This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P. Q.

A NOTE ON AN OSCILLATION CRITERION FOR AN EQUATION WITH A FUNCTIONAL ARGUMENT

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1. It might be thought that, as far as the oscillation of solutions is concerned, the behaviour of

$$y''(t) + a(t)y(t) = 0$$

and

$$y''(t) + a(t)y(t - \alpha(t)) = 0$$

would be the same as long as $t - \alpha(t) \to \infty$ as $t \to \infty$. To motivate the theorem presented in this note, we show first that this is not the case. Consider the above equation with $\alpha(t) = 3t/4$, $a(t) = 1/2t^2$, i.e.

$$y''(t) + \frac{1}{2t^2} y(t/4) = 0.$$

This equation has the non-oscillatory solution $y(t) = t^{1/2}$ although all solutions of

$$y''(t) + \frac{1}{2t^2} y(t) = 0$$

are oscillatory [1, p. 121].

2. In this section we consider the differential equation

$$y''(t) + a(t)\, f(y(t),\ y(g(t))) = 0,$$

where $\lim_{t \to \infty} g(t) = +\infty$. A solution, not eventually identically zero, is said to be oscillatory if it has zeros for arbitrarily large t. We prove below a simple theorem which shows that one of the standard "strong" sufficient conditions for oscillation of all solutions of equations without the functional argument remains valid. In the following section we modify the above example to show that for the problem considered this is not a strong condition.

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Concerning the coefficients in (1) we assume:

(a) $a(t) \geq 0$;

(b) $\lim_{t \to \infty} g(t) = +\infty$;

(c) $f(y, w)$ is non-decreasing in y and w;

(d) when y and w are of one sign, f has that sign.

THEOREM. Let (a) - (d) hold. If $\int^{\infty} a(t)dt = +\infty$, all solutions of (1), valid for large t, are oscillatory.

Proof. Let $y(t)$ be a non-oscillatory solution of (2) valid on (a, ∞). Then $y(t)$ is eventually of one sign; we assume $y(t) > 0$ for $t \geq t_0$ (a similar proof follows if $y(t) < 0$ for $t \geq t_0$). Choose t_1 such that $g(t) > t_0$ for $t > t_1$. For $t > t_1$, $y(t)$ is concave downward, for, by (a) and (d),

$$y''(t) = -a(t)f(y(t), y(g(t))) < 0.$$

If $y'(t) \leq 0$ for any $t > t_1$, it follows immediately that $y(t)$ has a zero in (t_0, ∞), and hence it may be assumed that $y'(t) > 0$ for $t > t_1$.

Choose t_2 such that $g(t) > t_1$ for $t > t_2$. Then, in view of the monotonicity of y and f, $f(y(t_1), y(t_1)) \leq f(y(t), y(g(t)))$ for $t > t_2$, or $y(t)$ satisfies

$$y''(t) + a(t)f(y(t_1), y(t_1)) \leq 0.$$

An integration yields

$$y'(t) - y'(t_2) + f(y(t_1), y(t_1)) \int_{t_2}^{t} a(s)ds \leq 0$$

which, in view of the integral condition, produces a contradiction.

3. In this section we modify the example to show that the integral condition is not as severe in the case considered here as it is without the modified argument. Consider the equation

$$y''(t) + t^{-(1+\gamma)} y(g(t)) = 0 .$$

For small $\gamma > 0$, choose $\alpha < 1$ such that $\alpha + \gamma - 1 > 0$ and take

$$g(t) = \left[(1 - \alpha)\alpha t^{\alpha+\gamma-1} \right]^{1/\alpha} .$$

$\lim_{t \to \infty} g(t) = + \infty$, and $y(t) = t^{\alpha}$ is a non-oscillatory solution.

REFERENCE

1. R. Bellman, Stability Theory of Differential Equations. (McGraw Hill, 1953).

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