathbb{E}[Y \mathbf{1}_{\{W \le w\}}]\} - a_T^{-1}\{S_{N(T)} - \lambda^{-1} N(T)\} F_0(w) \qquad (3.3a) \\ &\quad - a_T^{-1}\{S_{N(T)} - T\}\{\mathbf{1}_{\{W_{N(T)} \le w\}} - \lambda \mathbb{E}[Y \mathbf{1}_{\{W \le w\}}]\}. \qquad (3.3b) \end{aligned}$$

The term in (3.3b) is $o_P(1)$, uniformly with respect to $w \in \mathbb{R}$. Define $U_i = G^*(W_i)$ and $U = G^*(W)$. Define the sequence of bivariate processes S_n on $I = [0, 1]$ by

$$S_n(t) = a_n^{-1} \sum_{i=1}^{n} (Y_i [\mathbf{1}_{\{U_i \le t\}}, 1]' - \mathbb{E}[Y [\mathbf{1}_{\{U_i \le t\}}, 1]']),$$

where x' denotes the transpose of a vector $x \in \mathbb{R}^2$. Then the term in (3.3a) can be expressed as the scalar product $[1, -F_0(w)] S_{N(T)}(G^*(w))$. Using that $N(T)/T$ converges almost surely to λ, we can relate the asymptotic behavior of $S_{N(T)}$ to that of S_n. The latter is obtained by applying Theorem 1.2. The fact that the mapping $(y, w) \mapsto y[\mathbf{1}_{[G^*(w),1]}, \mathbf{1}_{[0,1]}]'$ is continuous from $(0, \infty) \times \mathbb{R}$ to $\mathcal{D}_I^2$ and the convergence (3.2) imply that the distribution of $Y[\mathbf{1}_{\{U \le t\}}, 1]'$ is regularly varying with index α in $\mathcal{D}_I^\ell$ with $\ell = 2$ and ν defined by $\nu(\cdot) = \mathbb{P}((\mathbf{1}_{[U^*,1]}, \mathbf{1}_{[0,1]})' \in \cdot)$ where U^* is uniformly distributed on $[0, 1]$. Conditions (1.4), (1.5), and (1.6) then follow by Remark 1.2. Next, we must prove the asymptotic negligibility condition (1.7). It suffices to prove it for the first marginal $X = Y \mathbf{1}_{[U,1]}$. For $\varepsilon > 0$ and $n \ge 1$, define $G_{n,\varepsilon}(t) = n a_n^{-2} \varepsilon^\alpha \mathbb{E}[Y^2 \mathbf{1}_{\{Y \le a_n \varepsilon\}} \mathbf{1}_{\{U \le t\}}]$. It follows that (2.15) and (2.16) hold with $p = 2$, $\gamma = 1$, and $F_n = \sup_{0 < \varepsilon \le 1} G_{n,\varepsilon} = G_{n,1}$. By (3.2) and Karamata's theorem, we have $\lim_{n \to \infty} G_{n,1}(t) = t$. Therefore, Corollary 2.2 yields (1.7). By Theorem 1.2, the previous steps imply that S_n converges weakly in $(\mathcal{D}, J_1)$ to a bivariate stable process which can be expressed as $[Z, Z(1)]$, where Z is a totally skewed to the right α-stable Lévy process.

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