FUBINI THEOREM FOR A CERTAIN TYPE OF INTEGRAL

CHARLES A. HAYES, JR.

1. Introduction. In **(1)**, the writer defined a process of integration that leads to a kind of Riemann integral under certain rather general conditions. The purpose of this paper is to show how it is possible to use the process of integration of **(1)** to obtain integrals in a product space that satisfy a Fubini theorem. In this connection, we define a class of integrands that are the analogues of continuous functions in the product space, establish some of their properties, and finally arrive at a Fubini theorem for this class.

2. Notation, terminology, and background material. If $\mathbf{F}$ is a family of sets, we let $\cup\mathbf{F}$ denote its *union*, i.e., the set of points belonging to at least one member of $\mathbf{F}$, and we let $\cap\mathbf{F}$ denote its intersection, i.e., the set of points belonging to each member of $\mathbf{F}$. In case $\mathbf{F}_\beta$ is a set for each β in an index set B, we let $\cup_{\beta\in B}\mathbf{F}_\beta$ and $\cap_{\beta\in B}\mathbf{F}_\beta$ denote, respectively, the union and intersection of the appropriate family. We say that a family $\mathbf{F}$ *covers* a set A if and only if $A \subset \cup\,\mathbf{F}$. If A and B are sets, then by $A - B$ we mean the set of those points that are in A but not in B. We denote the empty set by $\emptyset$.

We allow real-valued functions to take on the values $+\infty$ or $-\infty$. We agree that $0(+\infty) = 0(-\infty) = 0$ and $c(+\infty) = +\infty$, $c(-\infty) = -\infty$ if $c > 0$ (the signs are reversed if $c < 0$). If A is a set of real numbers, we denote by $\sup A$ and $\inf A$ the supremum and infimum, respectively, of A. We keep in mind that $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$. If F is a real-valued function and A is a subset of its domain, we use the notations $\sup_{x\in A} F(x)$ and $\inf_{x\in A} F(x)$ to denote, respectively, the supremum and infimum of the values taken by F on the set A. We note that empty sums are zero.

We say that Ω is an *outer measure on* S if and only if the domain of Ω is the set of all subsets of S and

$$0 \leqslant \Omega(A) \leqslant \textstyle\sum_{\beta\in\mathbf{F}}\Omega(\beta)$$

whenever $\mathbf{F}$ is a finite or countably infinite family and $A \subset \cup\,\mathbf{F} \subset S$. This is equivalent to the usual definition, namely $\Omega(\emptyset) = 0$; $\Omega(A) \leqslant \Omega(B)$ whenever $A \subset B \subset S$; and $\Omega(\cup\,\mathbf{F}) \leqslant \sum_{\beta\in\mathbf{F}}\Omega(\beta)$ whenever $\mathbf{F}$ is a finite or countably infinite class of subsets of S.

We say that the set $A \subset S$ is Ω-*measurable* if and only if Ω is an outer measure on S and $\Omega(E) = \Omega(E \cap A) + \Omega(E - A)$ whenever $E \subset S$. It is well known that if Ω is an outer measure on S, then the class of all Ω-measurable sets is a

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σ-algebra that includes all sets E for which $\Omega(E) = 0$ (in particular, the null set) and their complements (in particular, S); cf. (**2**, p. 87).

If $\mathbf{F}_0$ is any non-empty family of subsets of a given set S, and g is any non-negative function whose domain is $\mathbf{F}_0$, then for any set $A \subset S$ we let $\mathfrak{M}(\mathbf{F}_0; A)$ denote the family of all finite or countably infinite subfamilies $\mathbf{G}$ of $\mathbf{F}_0$ that cover A ($\mathfrak{M}(\mathbf{F}_0; A)$ may happen to be empty), and define the function $\bar{g}$ so that

$$\bar{g}(A) = \inf_{\mathbf{G} \in \mathfrak{M}(\mathbf{F}_0;\, A)} \sum_{\beta \in \mathbf{G}} g(\beta).$$

It is well known that $\bar{g}$ is an outer measure on S (**2**, pp. 90–91).

The technique just described is used in (**1**) to define a kind of integral, as follows. Taking a non-empty family $\mathbf{F}_0$ of subsets of S as above, we assume that $\mathfrak{M}(\mathbf{F}_0; S) \neq \emptyset$ and consider a finite-valued, non-negative-valued function ϕ defined on $\mathbf{F}_0$. For any function f bounded and non-negative on S, we define f^* on $\mathbf{F}_0$ so that $f^*(\beta) = \phi(\beta) \sup_{x \in \beta} f(x)$ whenever $\emptyset \neq \beta \in \mathbf{F}_0$, and $f^*(\emptyset) = 0$ if $\emptyset \in \mathbf{F}_0$. Subjecting f^* to the process described in the preceding paragraph, we obtain, in conformity with the notation there employed, an outer measure $\bar{f}^*$ on S. We call this outer measure the *integral* of f and denote its value on an arbitrary set $A \subset S$ by $\int_A f$. For unbounded functions that are non-negative throughout S, we use the chopped-off functions $f^{(n)}$, $n = 1, 2, \ldots$ and define $\int_A f = \lim_n \int_A f^{(n)}$; this leads again to an outer measure on S. Thus defined, each such integral defines its own class of $\int f$-measurable subsets of S. We let $\mathbf{S}_\phi$ denote the intersection of all these classes; evidently each of these integrals is completely additive on $\mathbf{S}_\phi$, which is itself a completely additive class of subsets of S. However, at this point we can be sure only that $\mathbf{S}_\phi$ includes $\emptyset$ and S as members. For functions of variable sign on S we introduce the positive and negative parts of f, namely f_+ and f_-, and define $\int_A f = \int_A f_+ - \int_A f_-$ whenever $A \subset S$ and at least one term on the right side is finite. If both are finite when $A = S$, we say that f is *ϕ-summable.* An integral that plays a special role in this theory is $\int K_S$, where K_S is the characteristic function of S. We denote this integral by $\bar{\phi}$ and its value for any set $A \subset S$ by $\bar{\phi}(A)$. This is in conformity with the notation of the preceding paragraph.

Without further restrictions on $\mathbf{F}_0$ and ϕ, the integrals just defined have very few useful properties. We therefore assume the following: Whenever $\alpha \in \mathbf{F}_0$ and $\beta \in \mathbf{F}_0$, then $(\alpha - \beta) \in \mathbf{F}_0$ (and hence $\alpha \cap \beta \in \mathbf{F}_0$); also

$$\phi(\alpha) \geqslant \phi(\alpha \cap \beta) + \phi(\alpha - \beta).$$

This restriction ensures that $\mathbf{S}_\phi$ contains $\mathbf{S}(\mathbf{F}_0)$, the smallest σ-algebra of sets containing $\mathbf{F}_0$; in fact $\mathbf{S}_\phi$ consists of all sets of the form $D - N$, where $D \in \mathbf{S}(\mathbf{F}_0)$ and $\bar{\phi}(N) = 0$. Thus our integrals are all completely additive on a large family of sets. It turns out (**1**, §5) that finite additivity with respect to the integrand is ensured for a class of functions possessing a kind of continuity property. Thus the integrals behave like a kind of Riemann integral in this respect. In order to achieve the more general Lebesgue theorems, $\mathbf{F}_0$ and ϕ have to satisfy

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additional conditions, weaker than the usual ones, however. These matters are considered in **(1**, §6**)**.

3. A kind of product measure. For the remainder of this paper we assume that S and T are fixed sets, $\mathbf{F}_0$ and $\mathbf{G}_0$ are non-empty classes of subsets of S and T respectively, ϕ and ψ are finite-valued functions non-negative on $\mathbf{F}_0$ and $\mathbf{G}_0$, respectively, such that whenever $\alpha \in \mathbf{F}_0$, $\beta \in \mathbf{F}_0$, $\alpha' \in \mathbf{G}_0$, and $\beta' \in \mathbf{G}_0$, the following hold:

(i) $(\alpha - \beta) \in \mathbf{F}_0$ (and hence $\alpha \cap \beta \in \mathbf{F}_0$);

$$\phi(\alpha) \geqslant \phi(\alpha \cap \beta) + \phi(\alpha - \beta);$$

(ii) $(\alpha' - \beta') \in \mathbf{G}_0$ (and hence $\alpha' \cap \beta' \in \mathbf{G}_0$);

$$\psi(\alpha') \geqslant \psi(\alpha' \cap \beta') + \psi(\alpha' - \beta').$$

We also assume that $\mathfrak{M}(\mathbf{F}_0; S) \neq \emptyset$, $\mathfrak{M}(\mathbf{G}_0; T) \neq \emptyset$. Since $\mathbf{G}_0$ and ψ satisfy the same kind of conditions imposed upon $\mathbf{F}_0$ and ϕ at the end of §2, then the class $\mathbf{S}_\psi$ and integrals defined with respect to ψ and $\mathbf{G}_0$ have the same properties as $\mathbf{S}_\phi$ and integrals with respect to ϕ and $\mathbf{F}_0$ described at the end of §2.

We define the product space $U = S \times T$ and the family

$$\mathbf{H}_0 = \{\gamma | \gamma = \alpha \times \beta \text{ for some } \alpha \in \mathbf{F}_0 \text{ and some } \beta \in \mathbf{G}_0\}$$

of subsets of U. Clearly $\mathfrak{M}(\mathbf{H}_0; U) \neq \emptyset$. We define the function μ on $\mathbf{H}_0$ so that $\mu(A \times B) = \phi(A)\psi(B)$ whenever $A \times B \in \mathbf{H}_0$. In this section we shall define the measure $\bar{\mu}$ on U in terms of μ and $\mathbf{H}_0$ in the same manner as $\bar{\phi}$ and $\bar{\psi}$ were defined in terms of ϕ and $\mathbf{F}_0$, ψ and $\mathbf{G}_0$, respectively by the procedure of §2. We propose to establish a key relationship between $\bar{\mu}$, $\bar{\phi}$, and $\bar{\psi}$. Since μ and $\mathbf{H}_0$ do not satisfy relations corresponding to (i) or (ii), we cannot take over directly all the results found in **(1)**, but certain similar results emerge in due course, as will be seen.

3.1. LEMMA. *If $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $C = A \times B$, $\mathbf{F}$ and $\mathbf{G}$ are disjoint finite or countably infinite subfamilies of $\mathbf{F}_0$ and $\mathbf{G}_0$, respectively, such that $A = \cup \mathbf{F}$, $B = \cup \mathbf{G}$,*

$$\mathbf{H} = \{\gamma | \gamma = \alpha \times \beta \textit{ for some } \alpha \in \mathbf{F} \textit{ and some } \beta \in \mathbf{G}\},$$

and f is real-valued, bounded, and non-negative on U, then $f^*(C) \geqslant \sum_{\gamma \in \mathbf{H}} f^*(\gamma)$. In particular, $\mu(C) \geqslant \sum_{\gamma \in \mathbf{H}} \mu(\gamma)$.

Proof. From **(**1, Lemma 2.2**)** it follows that

$$(1) \qquad \phi(A) \geqslant \textstyle\sum_{\alpha \in \mathbf{F}} \phi(\alpha), \qquad \psi(B) \geqslant \textstyle\sum_{\beta \in \mathbf{G}} \psi(\beta).$$

The statement of this lemma is obviously true if $A \times B = \emptyset$, because of our definition of f^* in §2. Thus we assume $A \times B \neq \emptyset$, and we obtain from (1):

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(2) $$f^*(C) = \mu(A \times B) \sup_{z \in C} f(z) = \phi(A)\psi(B) \sup_{z \in C} f(z)$$
$$\geqslant \Big(\sum_{\alpha \in \mathbf{F}} \phi(\alpha)\Big)\Big(\sum_{\beta \in \mathbf{G}} \psi(\beta)\Big) \sup_{z \in C} f(z)$$
$$= \sum_{\alpha \times \beta \in \mathbf{H}} \phi(\alpha)\psi(\beta) \sup_{z \in C} f(z) = \sum_{\gamma \in \mathbf{H}} \mu(\gamma) \sup_{z \in C} f(z).$$

Now for any set $\alpha \times \beta \neq \emptyset$ contained in $A \times B$, we have

$$\mu(\alpha \times \beta) \sup_{z \in (A \times B)} f(z) \geqslant \mu(\alpha \times \beta) \sup_{z \in (\alpha \times \beta)} f(z) = f^*(\alpha \times \beta);$$

if $\alpha \times \beta = \emptyset$, then $f^*(\alpha \times \beta) = 0$ by definition. Hence from (2) we see that

$$f^*(C) \geqslant \textstyle\sum_{\gamma \in \mathbf{H}} f^*(\gamma).$$

If we take the function $f = K_U$, the characteristic function of U, we obtain the special case $\mu(C) \geqslant \sum_{\gamma \in \mathbf{H}} \mu(\gamma)$.

The next lemma expresses some facts that we shall have occasion to use repeatedly later on.

3.2. Lemma. *If f is a function non-negative on S and $\mathbf{H}$ is a finite subfamily of $\mathbf{H}_0$, then there exist finite disjoint subfamilies $\mathbf{F}' \subset \mathbf{F}_0$, $\mathbf{G}' \subset \mathbf{G}_0$, and $\mathbf{H}' \subset \mathbf{H}_0$ such that*

(i) $\mathbf{H}' = \{\gamma' | \gamma' = \alpha' \times \beta'$ *for some* $\alpha' \in \mathbf{F}'$ *and some* $\beta' \in \mathbf{G}'\}$;

(ii) *each member of $\mathbf{H}$ is the union of a finite subfamily of $\mathbf{H}'$; more specifically, if $\gamma \in \mathbf{H}$ and α and β are such members of $\mathbf{F}_0$ and $\mathbf{G}_0$, respectively, that $\gamma = \alpha \times \beta$, then there exist families $\mathbf{F}_\gamma'' \subset \mathbf{F}'$, $\mathbf{G}_\gamma'' \subset \mathbf{G}'$, and $\mathbf{H}_\gamma'' \subset \mathbf{H}'$ such that $\alpha = \bigcup \mathbf{F}_\gamma''$, $\beta = \bigcup \mathbf{G}_\gamma''$, and $\gamma = \bigcup \mathbf{H}_\gamma''$, where*

$$\mathbf{H}_\gamma'' = \{\gamma' | \gamma' = \alpha' \times \beta' \text{ for some } \alpha' \in \mathbf{F}_\gamma'' \text{ and some } \beta' \in \mathbf{G}_\gamma''\};$$

(iii) $\bigcup \mathbf{H} = \bigcup \mathbf{K}'$, *where* $\mathbf{K}' = \bigcup_{\gamma \in \mathbf{H}} \mathbf{H}_\gamma'' \subset \mathbf{H}'$;

(iv) $\sum_{\gamma \in \mathbf{H}} f^*(\gamma) \geqslant \sum_{\gamma \in \mathbf{K}'} f^*(\gamma')$; *in particular,*

$$\textstyle\sum_{\gamma \in \mathbf{H}} \mu(\gamma) \geqslant \sum_{\gamma' \in \mathbf{K}'} \mu(\gamma');$$

(v) *if $\bigcup \mathbf{H} \subset C \in \mathbf{H}_0$, then the families $\mathbf{F}'$ and $\mathbf{G}'$ may be so chosen that $C - \bigcup \mathbf{H}$ is the union of a finite subfamily of $\mathbf{H}'$, specifically, $\mathbf{H}' - \mathbf{K}'$.*

Proof. We arrange the members of $\mathbf{H}$ in a finite sequence Γ with n different values and determine corresponding sequences A and B with values in $\mathbf{F}_0$ and $\mathbf{G}_0$, respectively, such that $\Gamma_i = A_i \times B_i$, $i = 1, 2, \ldots, n$. Next we consider the totality of sets (at most 2^n) obtained from

(1) $$V_1 \cap V_2 \cap \ldots \cap V_n$$

by replacing $V_1, V_2, \ldots, V_n$, by A_1 or $S - A_1, A_2$ or $S - A_2, \ldots, A_n$ or $S - A_n$, respectively. It is clear that the resulting sets are pairwise disjoint and comprise a finite family $\mathbf{F}' \subset \mathbf{F}_0$. In similar fashion, replacing $V_1, V_2, \ldots, V_n$ by B, or $T - B_1, B_2$ or $T - B_2, \ldots, B_n$ or $T - B_n$, respectively, we obtain a finite disjoint family $\mathbf{G}' \subset \mathbf{G}_0$. It follows easily that (i), (ii), and (iii) are true. Also, by virtue of (ii) and Lemma 3.1, we see that for each set $\gamma \in \mathbf{H}$, $f^*(\gamma) \geqslant \sum_{\gamma'' \epsilon \mathbf{H}_{\gamma''}} f^*(\gamma'')$, and (iv) follows immediately.

To prove (v), we write $C = A \times B$, where $A \in \mathbf{F}_0$ and $B \in \mathbf{G}_0$, and intersect (1) with A in defining $\mathbf{F}'$, then with B in defining $\mathbf{G}'$. This modification yields the desired result.

3.3. Lemma. *If f is a function non-negative on S and $\mathbf{H}$ is a countably infinite subfamily of $\mathbf{H}_0$, then there exists a finite or countably infinite disjoint subfamily $\mathbf{H}'$ of $\mathbf{H}_0$ such that each member of $\mathbf{H}'$ is contained in exactly one member of $\mathbf{H}$, $\cup\mathbf{H}' = \cup\mathbf{H}$, and*

$$\sum_{\gamma \epsilon \mathbf{H}} f^*(\gamma) \geqslant \sum_{\gamma' \epsilon \mathbf{H}'} f^*(\gamma').$$

In particular,

$$\sum_{\gamma \epsilon \mathbf{H}} \mu(\gamma) \geqslant \sum_{\gamma' \epsilon \mathbf{H}'} \mu(\gamma').$$

Proof. We arrange $\mathbf{H}$ in the form of a sequence Γ and consider the disjoint sequence of sets A such that

$$A_1 = C_1, \qquad A_{n+1} = C_{n+1} - (C_1 \cup \ldots \cup C_n)$$

for each positive integer n. Since for each such n

$$C_{n+1} - (C_1 \cup \ldots \cup C_n) = C_{n+1} - ((C_1 \cap C_{n+1}) \cup \ldots \cup (C_n \cap C_{n+1}))$$

and since $C_1 \cap C_{n+1}, \ldots, C_n \cap C_{n+1}$ comprise a finite subfamily of $\mathbf{H}_0$ whose union is contained in C_1, we may apply part (v) of Lemma 3.2 to see that A_{n+1} may be represented as the union of a finite disjoint family $\mathbf{H}_{n+1} \subset \mathbf{H}_0$ with $\cup\mathbf{H}_{n+1} \subset C_{n+1}$ such that $f^*(C_{n+1}) \geqslant \sum_{\gamma \epsilon \mathbf{H}_{n+1}} f^*(\gamma)$. We let $\mathbf{H}_1$ denote the family consisting of the set C_1, define $\mathbf{H}' = \bigcup_{n=1}^{\infty} \mathbf{H}_n$, and note that $\mathbf{H}'$ satisfies the conditions required above.

3.4. Lemma. *$\mathbf{S}_\mu$ contains the smallest σ-algebra that includes $\mathbf{H}_0$.*

Proof. We consider an arbitrary function f non-negative on U. It will be sufficient to show that the class of $\bar{f}^*$-measurable sets includes $\mathbf{H}_0$, for then it must include the smallest σ-algebra containing $\mathbf{H}_0$; and so then will $\mathbf{S}_\mu$.

We take an arbitrary set $E \subset U$ and an arbitrary set $C \in \mathbf{H}_0$; there exist sets $A \in \mathbf{F}_0$ and $B \in \mathbf{G}_0$ such that $C = A \times B$. Next we select an arbitrary family $\mathbf{H} \in \mathfrak{M}(\mathbf{H}_0; E)$; for any member $\gamma \in \mathbf{H}$, we select sets $\alpha \in \mathbf{F}_0$, $\beta \in \mathbf{G}_0$ such that $\gamma = \alpha \times \beta$. We now let

$$\gamma_1 = (\alpha \cap A) \times (\beta \cap B), \qquad \gamma_2 = (\alpha \cap A) \times (\beta - B),$$

$$\gamma_3 = (\alpha - A) \times (\beta \cap B), \qquad \gamma_4 = (\alpha - A) \times (\beta - B).$$

Clearly these four sets are disjoint members of $\mathbf{H}_0$, $\gamma_1 = \gamma \cap C$, and $\gamma_2 \cup \gamma_3 \cup \gamma_4 = \gamma - C$. We let $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$ denote, respectively, the families of all sets γ_1, γ_2, γ_3, and γ_4 thus obtained from all sets $\gamma \in \mathbf{H}$. From our observations, we see that

$$\mathbf{H}_1 \in \mathfrak{M}(\mathbf{H}_0; E \cap C), \qquad (\mathbf{H}_2 \cup \mathbf{H}_3 \cup \mathbf{H}_4) \in \mathfrak{M}(\mathbf{H}_0; E - C).$$

Also, because of Lemma 3.1, we have

$$f^*(\gamma) \geqslant f^*(\gamma_1) + f^*(\gamma_2) + f^*(\gamma_3) + f^*(\gamma_4)$$

whenever $\gamma \in \mathbf{H}$. Hence

$$\sum_{\gamma \in \mathbf{H}} f^*(\gamma) \geqslant \sum_{\gamma_1 \in \mathbf{H}_1} f^*(\gamma_1) + \Big(\sum_{\gamma_2 \in \mathbf{H}_2} f^*(\gamma_2) + \sum_{\gamma_3 \in \mathbf{H}_3} f^*(\gamma_3) + \sum_{\gamma_4 \in \mathbf{H}_4} f^*(\gamma_4)\Big)$$
$$\geqslant \bar{f}^*(E \cap C) + \bar{f}^*(E - C).$$

From the arbitrary nature of $\mathbf{H}$ in this inequality, we infer that $\bar{f}^*(E) \geqslant \bar{f}^*(E \cap C) + \bar{f}^*(E - C)$. The reverse inequality holds since $\bar{f}^*$ is an outer measure on U, and so C is $\bar{f}^*$-measurable.

3.5. Theorem. *$\mathbf{S}_\mu$ consists of all sets of the form $D - N$ where D belongs to the smallest σ-algebra containing $\mathbf{H}_0$ and N is a set such that $\bar{\mu}(N) = 0$.*

Proof. This follows exactly the lines of proof of (**1**, Theorem 4.2), using Lemma 3.4 above, and consequently is not given here.

3.6. Theorem. *If $A \in \mathbf{F}_0$ and $B \in \mathbf{G}_0$, then $\bar{\mu}(A \times B) = \bar{\phi}(A)\,\bar{\psi}\cdot(B)$.*

Proof. Given $\epsilon > 0$, we select such families $\mathbf{F} \in \mathfrak{M}(\mathbf{F}_0; A)$ and $\mathbf{G} \in \mathfrak{M}(\mathbf{G}_0; B)$ that

$$(1) \qquad \textstyle\sum_{\alpha \in \mathbf{F}} \phi(\alpha) \leqslant \bar{\phi}(A) + \epsilon, \qquad \sum_{\beta \in \mathbf{G}} \psi(\beta) \leqslant \bar{\psi}(B) + \epsilon,$$

let $\mathbf{H} = \{\gamma | \gamma = \alpha \times \beta$ for some $\alpha \in \mathbf{F}$ and some $\beta \in \mathbf{G}\}$, note that $\mathbf{H} \in \mathfrak{M}(\mathbf{H}_0; A \times B)$, and with the help of (1) observe that

$$\bar{\mu}(A \times B) \leqslant \sum_{\gamma \in \mathbf{H}} \mu(\gamma) = \sum_{(\alpha \times \beta) \in \mathbf{H}} \phi(\alpha)\psi(\beta)$$
$$= \Big(\sum_{\alpha \in \mathbf{F}} \phi(\alpha)\Big)\Big(\sum_{\beta \in \mathbf{G}} \psi(\beta)\Big) \leqslant (\bar{\phi}(A) + \epsilon)(\bar{\bar{\psi}}(B) + \epsilon).$$

From this relation we infer that $\bar{\mu}(A \times B) \leqslant \bar{\phi}(A) \cdot \bar{\psi}(B)$.

Since $\bar{\phi}$ and $\bar{\psi}$ are completely additive on the σ-algebras $\mathbf{S}_\phi$ and $\mathbf{S}_\psi$, respectively, then there exists a function χ defined and completely additive on a σ-algebra that includes $\mathbf{H}_0$, and satisfies the relation $\chi(\alpha \times \beta) = \bar{\phi}(\alpha)\bar{\psi}(\beta)$ whenever $\alpha \in \mathbf{S}_\phi$ and $\beta \in \mathbf{S}_\psi$ (**3**, pp. 223–234); by virtue of Lemma 3.4, this holds in particular whenever $\alpha \in \mathbf{F}_0$ and $\beta \in \mathbf{G}_0$.

Given any number $\epsilon > 0$, we select a family $\mathbf{H} \in \mathfrak{M}(\mathbf{H}_0; A \times B)$ such that

$$(2) \qquad \textstyle\sum_{\gamma \in \mathbf{H}} \mu(\gamma) \leqslant \bar{\mu}(A \times B) + \epsilon.$$

Owing to Lemma 3.3, there is no loss of generality in assuming that $\mathbf{H}$ is a disjoint family. By intersecting each member of $\mathbf{H}$ with $A \times B$, we obtain a family belonging to $\mathfrak{M}(\mathbf{H}_0; A \times B)$, each of whose members is a subset of $A \times B$. Since, by virtue of Lemma 3.2, $\mu(\alpha \cap \beta) \leqslant \mu(\alpha)$ holds whenever $\alpha \in \mathbf{H}_0$ and $\beta \in \mathbf{H}_0$, it follows that the inequality corresponding to (1) holds for the new family. Thus we may assume without loss of generality that the members of $\mathbf{H}$ are all subsets of $A \times B$, so that $A \times B = \cup\mathbf{H}$.

From (2) and the countable additivity of χ, we now see that

$$\bar{\phi}(A)\bar{\psi}(B) = \chi(A \times B) = \sum_{\gamma\epsilon\mathbf{H}} \chi(\gamma) = \sum_{\alpha\times\beta\epsilon\mathbf{H}} \bar{\phi}(\alpha)\bar{\psi}(\beta) \leqslant \sum_{\alpha\times\beta\epsilon\mathbf{H}} \phi(\alpha)\psi(\beta)$$

$$= \sum_{\gamma\epsilon\mathbf{H}} \mu(\gamma) \leqslant \bar{\mu}(A \times B) + \epsilon.$$

Hence $\bar{\phi}(A)\,\bar{\psi}(B) \leqslant \bar{\mu}(A \times B)$ and the proof is complete.

4. Some properties of a general kind of continuity. In this section we shall consider classes of functions defined on the spaces S, T, and U that have properties analogous to functions that are continuous almost everywhere on an interval on the real line. Some properties of these functions were established in (**1**, §5). We shall give a definition of this kind of continuity in terms of a space V and a function ν that may be thought of as representing S, T, or U and ϕ, ψ, or μ, respectively. If $D \subset V$, we shall write $\mathfrak{M}(D)$ to stand for $\mathfrak{M}(\mathbf{F}_0; D)$, $\mathfrak{M}(\mathbf{G}_0; D)$, or $\mathfrak{M}(\mathbf{H}_0; D)$, according to whether V stands for S, T, or U, respectively.

We shall be considering functions f defined on U, integrals of such functions over portions of U, and also iterated integrals of these functions. To keep matters straight, we shall introduce dummy variables and write, for example, $\int_C f(x, y)\, d\mu(x, y)$ to denote the integral of f with respect to μ over a set $C \subset U$. Iterated integrals over a set $C = A \times B \subset U$ will be expressed by $\int_A(\int_B f(x, y)d\phi(x))d\psi(y)$ or a similar expression with variables reversed. Also, when we have occasion to consider the function obtained from f by holding y fixed in T, we shall denote it by $f(\ , y)$; its value at a point $x \in S$ will be $f(x, y)$. Similarly, $f(x,\ )$ will denote the function obtained from f when x is held fixed in S.

4.1. *Definition.* If f is a function real-valued on a set β, then we define the *oscillation of f on β*, written $\Omega(f, \beta)$, by

$$\Omega(f, \beta) = \sup_{x\epsilon\beta} f(x) - \inf_{x\epsilon\beta} f(x) \qquad \text{if } \beta \neq \emptyset;$$

$$\Omega(f, \beta) = 0 \text{ if } \beta = \emptyset.$$

4.2. *Definition.* If $D \subset V$, then we agree to denote by $\mathbf{C}(D)$ the class of all functions f real-valued and bounded on D with the following property: for each $\epsilon > 0$, there exists a family $\mathbf{K} \in \mathfrak{M}(D)$ with a finite subfamily $\mathbf{K}'$ such that (i) $\Omega(f, D \cap \beta) < \epsilon$ whenever $\beta \in \mathbf{K}'$; (ii) $\sum_{\beta\epsilon(\mathbf{K}-\mathbf{K}')} \nu(\beta) < \epsilon$.

Owing to (**1**, Lemma 2.2), Lemma 3.2 of this paper, and the fact that the oscillation of a function on a given set is never less than its oscillation on a

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subset, it follows that the family $\mathbf{K}$ occurring in Definition 4.2 may always be taken as disjoint. Also, if D belongs to $\mathbf{F}_0$, $\mathbf{G}_0$, or $\mathbf{H}_0$ as the case may be, the members of $\mathbf{K}$ may be taken as subsets of D, since each member of $\mathbf{K}$ can be intersected with D to obtain a family of the required kind. We shall frequently use these facts.

4.3. LEMMA. *If $A \in \mathbf{F}_0$, g is a real-valued function bounded on A, and $g \notin \mathbf{C}(A)$, then there exists $\epsilon > 0$ such that if $\mathbf{F}$ is any finite subfamily of $\mathbf{F}_0$ with $\cup \mathbf{F} \subset A$ and $\Omega(g, \alpha) < \epsilon$ whenever $\alpha \in \mathbf{F}$, then $\bar{\phi}(A - \cup\mathbf{F}) \geqslant \epsilon$.*

Proof. Since $g \notin \mathbf{C}(A)$, it follows that there exists $\epsilon > 0$ such that whenever $\mathbf{K} \in \mathfrak{M}(A)$ and $\mathbf{K}'$ is a finite subfamily of $\mathbf{K}$ for which $\Omega(g, \alpha) < \epsilon$ whenever $\alpha \in \mathbf{K}'$, then

$$(1) \qquad \sum_{\alpha \in (\mathbf{K}-\mathbf{K}')} \phi(\alpha) \geqslant \epsilon.$$

We take an arbitrary family $\mathbf{F}$ satisfying our hypotheses and let $K = \cup\mathbf{F}$, pick an arbitrary family $\mathbf{G} \in \mathfrak{M}(A - K)$, define $\mathbf{K} = \mathbf{G} \cup \mathbf{F}$, note that $\mathbf{K} \in \mathfrak{M}(A)$, observe with the help of (1) that

$$\sum_{\alpha \in \mathbf{G}} \phi(\alpha) \geqslant \sum_{\alpha \in (\mathbf{K}-\mathbf{F})} \phi(\alpha) \geqslant \epsilon,$$

and so conclude that $\bar{\phi}(A - K) \geqslant \epsilon$, as required.

4.4. THEOREM. *If $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $C = A \times B$, and $f \in \mathbf{C}(C)$, then*

(i) *$f(\ ,y) \in \mathbf{C}(A)$ for $\bar{\psi}$-almost all $y \in B$ and $f(x,\ ) \in \mathbf{C}(B)$ for $\bar{\phi}$-almost all $x \in A$;*

(ii) *$\int_A f(x,\ )d\phi(x) \in \mathbf{C}(B)$ and $\int_B f(\ ,y)d\psi(y) \in \mathbf{C}(A)$.*

Proof. (i) We shall prove only the first part of (i), since the second part will then be obvious.

We let $E = \{y | y \in B \text{ and } f(\ ,y) \notin \mathbf{C}(A)\}$, and for each positive integer n we let D_n be the set of points $y \in E$ such that if $\mathbf{F}$ is any finite subfamily of $\mathbf{F}_0$ with $\cup\mathbf{F} \subset A$ and $\Omega(f(\ ,y),\alpha) < 1/n$ for each $\alpha \in \mathbf{F}$, then $\bar{\phi}(A - \cup\mathbf{F}) \geqslant 1/n$. In the light of Lemma 4.3 we see that $E = \cup_{n=1}^{\infty} D_n$. We shall show that $\bar{\psi}(D_n) = 0$ for each positive integer n and therefore $\bar{\psi}(E) = 0$. To this end we take an arbitrary positive integer n and $\epsilon > 0$. We may and do assume $\epsilon < 1$.

Since $f \in \mathbf{C}(C)$, there exists a family $\mathbf{L} \in \mathfrak{M}(C)$ with a finite subfamily $\mathbf{H}$ such that

$$(1) \qquad \Omega(f, \gamma) < \epsilon/n \text{ whenever } \gamma \in \mathbf{H}; \qquad \sum_{\gamma \in (\mathbf{L}-\mathbf{H})} \mu(\gamma) < \epsilon/n.$$

In accordance with the observations following Definition 4.2, we may assume that $\mathbf{L}$ is disjoint and that all its members are contained in C; i.e., $C = \cup\mathbf{L}$.

We let $Q = C - \cup\mathbf{H}$, note that $Q = \cup(\mathbf{L} - \mathbf{H})$ so that $(\mathbf{L} - \mathbf{H}) \in \mathfrak{M}(Q)$, and use the second relation in (1) to infer that

$$(2) \qquad \bar{\mu}(Q) < \epsilon/n.$$

We next determine disjoint families $\mathbf{F}' \subset \mathbf{F}_0$, $\mathbf{G}' \subset \mathbf{G}_0$, $\mathbf{H}' \subset \mathbf{H}_0$, and $\mathbf{K}' \subset \mathbf{H}'$ satisfying the conditions (i)–(v) inclusive of Lemma 3.2. We see that $A = \cup\mathbf{F}'$, $B = \cup\mathbf{G}'$, $C = \cup\mathbf{H}'$, $\cup\mathbf{H} = \cup\mathbf{K}'$, and $Q = \cup(\mathbf{H}' - \mathbf{K}')$. Every member γ' of $\mathbf{K}'$ is a subset of some set $\gamma \in \mathbf{H}$; hence, by (1),

$$(3) \qquad \Omega(f, \gamma') \leqslant \Omega(f, \gamma) < \epsilon/n < 1/n$$

for each such set γ'.

We now define K_β to be the union of those sets $\alpha \in \mathbf{F}'$ for which $\beta \in \mathbf{G}'$ and $\alpha \times \beta \subset Q$. We let $\mathbf{G}''$ be the subset of $\mathbf{G}'$ for which $\beta \in \mathbf{G}''$ if and only if $\bar{\phi}(K_\beta) < /1n$. Since $\cup_{\beta\epsilon\mathbf{G}'} K_\beta \subset Q$, then, with the help of (2), we have

$$(4) \qquad \sum_{\beta\epsilon(\mathbf{G}'-\mathbf{G}'')} \bar{\mu}(K_\beta \times \beta) \leqslant \bar{\mu}(Q) < \epsilon/n.$$

Also, in the light of Theorem 3.6 and the complete additivity of μ, we have, for each $\beta \in (\mathbf{G}' - \mathbf{G}'')$, $\bar{\mu}(K_\beta \times \beta) = \bar{\phi}(K_\beta)\bar{\psi}(\beta) \geqslant \bar{\psi}(\beta)/n$; and hence from (4), we conclude that

$$(5) \qquad \bar{\psi}(\cup(\mathbf{G}' - \mathbf{G}'')) = \sum_{\beta\epsilon(\mathbf{G}'-\mathbf{G}'')} \bar{\psi}(\beta) < \epsilon.$$

Next, we let $\mathbf{G}'_n$ denote the subfamily of $\mathbf{G}'$ for which $\beta \in \mathbf{G}'_n$ if and only if $\beta \cap D_n \neq \emptyset$. For each such β there exists $y_\beta \in \beta \cap D_n$. We let $\mathbf{F}'_\beta$ denote those members α' of $\mathbf{F}'$ for which $\Omega(f(\ ,y_\beta),\alpha') \geqslant 1/n$. Since clearly $\Omega(f,(\alpha' \times \beta)) \geqslant \Omega(f(\ ,y_\beta),\alpha')$ whenever $\alpha' \in \mathbf{F}'_\beta$, it follows from (3) that $\alpha' \times \beta \notin \mathbf{K}'$, whence $\alpha' \times \beta \in (\mathbf{H}' - \mathbf{K}')$ and $\alpha' \times \beta \subset K_\beta$ for each such α', and so $\cup\mathbf{F}'_\beta \subset K_\beta$. Since $y_\beta \in D_n$ and $\Omega(f(\ ,y_\beta), \alpha') < 1/n$ whenever $\alpha' \in (\mathbf{F}' - \mathbf{F}_\beta')$, it follows from our definition of D_n that $\bar{\phi}(K_\beta) \geqslant \bar{\phi}(\cup\mathbf{F}_\beta') \geqslant 1/n$, whence $\beta \in (\mathbf{G}' - \mathbf{G}'')$, $\mathbf{G}'_n \subset (\mathbf{G}' - \mathbf{G}'')$; hence $\bar{\psi}(D_n) \leqslant \bar{\psi}(\cup\mathbf{G}_n') < \epsilon$ because of (5), and finally $\bar{\psi}(D_n) = 0$. This completes the proof of part (i).

(ii). We let $M > 0$ denote an upper bound for the values taken by $|f|$ on C. According to our definition of integration given in §2, we must assume that f is defined throughout U, but it also follows, since $C \in \mathbf{H}_0$, that the values taken by f outside C have no effect upon the values of either $\int_C f(x, y)d\mu(x, y)$, $\int_A f(x, y)d\phi(x)$ for $y \in B$, or $\int_B f(x, y)d\psi(y)$ for $x \in A$. We shall prove only the first part of (ii), since the second part is entirely analogous. For convenience we define F on B so that $F(y) = \int_A f(x, y)d\phi(x)$ for each $y \in B$. We also choose a positive integer n such that $1/n < \epsilon/(2M + \bar{\phi}(A))$.

Recalling the notation introduced in (i), we take an arbitrary set $\beta \in \mathbf{G}''$ and any point $y' \in \beta$. For $\alpha \in \mathbf{F}'$ we define

$$m_\alpha = \inf_{(x,\ y)\,\epsilon(\alpha\times\beta)} f(x, y), \qquad M_\alpha = \sup_{(x,\ y)\,\epsilon(\alpha\times\beta)} f(x, y)$$

if $\alpha \times \beta \neq \emptyset$; $m_\alpha = M_\alpha = 0$ if $\alpha \times \beta = \emptyset$. Then $m_\alpha \leqslant f(x, y') \leqslant M_\alpha$ holds for each $x \in \alpha$, and so by (**1**, Lemma 5.1),

$$m_\alpha\, \bar{\phi}(\alpha) \leqslant \int_\alpha f(x, y')d\phi(x) \leqslant M_\alpha\, \bar{\phi}(\alpha)$$

for each such set α. Setting $v = \sum_{\alpha\epsilon\mathbf{F}'} m_\alpha \bar{\phi}(\alpha)$ and $V = \sum_{\alpha\epsilon\mathbf{F}'} M_\alpha \bar{\phi}(\alpha)$, it follows that

$$(6) \qquad v \leqslant \sum_{\alpha\epsilon\mathbf{F}'} \int_\alpha f(x, y')d\phi(x) = F(y') \leqslant V.$$

Since y' is an arbitrary point of β and v and V do not depend upon y', then v and V are a lower and an upper bound, respectively, for the values taken by F on β.

Now, in the sums making up the values v and V, the terms corresponding to sets α for which $\alpha \cap K_\beta = \emptyset$ clearly satisfy the relationship $\alpha \times \beta \in \mathbf{K}'$, so that by (3) $0 \leqslant M_\alpha - m_\alpha < \epsilon/n$; for the remaining sets α occurring in these sums, whose union is K_β, we have $0 \leqslant M_\alpha - m_\alpha \leqslant 2M$. Therefore

$$V - v \leqslant \frac{\epsilon}{n} \sum_{\alpha\epsilon\mathbf{F}'} \bar{\phi}(\alpha) + 2M\bar{\phi}(K_\beta) < \frac{\epsilon}{n}\,\bar{\phi}(A) + \frac{2M}{n} < \epsilon$$

because of our choice of n. Hence

$$(7) \qquad \Omega(F, \beta) < \epsilon$$

whenever $\beta \in \mathbf{G}''$. Letting $B' = \bigcup(\mathbf{G}' - \mathbf{G}'')$, we see from (5) that $\bar{\psi}(B') < \epsilon$; hence there exists a family $\mathbf{G}''' \in \mathfrak{M}(B')$ such that

$$(8) \qquad \sum_{\beta\epsilon\mathbf{G}'''} \psi(\beta) < \epsilon.$$

By virtue of the relations (7) and (8) it follows that the family $(\mathbf{G}'' \cup \mathbf{G}''') \in \mathfrak{M}(B)$ satisfies the conditions that make F a member of $\mathbf{C}(B)$. This completes our proof.

4.5. Lemma. *If f is real-valued on S, $A \in \mathbf{F}_0$, and $f \in \mathbf{C}(A)$, then $\int_A fd\phi = \int_A fd\bar{\phi}$, where the right-hand integral may be interpreted as the usual Lebesgue integral of f with respect to $\bar{\phi}$. Analogous statements are valid if f is real-valued on T and $A \in \mathbf{G}_0$, or if f is real-valued on U and $A \in \mathbf{H}_0$.*

Proof. It is easily seen that the restriction of f to A is $\bar{\phi}$-measurable on A, and that we need prove our assertion only for a function f non-negative on A. Accordingly, we assume $f \geqslant 0$ on A. We let M denote a positive upper bound for f on A and assume $\epsilon > 0$. We select a family $\mathbf{F} \in \mathfrak{M}(A)$ with a finite subfamily $\mathbf{F}'$ such that

$$(1) \qquad \Omega(f, \alpha) < \epsilon \text{ whenever } \alpha \in \mathbf{F}'; \qquad \sum_{\alpha\epsilon(\mathbf{F}-\mathbf{F}')} \phi(\alpha) < \epsilon.$$

As usual, we may assume $\mathbf{F}$ to be disjoint and that its members are all subsets of A; i.e., $A = \bigcup\mathbf{F}$.

We let $m_\alpha = \inf_{x\epsilon\alpha} f(x)$, $M_\alpha = \sup_{x\epsilon\alpha} f(x)$ whenever $\emptyset \neq \alpha \in \mathbf{F}$; $m_\alpha = M_\alpha = 0$ if $\alpha = \emptyset$. Then, by (**1**, Lemma 5.1) and the properties of ordinary Lebesgue integrals, we have, for each $\alpha \in \mathbf{F}'$,

$$(2) \qquad m_\alpha \bar{\phi}(\alpha) \leqslant \int_\alpha fd\phi \leqslant M_\alpha \bar{\phi}(\alpha); \qquad m_\alpha \bar{\phi}(\alpha) \leqslant \int_A fd\bar{\phi} \leqslant M_\alpha \bar{\phi}(\alpha).$$

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From (1) and (2), we see that

$$(3) \qquad |\textstyle\int_\alpha f d\phi - \int_\alpha f d\bar{\phi}| \leqslant (M_\alpha - m_\alpha)\bar{\phi}(\alpha)$$

for each such α. In case $\alpha \in (\mathbf{F} - \mathbf{F}')$, it is obvious that

$$(4) \qquad |\textstyle\int_\alpha f d\phi - \int_\alpha f d\bar{\phi}| \leqslant M\,\bar{\phi}(\alpha).$$

From (1), it follows that $\bar{\mu}(\bigcup(\mathbf{F} - \mathbf{F}')) < \epsilon'$. Putting (3) and (4) together and using the additivity of the integrals concerned, we obtain

$$|\textstyle\int_A f d\phi - \int_A f d\bar{\phi}| \leqslant \sum_{\alpha \epsilon \mathbf{F}} |\int_\alpha f d\phi - \int_\alpha f d\bar{\phi}|$$

$$< \epsilon\bar{\phi}(\bigcup \mathbf{F}') + M\bar{\mu}(\bigcup(\mathbf{F} - \mathbf{F}')) < \epsilon\bar{\phi}(A) + \epsilon M,$$

whence it follows that $\int_A f d\phi = \int_A f d\bar{\phi}$.

4.6. Corollary. *Under the hypotheses of Lemma* 4.5, *if* $A \supset A' \in \mathbf{S}_\phi$, *then* $\int_{A'} f d\phi = \int_{A'} f d\bar{\phi}$. *Corresponding results hold in the spaces* T *and* U.

Proof. As shown in (**1**, Theorem 4.2), it is possible to write A' in the form $A' = D - N$, where D is the intersection of a decreasing sequence E of sets, each term in the sequence being the union of a countable disjoint subfamily of $\mathbf{F}_0$ whose union is contained in A, and N is a set for which $\bar{\phi}(N) = 0$. From the measure-theoretic properties of the integrals concerned and the fact that $\int_N f d\phi = \int_N f d\bar{\phi} = 0$ (**1**, Corollary 3.2), the desired conclusion follows.

4.7. Theorem. *If* $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $C = A \times B$, *and* $f \in \mathbf{C}(C)$, *then*

$$\int_C f(x, y) d\mu(x, y) = \int_A (\int_B f(x, y) d\psi(y)) d\phi(x)$$
$$= \int_B (\int_A f(x, y) d\phi(x)) d\psi(y).$$

Proof. It is clearly necessary to prove only the first relation above. As always, we require that functions being integrated be defined throughout the space in question, but inasmuch as $C \in \mathbf{H}_0$, it makes no difference to the validity of our theorem what the values of f outside of C may be; thus we may assume $f(x, y) = 0$ for all $(x, y) \in (U - C)$.

We apply Lemma 4.5 to see that $\int_C f d\mu = \int_C f d\bar{\mu}$, where the right-hand integral is the Lebesgue integral of the $\bar{\mu}$-measurable function f, and from Theorem 3.6, we see that well-known Fubini theorems on product measures may be applied to infer that

$$(1) \qquad \int_C f d\mu = \int_C f d\bar{\mu} = \int_A (\int_B f(x, y) d\bar{\psi}(y)) d\bar{\phi}(x).$$

We define F and G so that

$$F(x) = \int_B f(x, y) d\psi(y) \qquad \text{and} \qquad G(x) = \int_B f(x, y) d\bar{\psi}(y)$$

for each $x \in S$ for which the appearing Lebesgue integral is defined, i.e., for $\bar{\phi}$-almost all $x \in S$; elsewhere in S we may define $G(x) = 0$.

According to Theorem 4.4(i), $f(x, \ ) \in \mathbf{C}(A)$ for $\bar{\phi}$-almost all $x \in A$; hence, by Lemma 4.5, $F(x) = G(x)$ for $\bar{\phi}$-almost all $x \in A$.

Our weak hypotheses on $\mathbf{F}_0$ do not permit us to infer, in general, that the $\bar{\phi}$-almost everywhere equality of two functions ensures the equality of their integrals with respect to ϕ, which is supposed defined only on $\mathbf{F}_0$. However, because of the properties of ordinary Lebesgue integrals, we can say that $\int_A F(x)d\bar{\phi}(x) = \int_A G(x)d\bar{\phi}(x)$; combining this with (1), we obtain

$$\int_C f(x, y)d\mu(x, y) = \int_C F(x)d\bar{\phi}(x). \tag{2}$$

Since $F \in \mathbf{C}(A)$ in accordance with Theorem 4.4(ii), we may use Lemma 4.5 again to infer that

$$\int_A F(x)d\phi(x) = \int_A F(x)d\bar{\phi}(x). \tag{3}$$

Combining (2) and (3) yields the desired result.

4.8. Corollary. *If $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $C = A \times B$, $f \in \mathbf{C}(C)$, and $g \in \mathbf{C}(C)$, then*

$$\begin{aligned}\int_B[\int_A (f(x, y) + g(x, y))d\phi(x)]d\psi(y) &= \int_C (f(x, y) + g(x, y))d\mu(x, y)\\ &= \int_C f(x, y)d\mu(x, y) + \int_C f(x, y)d\mu(x, y)\\ &= \int_B (\int_A f(x, y)d\phi(x))d\psi(y) + \int_B (\int_A f(x, y)d\phi(x))d\bar{\psi}(y)\\ &= \int_B [\int_A f (x, y)d\phi(x) + \int_A g(x, y)d\phi(x)]d\psi(y).\end{aligned}$$

Similar results, obtained by reversing the order of integration, are valid.

The results obtained thus far in §4 show that when our definition of integration is applied to functions of the class $\mathbf{C}(C)$, where $C \in \mathbf{H}_0$, the resulting integrals have many of the same properties as ordinary Riemann integrals. The analogy is carried a step farther in the following theorem.

4.9. Theorem. *If $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $C = A \times B$, and u is a sequence of functions such that $u_n \in \mathbf{C}(C)$ for each positive integer n, and u converges uniformly on C to a function f, then $f \in \mathbf{C}(C)$ and*

$$\begin{aligned}\int_B (\int_A f(x, y)d\phi(x))d\psi(y) &= \lim_n \int_B (\int_A u_n(x, y)d\phi(x))d\psi(y)\\ &= \lim_n \int_C u_n(x, y)d\mu(x, y) = \int_C f(x, y)d\mu(x, y).\end{aligned}$$

Similar results with the order of integration reversed also hold.

Proof. We shall establish only the equations expressed above; the corresponding ones with the variables reversed may be proved similarly. As usual, we must assume that u_n and f are defined throughout U for $n = 1, 2, \ldots$; but since $C \in \mathbf{H}_0$, it makes no difference what values these functions have outside of C. We may assume they are all identically zero whenever $(x, y) \in (U - C)$.

That $f \in \mathbf{C}(C)$ follows from Definition 5.2, the fact that $\Omega(f, \gamma) \leqslant \Omega(u_n, \gamma) + \Omega(f - u_n, \gamma)$ whenever $\gamma \in \mathbf{H}_0$ and n is a positive integer, and the uniform convergence of the sequence u to f.

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We define the sequence U and the function F by

$$U_n(y) = \int_A u_n(x, y)d\phi(x), \qquad F(y) = \int_A f(x, y)d\phi(x)$$

for each $y \in T$. If ϵ is an arbitrary positive number, then it follows from the uniform convergence of the sequence u to f, and (**1**, Lemmas 5.13 and 5.1) that

$$|U_n(y) - F(y)| \leqslant 2\int_A |u_n(x, y) - f(x, y)|d\phi(x) \leqslant 2\epsilon\bar{\phi}(A)$$

for all sufficiently large values of n and all $y \in B$. Therefore, the sequence U converges uniformly on B to F. Now, applying (**1**, Corollary 5.14) and Corollary 4.7 above, we obtain

$$\int_B F(y)d\psi(y) = \lim_n \int_B U_n(y)d\psi(y) = \lim_n \int_B \left(\int_A u_n(x, y)d\phi(x)\right)d\psi(y)$$
$$= \lim_n \int_C u_n(x, y)d\mu(x, y) = \int_C f(x, y)d\mu(x, y).$$

We conclude with a few observations. If $A \in \mathbf{F}_0$, $B \in \mathbf{G}_0$, $D = A \times B \in \mathbf{H}_0$, $E \in \mathbf{H}_0$, and $g = K_{D \cap E}$, then it is easily checked that $g \in \mathbf{C}(D)$. Hence, by virtue of (**1**, Corollary 5.10) and Corollary 4.8 above, it follows that if n is a positive integer, s_i is a real number, and $C_i \in \mathbf{H}_0$ for $i = 1, 2, \ldots, n$, and

$$f(x, y) = \sum_{i=1}^{n} s_i K_{C_i}(x, y)$$

for each $(x, y) \in U$, then $f \in \mathbf{C}(D)$ and

$$\sum_{i=1}^{n} s_i \bar{\mu}(C_i \cap D) = \int_D f(x, y)d\mu(x, y) = \int_B \left(\int_A f(x, y)d\phi(x)\right)d\psi(y)$$
$$= \int_A \left(\int_B f(x, y)d\psi(y)\right)d\phi(x).$$

In particular, if $s_i = 1$ for $i = 1, 2, \ldots, n$, $E = \bigcup_{i=1}^{n} C_i$, and the sequence of sets C is pairwise disjoint, then

$$\bar{\mu}(E \cap D) = \int_D K_E(x, y)d\mu(x, y) = \int_B \left(\int_A K_E(x, y)d\phi(x)\right)d\psi(y)$$
$$= \int_A \left(\int_B K_E(x, y)d\psi(y)\right)d\phi(x).$$

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University of California,
Davis, California