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MAXIMAL LEFT IDEALS AND IDEALIZERS IN MATRIX RINGS

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Let R be a ring with identity, $M_n(R)$ the ring of $n \times n$ matrices over R. The lattice of two-sided ideals of R is carried via $A \to M_n(A)$ to form the lattice of two-sided ideals of $M_n(R)$. We wish to study the more complex left ideal structure of $M_n(R)$. For example, if K is a commutative field, then $M_n(K)$ has non-trivial left ideals. In particular $M_n(K)$ has the maximal left ideal consisting of all matrices with some designated column zero. Or for any ring with maximal left ideal M, $M_n(R)$ has the maximal left ideal consisting of all matrices with some column's entries from M. In Theorem 1.2 we characterize the maximal left ideals of $M_n(R)$ in terms of those of R. We briefly study some contraction properties of maximal left ideals in matrix rings. For R commutative we "count" the maximal left ideals of $M_n(R)$ and describe the idealizer of any such ideal; in the case where K is a field we see that the collection of maximal left ideals of $M_n(K)$ can be naturally identified with $P^{n-1}(K)$ (projective space).

In Section 3 we define two maximal left ideals M and N of R to be *conjugate* if $M = pNp^{-1}$ for some unit p of R, then study the lifting of conjugacy from R to $M_n(R)$. For example, in Proposition 3.3 we show that if M is two-sided maximal ideal of R, then all maximal left ideals of $M_n(R)$ which lie over M are conjugate. In particular, all maximal left ideals of $M_n(R)$ are conjugate when R is a local ring.

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Notation. "Ideal (module)" will always mean left ideal (module); M and N will be generic symbols for maximal left ideals. The elements of R^n will be thought of as $n \times 1$ columns but written as transposed rows: $u = (u_1, u_2, \ldots, u_n)'$. For a matrix X we shall let X_i denote the ith row; e_{ij} denotes the matrix having 1 in the (i, j)-position and 0 elsewhere; e_i denotes the $n \times 1$ column with 1 in the ith position and 0 elsewhere. R will be considered a subring of $M_n(R)$ via the natural imbedding $r \to \operatorname{diag}(r, r, \ldots, r)$. We shall let $\operatorname{Max}(R)$ denote the collection of maximal (left) ideals of R.

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1. The maximal left ideals of $M_n(R)$. Let A be a left ideal of R, let $u = (u_1, \ldots, u_n)' \in R^n$ and consider the $M_n(R)$-linear maps

$$M_n(R) \xrightarrow{f} R^n \xrightarrow{g} (R/A)^n \cong R^n/A^n$$

where $f(X) = Xu$ and g is the natural surjection reducing mod A. Denote by $D(A:u)$ the kernel of this composition. Thus

$$D(A:u) = \{X \in M_n(R) | Xu \in A^n\}$$
$$= \{X \in M_n(R) | X_i u \in A, i = 1, 2, \ldots, n\}.$$

This is a left ideal of $M_n(R)$ which is proper unless $u \in A^n$.

If M is maximal in R and $u \in R^n$, $u \notin M^n$, then $(R/M)^n$ is a simple $M_n(R)$-module. Therefore $g \circ f$ is onto, $M_n(R)/D(M:u) \cong (R/M)^n$ and $D(M:u)$ is a maximal left ideal of $M_n(R)$.

Examples. 1.1 (1)

$$D(0:e_i) = \begin{bmatrix} R \ldots 0 \ldots R \\ \cdot \quad\ \cdot \quad\ \cdot \\ \cdot \quad\ \cdot \quad\ \cdot \\ \cdot \quad\ \cdot \quad\ \cdot \\ R \ldots 0 \ldots R \end{bmatrix}$$

i.e., zeros in the ith column. This is maximal if and only if R is a division ring.

(2) If K is a field, $M = 0$, $u = (1, -1)'$, then

$$D(0:u) = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} | a - c = 0 \text{ and } b - d = 0 \right\}$$
$$= \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} | a, b \in K \right\}$$

is maximal in $M_2(K)$.

(3) Similarly in the ring of integers $\mathbf{Z}$, let $M = p\mathbf{Z}$ for p a prime and let $u = (1, -1)'$. Then

$$D(pZ:u) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a \equiv b \ (\mathrm{mod}\ p) \text{ and } c \equiv d \ (\mathrm{mod}\ p) \right\}$$

is maximal in $M_2(\mathbf{Z})$.

THEOREM 1.2. *The collection of $D(M:u)$, for $M \in \mathrm{Max}(R)$ and $u \in R^n - M^n$, gives all maximal left ideals of $M_n(R)$.*

Proof. Let M' be a maximal ideal of $M_n(R)$; so $M_n(R)/M'$ is a simple $M_n(R)$-module. By the Morita equivalence, $M_n(R)/M' \cong E^n$ for E a simple left R-module. Thus $E \cong R/M$ for some maximal ideal M of R.

The isomorphism

$$M_n(R)/M' \to (R/M)^n \cong R^n/M^n$$

maps $1 + M'$ to some $u + M^n$, and it is easily checked that $M' = D(M:u)$.

Example 1.3. It may happen that $D(A:u)$ is maximal even though A is not maximal in R. Let $R = \mathbf{Z}_6[x]$, let A be the principal ideal generated by x, let M be the ideal generated by 2 and x, and let $u = (3 + x, 0)'$ in R^2. Then M is maximal in R and $D(A:u) = D(M:u)$ is maximal in $M_2(R)$.

For a left R-module E with submodule F and for $x \in E$, let $(F:x)$ denote the left ideal $\{r \in R | rx \in F\}$. This is a proper ideal if and only if $x \notin F$; if F is a maximal submodule of E and $x \notin F$, then $(F:x)$ is a maximal left ideal of R and $R/(F:x) \cong E/F$. In particular, if M is a maximal ideal of R, $u \in R - M$, then in $M_1(R) = R$ we have $D(M:u) = (M:u)$. This also provides alternate visualizations of $D(A:u)$ in any $M_n(R)$; namely, for A a left ideal of R, $u \in R^n$,

$$\begin{aligned} D(A:u) &= (A^n:u) \text{ computed in the } M_n(R)\text{-module } R^n,\\ &= (0:u + R^n) \text{ computed in } R^n/A^n,\\ &= (M_n(A):U) \text{ in the module } M_n(R), \end{aligned}$$

where U is the $n \times n$ matrix having u down the first column and zeros elsewhere. Note that for $n = 1$ the theorem restates the obvious:

$$\operatorname{Max}(R) = \{(M:u) | M \in \operatorname{Max}(R), u \in R - M\}.$$

Example 1.4. Let R be a matrix ring; say $R = M_n(S)$. For N a maximal ideal of S and $u \in S^n - N^n$ we have the maximal ideal $M' = D(N:u)$ of R. If X is any matrix in $R - M'$, then $(M':X)$ is a maximal left ideal of $R = M_n(S)$, and the theorem tells us its form. In fact $(M':X) = D(N:Xu)$.

Example 1.5. (Nested matrix rings). For m, $n \geqq 1$, $M_m(M_n(R)) \cong M_{mn}(R)$. For A a left ideal of R and $u \in R^n - A^n$, $D(A:u)$ is a proper left ideal of $M_n(R)$. Then for $U \in M_n(S)^m - D(A:u)^m$, we should be able to identify the image of the proper ideal $D(D(A:u):U)$ in $M_{mn}(R)$. Regarding U as an $mn \times n$ matrix over R, we see that

$$Uu \in R^{mn} - A^{mn} \text{ and } D(D(A:u):U) = D(A:Uu).$$

(The previous example was just a disguised special case of this one.)

For a left ideal of R, let $I(A) = \{r \in R | Ar \subseteq A\}$ denote the *idealizer* of A. (This was introduced by Ore, studied in [**5**] by Robson, [**2**] by Goldie and recently in [**4**] and [**6**] by Kruase and Teply.) $I(A)$ is the largest subring of R in which A sits as a two-sided ideal, so $I(A) = R$ if

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and only if A is a two-sided ideal of R. If M is a maximal left ideal of R, then M is maximal in $I(M)$ and the *eigen-ring* $I(M)/M$ is a division ring [**2**]. Moreover, $M_n(M)$ is a left ideal of $M_n(R)$ and $I(M_n(M)) = M_n(I(M))$. In this case the eigen-ring

$$I(M_n(M))/M_n(M) = M_n(I(M))/M_n(M) \cong M_n(I(M)/M)$$

is a simple artinian ring.

Let A be an ideal of R. Note that a left ideal of $M_n(R)$ contains A if and only if it contains $M_n(A)$, and one might guess that the ideals $D(A:u)$ would fall into this category. Such is not always the case and in fact we can say precisely how $M_n(A)$ is related to the $D(A:u)$. Let C be the center of R and B be the transporter ideal $B = (A:R) = \{r \in R | rR \subseteq A\}$; we then have

$$M_n(A \cap C) \subseteq M_n(A \cap B) \subseteq \cap_u D(A:u) \subseteq M_n(A),$$

where the intersection is taken over all $u \in R^n$. Moreover, $M_n(A) \subseteq D(A:u)$ if and only if each $u_i \in I(A)$; so it can be shown that $M_n(A) = \cap_u D(A:u)$ if and only if A is two-sided. But for M maximal and not two-sided, it may even happen that $\cap_u D(M:u) = 0$. (Example: let $R = M_2(\mathbf{Z})$, $M = D(0:(0,1)')$ for $n = 2$.)

We catalog a few other easily-proved properties.

LEMMA 1.6. (a) *For $M \in \operatorname{Max}(R)$, $(M:u) = M$ if and only if $u \in I(M) - M$.*

(b) *If $A \subseteq B$ are ideals of R and $u \in R^n$, then $D(A:u) \subseteq D(B:u)$.*

(c) *If $A = \cap A_i$ is the intersection of a collection of ideals of R, then $D(A:u) = \cap D(A_i:u)$.*

(d) *For A an ideal of R and $u \in R^n$,*

$$D(A:u) \cap R = \bigcap_{i=1}^{n} (A:u_i).$$

It is natural to ask whether the maximal ideals of $M_n(R)$ lie over (and thus contract to) maximal ideals of R. By (a) and (d) above

COROLLARY 1.7. *If M is two-sided then $D(M:u)$ contracts to M.*

S. H. Brown [**1**] calls a ring *left quasi-duo* if every maximal left ideal is two-sided. Any local ring is left quasi-duo. Certainly every left duo ring is left quasi-duo. The ring of 2×2 lower triangular matrices over a division ring is a left quasi-duo ring which is not left duo. Note that $M_n(R)$ is never left quasi-duo if $n \geqq 2$.

PROPOSITION 1.8. *Let $n \geqq 2$. Every maximal left ideal of $M_n(R)$ contracts to a maximal left ideal of R if and only if R is left quasi-duo.*

Proof. If some $M \in \operatorname{Max}(R)$ is not two-sided, let $r \in R - I(M)$ and let $u = (1, r, 0, \ldots, 0)' \in R^n - M^n$. Then by (d) above

$$D(M:u) \cap R = (M:1) \cap (M:r) = M \cap (M:r).$$

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By (a) above this contraction is not maximal. The converse follows from the preceding corollary.

In any case, by Lemma 1.6 (d) the contraction of a maximal ideal of $M_n(R)$ is always an intersection of maximal ideals of R.

Let $\mathbf{Z}$ be the ring of integers and let $\mathbf{Q}$ be the rational field. Noting that the natural maximal left ideals $D(0:e_i)$ of $M_n(\mathbf{Q})$ do not contract to maximal ideals of $M_n(\mathbf{Z})$, we are led to ask if this behavior is typical. We first look at a more general situation.

Let R be commutative and S a multiplicative subset of R ($0 \notin S$) which is contained in the set of non-zero divisors of R. Then R can be considered a subring of the ring of fractions $S^{-1}R$ (which is itself a subring of the classical ring of quotients of R). For A' an ideal of $S^{-1}R$ let $A = A' \cap R$ denote its contraction to R.

LEMMA 1.9. *Let P' be a prime ideal of $S^{-1}R$ and let $u = (u_1/s_1, u_2/s_2, \ldots, u_n/s_n)' \in (S^{-1}R)^n$. If some entry of u is not in P', then $D(P':u)$ is a proper left ideal of $M_n(S^{-1}R)$. Its contraction to $M_n(R)$ is $D(P:su)$, where $s = s_1 \cdot s_2 \cdots s_n$.*

Proof. Note that s is a unit of $S^{-1}R$ and is thus not in P'. Since some $u_i/s_i \notin P'$, then $s \cdot u_i/s_i \notin P'$. Certainly $u_i s/s_i \in R$, but $su_i/s_i \notin P = P' \cap R$. Hence $su \in R^n - P^n$ and so $D(P:su)$ is a proper left ideal of $M_n(R)$. It also follows that $D(P:su)$ is independent of the choice of representatives of the entries of u.

Let $X \in D(P:su)$. Then each $X_i(su) \in P = P' \cap R$, and so, by the commutativity of R, each $s(X_iu) \in P'$. Since P' is prime and $s \in P'$, this forces each $X_iu \in P'$. That is, $X \in D(P':u)$. Since we are only concerned with $X \in M_n(R)$, and these steps are reversible even without the primality assumption, we have

$$D(P':u) \cap M_n(R) = D(P:su).$$

Now let R be an integral domain and let S be the set of non-zero elements of R.

PROPOSITION 1.10. *If R is an integral domain and K its field of fractions, then no maximal left ideal of $M_n(K)$ contracts to a maximal left ideal of $M_n(R)$.*

Proof. $D(0:u) \cap M_n(R) = D(0:su)$ is not maximal in $M_n(R)$.

Note that R as a subring of K trivially has the property described in the proposition, since the maximal ideal 0 of K contracts to the non-maximal ideal 0 of R, whereas in the matrix ring case the (non-zero) maximal ideals of $M_n(K)$ all contract to non-zero, non-maximal ideals of $M_n(R)$.

2. Equality of $D(M:u)$ and $D(M:v)$. We would like to know when $D(M:u) = D(M:v)$ for $u = (u_1, \ldots, u_n)'$ and $v = (v_1, \ldots, v_n)' \in R^n - M^n$. This is certainly true if $u \equiv v(\text{mod } M)$ (i.e. each $u_j - v_j \in M$), but we have a weaker condition involving the idealizer of M.

PROPOSITION 2.1. *If $u, v \in R^n - M^n$ and $v \equiv uc(\text{mod } M)$ for some $c \in I(M)$, then $D(M:u) = D(M:v)$.*

Proof. Since these are maximal ideals, it suffices to show one inclusion. Let $v = uc + m$ where $m \in M^n$, and let $X \in D(M:u)$, so that each $X_i u \in M$. Then

$$X_i v = X_i uc + X_i m.$$

But $c \in I(M)$ forces $(X_i u)c \in M$, and certainly $X_i m \in M$. Thus each $X_i v \in M$, so $X \in D(M:v)$ and $D(M:u) \subseteq D(M:v)$.

Note that for $c \in I(M) - M$ the coset $c + M$ is invertible in the division ring $I(M)/M$. Thus $v \equiv uc$ (mod M) for some $c \in I(M) - M$ if and only if $u \equiv vc$ (mod M) for some $c \in I(M) - M$.

The preceding proposition gives a natural sufficient condition for the desired equality. We have no unrestricted necessary and sufficient condition, but we can show that in many cases the above condition is necessary. First we note that if A is any left ideal of R and $D(A:u) = D(A:v)$, then the n-tuples u and v must behave alike (with respect to A) at each coordinate.

LEMMA 2.2. *Let $D(A:u) = D(A:v)$ and let $i \in \{1, 2, \ldots, n\}$.*
Then (a) *$u_i \in A$ if and only if $v_i \in A$;*
(b) *$u_i \in I(A) - A$ if and only if $v_i \in I(A) - A$; and*
(c) *$u_i \in R - I(A)$ if and only if $v_i \in R - I(A)$.*

Proof. (a) Let $u_i \in A$. Then $e_{1i}u = (u_i, 0, \ldots, 0)' \in A^n$; that is, $e_{1i} \in D(A:u) = D(A:v)$. Hence $e_{1i}v = (v_i, 0, \ldots, 0)' \in A^n$. Thus $v_i \in A$.

(b) Let $u_i \in I(A) - A$. By (a), $v_i \notin A$. Let $a \in A$. Then $ae_{1i}u = (au_i, 0, \ldots, 0)' \in A^n$, because $a \in A$ and $u_i \in I(A)$. Thus $ae_{1i} \in D(A:u) = D(A:v)$; so $ae_{1i}v = (av_i, 0, \ldots, 0)' \in A^n$. Hence $av_i \in A$. By the definition of idealizer, $v_i \in I(A)$.

(c) follows from (a) and (b).

PROPOSITION 2.3. *If each u_i and v_i is in $I(M)$, then $D(M:u) = D(M:v)$ if and only if $v \equiv uc(\text{mod } M)$ for some $c \in I(M) - M$.*

Proof. If u and v are in M^n, everything is trivial; so assume $u \notin M^n$.

First assume $D(M:u) = D(M:v)$. For $u_i \in I(M) - M$ there exists $w_i \in I(M) - M$ such that

$$u_i w_i + M = 1 + M = w_i u_i + M;$$

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say $u_i w_i = 1 + m_i$ and $w_i u_i = 1 + n_i$, with m_i and n_i in M. Fix k such that $u_k \in I(M) - M$. By the preceding lemma, part (b), v_k is also in $I(M) - M$. Thus $c = w_k v_k \in I(M) - M$. A brief calculation shows that $v_k - u_k c = -m_k v_k \in M$.

Now let j be any other index such that $u_j \in I(M) - M$, and let X be the matrix $w_k e_{1k} - w_j e_{1j}$. Then

$$Xu = (w_k u_k - w_j u_j, 0, \ldots, 0)' = (n_k - n_j, 0, \ldots, 0)' \in M^n.$$

Hence $X \in D(M:u) = D(M:v)$. This implies that

$$Xv = (w_k v_k - w_j v_j, 0, \ldots, 0)' \in M^n;$$

so

$$w_k v_k - w_j v_j = c - w_j v_j \in M^n.$$

Another quick calculation shows that $v_j - u_j c \in M$.

Finally, if u_j is in M, then v_j is also in M, by part (a) of the preceding lemma. So $v_j - u_j c \in M$.

Since $v_j - u_j c \in M$ for each index j, $v \equiv uc (\text{mod } M)$.

The converse was proved in Proposition 2.1, without the idealizer assumption on u and v.

Remark. Via the natural imbedding of R in $M_n(R)$, M can be considered as a subset of $M_n(R)$, where it generates the left ideal $M_n(M)$. So by the remarks preceding Lemma 1.6, we can restate Proposition 2.3 to say that if $D(M:u)$ and $D(M:v)$ contain M, then they are equal if and only if $v \equiv uc$ (mod M) for some $c \in I(M) - M$.

Remark. It may seem that these idealizer assumptions push everything inside $I(M)$, in which case we may as well assume initially that M is two-sided. However the ideal $D(M:u)$ is still being calculated in $M_n(R)$ and it is easy to find an example with all $u_i \in I(M)$ but $D(M:u)$ possessing an element which has none of its entries in $I(M)$.

It is often difficult to compute the idealizer of a left ideal. We can now describe the idealizer of $D(M:u)$ in $M_n(R)$ whenever u behaves nicely enough with respect to M.

COROLLARY 2.4. *If each $u_i \in I(M)$ (i.e., $M \subseteq D(M:u)$), then the idealizer of $D(M:u)$ is*

$$\{X \in M_n(R) | Xu \equiv uk \text{ (mod } M) \textit{ for some } k \in I(M)\}.$$

Proof. If $Xu \equiv uk$ (mod M) for $k \in I(M)$ and $Y \in D(M:u)$, then $(YX)u \equiv (Yu)k \in M^n$; so X is in the idealizer of $D(M:u)$.

Conversely suppose X is in the idealizer of $D(M:u)$ but not in $D(M:u)$ itself. Then we have

$$D(M:Xu) = (D(M:u):X) = D(M:u),$$

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the first equality by Example 1.4 and the second by Lemma 1.6(a). By the proposition, $Xu \equiv uk \pmod{M}$ for some $k \in I(M) - M$. On the other hand, if $X \in D(M{:}u)$, then $Xu \equiv u \cdot 0 \pmod{M}$.

Perhaps the nicest kind of non-commutative ring is a matrix ring over a commutative base ring. In this situation, Corollary 2.4 says that the idealizer of $D(M{:}u)$ consists of all matrices X which act on u like scalar multiplication mod M; i.e., X which have u as an eigenvector mod M. In particular, if K is a commutative field, then the idealizer of $D(0{:}e_j)$ in $M_n(K)$ consists of all matrices whose jth column is zero off the diagonal. We thus recover as a very special case the well-known result that in $M_2(K)$ the idealizer of $D(0{:}e_1) = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$ is the ring of 2×2 upper triangular matrices over K.

COROLLARY 2.5. *If M is two sided (e.g. if R is commutative), then $D(M{:}u) = D(M{:}v)$ if and only if $v \equiv uc$* (mod M) *for some $c \in R - M$.*

COROLLARY 2.6. *If all u_i and v_i are central in R (or even just central* mod M), *then $D(M{:}u) = D(M{:}v)$ if and only if $v \equiv uc$* (mod M) *for some $c \in I(M) - M$.*

COROLLARY 2.7. *$D(M{:}u) = D(M{:}e_i)$ if and only if $u_i \in I(M) - M$ and $u_k \in M$ for $k \neq i$.*

Again we point out what happens in the most special case.

COROLLARY 2.8. *If K is a commutative field, then $D(0{:}u) = D(0{:}v)$ in $M_n(K)$ if and only if $u = cv$ for some $c \neq 0$ in K.*

Remark. When $n = 1$ the proposition says that for $u, v \in I(M)$, $(M{:}u) = (M{:}v)$ if and only if $u \equiv vc \pmod{M}$ for some $c \in I(M) - M$. However, the restriction on u and v is not necessary for the equivalence, as will be shown by Corollary 2.11.

Let S be a ring, $R = M_n(S)$, let N be a maximal left ideal of S and let $w = (w_1, \ldots, w_n)' \in S^n - N^n$. Let $M' = D(N{:}w)$ in R and let $X = [x_{ij}]$ and Y be in R.

LEMMA 2.9. *If each $w_i \in I_S(N)$ and each $x_{ij} \in I_S(N)$ and $(M'{:}X) = (M'{:}Y)$, then $X \equiv YC$* (mod M') *for some $C \in I_R(M') - M'$ (where the idealizer subscript indicates the ring in which the idealizer is being computed).*

Proof. By Example 1.4, $(M'{:}X) = D(N{:}Xw)$; so by hypothesis $D(N{:}Xw) = D(N{:}Yw)$. The hypotheses also guarantee that the entries of Xw are in $I(N)$, and Lemma 2.2 then implies that the entries of Yw are in $I(N)$. Thus Proposition 2.3 implies the existence of $k \in I(N) - N$ such that $Xw \equiv Ywk \pmod{N}$.

Now for each $w_i \notin N$ there exist $y_i \in R$ and $n_i \in N$ such that $y_i w_i = 1 + n_i$. For each $i = 1, 2, \ldots, n$, define c_i to be 0 if $w_i \in N$ and $w_i k y_i$ if $w_i \notin N$. Let C be the diagonal matrix $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$. By direct computation,

$$Cw \equiv wk \pmod{N};$$

so

$$Xw \equiv Ywk \equiv YCw \pmod{N}.$$

Thus $(X - YC)w \in N^n$. We have $X - YC \in D(N:w) = M'$, and can conclude that $X \equiv YC \pmod{M'}$.

To show $C \in I_R(M')$, let $Z \in M' = D(N:w)$; we want $ZC \in M'$. But $ZCw \equiv Zwk \pmod{N}$, because $Cw \equiv wk \pmod{N}$; and $Zwk \equiv 0 \pmod{N}$, because $Zw \in N^n$ and $k \in I(N)$. Thus $Zc \in D(N:w) = M'$.

Finally, some $w_i \notin N$ and post-multiplication by the idealizer element k leaves the product $w_i k$ not in N. Thus $Cw \equiv wk \not\equiv 0 \pmod{N}$; that is $Cw \notin N^n$. So $C \notin D(N:w) = M'$.

COROLLARY 2.10. *If N is two-sided, then $(M':X) = (M':Y)$ if and only if $X \equiv YC \pmod{M'}$ for some $C \in I(M') - M'$.*

COROLLARY 2.11. *If R is a matrix ring over a commutative (or local or left quasi-duo) ring, then in R, $(M:u) = (M:v)$ if and only if $u \equiv vc \pmod{M}$ for some $c \in I(M) - M$.*

We conjecture from this that for any ring R and maximal left ideal M, $D(M:u) = D(M:v)$ in $M_n(R)$ if and only if $v \equiv uc \pmod{M}$ for some $c \in I(M) - M$.

Remark. If R is commutative, then $v \equiv uc \pmod{M}$ if and only if $u_i v_j \equiv u_j v_i \pmod{M}$ for all $i, j = 1, \ldots, n$. This can be interpreted as requiring that all 2×2 determinants $\begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix} \equiv 0 \pmod{M}$.

We can obtain a slight generalization involving this condition. Let R be commutative and let P be a prime ideal of R such that R/P is a valuation domain. Then in any $M_n(R)$, $D(P:u) = D(P:v)$ if and only if

$$u_i v_j \equiv u_j v_i \pmod{P} \text{ for all } i, j = 1, \ldots, n;$$

that is, if and only if $v \equiv uc \pmod{P}$ for some $c \in R - P$.

Example 2.12. In $M_2(\mathbf{Z})$ let $M = 5\mathbf{Z}$, $u = (2, 3)'$ and $v = (6, 24)'$. Then

$$\begin{vmatrix} 2 & 6 \\ 3 & 24 \end{vmatrix} \equiv 0 \pmod{5};$$

so $D(5\mathbf{Z}:u) = D(5\mathbf{Z}:v)$ in $M_2(\mathbf{Z})$.

Example 2.13. For K a commutative field, let $u = (c, d)'$ with $c \neq 0$ and $v = (1, c^{-1}d)'$. Then

$$\begin{vmatrix} c & 1 \\ d & c^{-1}d \end{vmatrix} = 0;$$

so $D(0:u) = D(0;v)$ in $M_2(K)$.

Let K be a commutative field. By the preceding example, the maximal left ideals of $M_2(K)$ are $D(0:u)$ for $u = (0, 1)'$ or $u = (1, c)'$, $c \in K$. Similarly, the maximal left ideals of $M_3(K)$ are indexed by $(0, 0, 1)'$, $(0, 1, a)'$ and $(1, b, c)'$ for $a, b, c \in K$. Thus $\mathrm{Max}(M_2(K))$ has $\mathrm{card}(K) + 1$ elements and can be naturally identified with the projective line $P^1(K)$ and $\mathrm{Max}(M_3(K))$ can be identified with the projective plane $P^2(K)$. Similarly $\mathrm{Max}(M_n(K))$ can be identified with $P^{n-1}(K)$. (This is analogous to the identification of $\mathrm{Max}(K[x])$ with K, for K an algebraically closed field.)

For R a commutative ring, M a maximal ideal of R, let $q_M = \mathrm{card}(R/M)$.

PROPOSITION 2.14. *Let R be commutative. Then $M_n(R)$ has $\sum_M \sum_{i=0}^{n-1} q_M{}^i$ maximal left ideals, where the outside sum is taken over $M \in$* Max (R).

Proof. If M and N are distinct maximal ideals of R, then by Corollary 1.7 the maximal left ideals of $M_n(R)$ lying over M are all distinct from those lying over N. And for fixed M, the function $M_n(R) \to M_n(R/M)$ reducing entries mod M sets up a one-to-one correspondence between those maximal left ideals of $M_n(R)$ lying over M and the maximal left ideals of $M_n(R/M)$ By the preceding remark $M_n(R/M)$ has $\sum_{i=0}^{n-1} q_M{}^i$ maximal left ideals.

Of course the sum above is infinite unless R is semi-local and each residual field is finite. In particular, if m is a positive integer then $\mathbf{Z}_m$ has one maximal ideal for each prime p dividing m (with residual field $\mathbf{Z}_p$); so $M_n(Z_m)$ has

$$\sum_{p|m} \sum_{i=0}^{n-1} p^i = \sum_{p|m} (p^n - 1)/(p - 1) = \sum_{p|m} \sigma(p^{n-1})$$

maximal left ideals (where σ is the sum of the divisors function and the outer sums are taken over the prime divisors of m).

3. Conjugate ideals. For p a unit of R let $i_p:R \to R$ be the inner (ring) automorphism $r \to prp^{-1}$. We say that two left ideals A and B of R are *conjugate* if $A = i_p(B) = pBp^{-1}$ for some unit p of R, and we then write $A \sim B$. This is certainly an equivalence relation on the collection of left ideals of R. If A and B are two-sided (e.g. if R is commutative) then $A \sim B$ if and only if $A = B$. If A and B are conjugate and one of them is maximal, so is the other. Since we are dealing with left ideals, $A \sim B$ if and only if $A = Bp$ for some unit p of R. Note that if M and N are conjugate maximal ideals, then R/M and R/N are isomorphic (simple) left R-modules.

Example 3.1. If K is a field and $D(0:e_i)$ the maximal left ideal of $M_n(K)$ having ith column zero, then for any i and j, $D(0:e_i) = D(0:e_j)\cdot P$ where P is the (invertible) $n \times n$ elementary matrix interchanging the ith and jth columns.

We want to investigate how conjugacy is propagated to matrix rings; i.e., if $M \sim N$ in R, is $D(M:u) \sim D(N:v)$ in $M_n(R)$? (Note that in the analogous situation for polynomial rings, if $A \sim B$ in R then $A[x] \sim B[x]$ in $R[x]$.) We also study the seemingly easier question: for a given maximal ideal M of R, are all $D(M:u)$ conjugate to one another in $M_n(R)$? If M satisfies this latter condition (for all $n \geqq 1$), we shall call M a c.p. *ideal.*

First we reduce the problem to working over a single maximal ideal.

LEMMA 3.2. *If M and N are conjugate maximal left ideals of R with $N = pMp^{-1}$ for p a unit of R and if $u \in R^n - N^n$, then $D(N:u) = D(M:up)$.*

Proof. Since $up \notin M^n$, $D(M:up)$ is proper and it suffices to show $D(N:u) \subseteq D(M:up)$. But if $X \in D(N:u)$, then each $X_iu \in N = Mp^{-1}$. Thus each $X_iup \in M$; so $X \in D(M:up)$.

The next easily-proved lemma is the basic tool for studying conjugacy of maximal ideals in $M_n(R)$.

LEMMA 3.3. *For $P \in GL_n(R)$, $P\cdot D(M:u)\cdot P^{-1} = D(M:Pu)$.*

Note that showing M c.p. is equivalent to showing that for any $u \in R^n - M^n$ we get $D(M:u) \sim D(M:e_1)$. To show this by writing $D(M:e_1) = P\cdot D(M:u)\,P^{-1} = D(M:Pu)$, it would thus be sufficient by Corollary 2.7 to show there exists $P \in GL_n(R)$ such that $P_1u \in I(M) - M$ and $P_iu \in M$ for $i \geqq 2$; i.e., to find an invertible matrix whose first row "pushes" u into the idealizer of M (but not into M) and whose other rows "push" u into M. On the other hand, to show conjugacy by writing

$$D(M:u) = P\cdot D(M:e_1)\cdot P^{-1} = D(M:Pe_1),$$

it would be sufficient to show that any u is congruent mod M to a column of an invertible matrix.

Recall that if $v \equiv uc \pmod{M}$ for some $c \in I(M) - M$, then $D(M:u) = D(M:v)$. Combining this with the previous lemma, we get the analogous result for conjugacy.

LEMMA 3.4. *Let $u, v \in R^n - M^n$. If $v \equiv Puc \pmod{M}$ for some $P \in GL_n(R)$ and $c \in I(M) - M$, then $D(M:u) \sim D(M:v)$.*

In particular, if v is a permutation of the entries of u, then $v = Pu$ with P the product of row-interchanging matrices, so $D(M:v) \sim D(M:u)$.

PROPOSITION 3.5. *If* (a) *some* $u_i \in I(M) - M$ *or* (b) *some* u_i *is congruent* mod M *to a unit of* R, *then* $D(M:u) \sim D(M:e_i)$.

Proof. By the preceding remark we may let $i = 1$ in either case.

(a) Let $b \in R$ and $m \in M$ be such that $bu_1 + m = 1$. Then for $i = 2, 3, \ldots, n$, we have

$$(u_ib)u_1 \equiv u_i \pmod M.$$

Let X be the $n \times n$ matrix having $(0, u_2b, u_3b, \ldots, u_nb)'$ as its first column and zeros elsewhere and let I denote the $n \times n$ identity matrix. Since $X^2 = 0$, $P = X + I$ is invertible (with inverse $I - X$). A direct computation shows that

$$Pe_1u_1 \equiv u \pmod M.$$

Thus $D(M:u) \sim D(M:e_1)$ by Lemma 3.4.

(b) Let u_1 be a unit of R and let $P \in GL_n(R)$ have u as its first column, the other diagonal elements unity and zeros elsewhere. Then $Pe_1 = u$; so $D(M:u) \sim D(M:e_1)$ by Lemma 3.4.

PROPOSITION 3.6. *If M and N are conjugate maximal left ideals of R and some u_i satisfies* (a) *or* (b) *above and some v_j satisfies* (a) *or* (b) *above* (*with respect to N*), *then* $D(M:u) \sim D(N:v)$.

Proof. Say $M = Np$ for a unit of R. By the preceding proposition and Lemma 3.2 it suffices to show $D(M:e_1) \sim D(N:e_1) = D(M:e_1p)$. Let $P = \operatorname{diag}(p, 1, 1, \ldots, 1)$ which is certainly invertible and satisfies $Pe_1 = e_1p$. Thus

$$D(N:e_1) = D(M:e_1p) = D(M:Pe_1),$$

which is conjugate to $D(M:e_1)$ by Lemma 3.3.

If M is a left ideal which is two-sided, then $I(M) = R$. We can then apply Proposition 3.5(a) to obtain a large class of c.p. ideals.

PROPOSITION 3.7. *Every two-sided maximal left ideal is* c.p.

Recall that $M \subseteq D(M:u)$ if and only if each $u_i \in I(M)$. Proposition 3.5(a) implies that for any maximal ideal M, all of the maximal left ideals of $M_n(R)$ which contain M are conjugate, even if M is not c.p.

Note that if M and N are two-sided non-conjugate (i.e., non-equal) maximal left ideals of R, then any proper $D(M:u)$ and $D(N:v)$ are non-conjugate in $M_n(R)$ (apply Lemma 1.6(d) and (a)). So, for example, if R is commutative (or just left quasi-duo), then $\operatorname{Max}(M_n(R))$ divides nicely into conjugacy classes and

$$\operatorname{Max}(M_n(R))/\sim \;= \operatorname{Max}(R).$$

COROLLARY 3.8. *If R is a local ring, then all maximal left ideals of $M_n(R)$ are conjugate.*

As a special case we recover the well-known result suggested in Example 3.1.

COROLLARY 3.9. *If K is a field, then all maximal left ideals of $M_n(K)$ are conjugate.*

The next proposition shows that the c.p. property propagates itself; this also shows that c.p. ideals do not have to be so nice as those found in Proposition 3.7 (i.e., two-sided).

PROPOSITION 3.10. *If $M \subseteq R$ is a* c.p. *ideal and $u \in R^n - M^n$, then $D(M:u)$ is a* c.p. *ideal of $M_n(R)$.*

Proof. By Example 1.5, in any $M_m(M_n(R)) \cong M_{mn}(R)$ we have

$$D(D(M:u) : U) = D(M:Uu) \sim D(M:Vu) = D(D(M:u) : V).$$

Thus if R is commutative or local or left quasi-duo, then every maximal left ideal of $M_n(R)$ is c.p. One consequence of this is that matrix rings over nice rings are themselves too nice to provide examples of non-c.p. ideals.

Corollary 3.8 suggests the question: What rings have all maximal left ideals conjugate? There is also the question which always suggests itself: If R has all maximal left ideals conjugate, does $M_n(R)$ have the same property? (More generally, is this a Morita property)? Noting that conjugacy (of maximal ideals) in R is equivalent to conjugacy in $R/J(R)$, it is straightforward to show that if R is left semi-perfect, then R has all maximal left ideals conjugate if and only if R is isomorphic to a matrix ring over a local ring. Perhaps this equivalence is always true without any initial assumptions on R. A matrix ring over a local ring also provides an example of R satisfying the second question.

A related question: Is $(M:u) \sim M$ for all $u \notin M$? (We're actually asking about M being c.p. at the $n = 1$ level.) Since $(M:u) = M$ for $u \in I(M) - M$, this is trivially true if M is two-sided. By the preceding proposition, any maximal ideal of a matrix ring arising from a c.p. ideal is itself c.p. Thus we can answer the question in the affirmative for some matrix rings.

PROPOSITION 3.11. *If R is a matrix ring over a left quasi-duo ring, then $(M:u) \sim M$ for any $M \in \mathrm{Max}(R)$, $u \notin M$.*

We conclude with one other non-commutative example. Let K be a division ring, let R be the left primitive ring of (square) countable column-finite matrices over K and let $K^{(N)}$ denote the direct sum of countably many copies of K. For $u = (u_1, u_2, \ldots)' \in K^{(N)}$, $u \neq 0$, let

$D(u) = \{X \in R | Xu = 0\}$. Then $D(u)$ is a maximal left ideal of R (known to Jacobson in 1946 [**3**, Section 3]). Moreover any $v \in R - D(u)$ is congruent mod $D(u)$ to a unit of R; so by Proposition 3.5 (b), $D(u)$ is a c.p. ideal. In addition, all $D(u)$, $u \neq 0$, are conjugate. That is, these are much like the ideals $D(0:u)$ in the finite matrix ring $M_n(K)$. However, these $D(u)$ are not the only maximal ideals of R. For

$$A' = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} | X \in M_n(K), n \geqq 1 \right\}$$

is a non-maximal left ideal of R that is not contained in any $D(u)$, $u \neq 0$. Furthermore, no maximal left ideal containing A' is conjugate to any $D(u)$; so R does not have all of its maximal left ideals conjugate.

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