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ASYMPTOTIC BEHAVIOUR OF EXTINCTION PROBABILITY OF INTERACTING BRANCHING COLLISION PROCESSES

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Abstract

Although the exact expressions for the extinction probabilities of the Interacting Branching Collision Processes (IBCP) were very recently given by Chen *et al.* [4], some of these expressions are very complicated; hence, useful information regarding asymptotic behaviour, for example, is harder to obtain. Also, these exact expressions take very different forms for different cases and thus seem lacking in homogeneity. In this paper, we show that the asymptotic behaviour of these extremely complicated and tangled expressions for extinction probabilities of IBCP follows an elegant and homogenous power law which takes a very simple form. In fact, we are able to show that if the extinction is not certain then the extinction probabilities $\{a_n\}$ follow an harmonious and simple asymptotic law of $a_n \sim kn^{-\alpha}\rho_c^n$ as $n \to \infty$, where k and α are two constants, ρ_c is the unique positive zero of the $C(s)$, and $C(s)$ is the generating function of the infinitesimal collision rates. Moreover, the interesting and important quantity α takes a very simple and uniform form which could be interpreted as the 'spectrum', ranging from $-\infty$ to $+\infty$, of the interaction between the two components of branching and collision of the IBCP.

Keywords: Markov branching process; interacting branching collision process; extinction probability; asymptotic behaviour

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1. Introduction

Due to the urgent need in analyzing practical models and developing corresponding challenging mathematical theory, the focus of research interests on branching models has been shifted from independent Markov branching processes into interacting branching systems; thus, the latter has attracted more and more extensive research attention. Many new interacting branching

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models have been posted and analyzed and the corresponding theory is also developing fast. For the traditional independent Markov branching processes, the good references are, among many others, [1], [3], [7], and the many references therein, while the references for interacting branching models can be found in, for example, [2], [5], [6], [8], [9], [10], [11], [12], [13], [14].

Very recently, Chen *et al.* [4] considered an important and challenging model of an interacting branching system, the so-called Interacting Branching Collision Process (IBCP), which consists of two strongly interacting components: the branching component and the collision component. Basic properties on uniqueness and extinction probabilities have been discussed and many important results have been obtained. In particular, they proved that there exists only one IBCP which is just the Feller minimal process for any given infinitesimal generator, the so-called q-matrix $\boldsymbol{Q}$. They also obtained a necessary and sufficient condition under which the IBCP will go to extinction with probability 1 and revealed all kinds of exact expressions of extinction probabilities when the extinction is not certain.

However, though given, some of these exact expressions are very complicated and it is difficult, for example, to obtain useful information about the asymptotic behaviour of the extinction probability from these exact expressions. The intuitive meanings of these complex extinction probabilities are also unclear. These disadvantages limit the applications of these obtained results in practical models. Hence, revealing simple asymptotic behaviour for these complex expressions is of great significance.

The basic aim of this paper is therefore to reveal the asymptotic behaviour for these complex expressions of extinction probabilities. We shall show that the asymptotic behaviour for these complicated extinction probabilities takes a very simple form.

This paper has only three sections. In Section 2, we report the main conclusions obtained in this paper. The proofs of these conclusions are given in Section 3. Compared with the previous two sections, Section 3 is a little bit lengthy which is, in fact, necessary. Indeed, a few separated theorems, dealing with different cases, are given in Section 3 in order to show the harmonious power law stated in Section 2. At the end of this paper, we use a simple example to demonstrate our elegant results.

2. Main results

Following Chen *et al.* [4], we define an IBCP as a continuous-time Markov chain on the state space $\mathbb{Z}_+$ whose transition function $\boldsymbol{P}(t) = (p_{ij}(t); i, j \in \mathbb{Z}_+)$ satisfies $\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{Q}$ where the interacting branching-collision infinitesimal q-matrix (henceforth referred to as an IBC q-matrix) $\boldsymbol{Q} = (q_{ij}; i, j \in \mathbb{Z}_+)$ is given by

$$q_{ij} = \begin{cases} \binom{i}{2} c_{j-i+2} + i b_{j-i+1} & \text{if } i \ge 1,\ j \ge i-2, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where

$$\begin{aligned} &c_0 > 0, \qquad c_j \ge 0 \quad (j \ne 2), \qquad \sum_{k=3}^{\infty} c_k > 0, \qquad 0 < \sum_{j \ne 2} c_j = -c_2 < \infty, \\ &b_0 > 0, \qquad b_j \ge 0 \quad (j \ne 1), \qquad \sum_{k=2}^{\infty} b_k > 0, \qquad 0 < \sum_{j \ne 1} b_j = -b_1 < \infty, \end{aligned} \tag{2.2}$$

together with the conventions $b_{-1} = 0$ and $\binom{1}{2} = 0$. In order to avoid discussing some degenerated and thus trivial cases, we also assume, through this paper, that $\sum_{k=0}^{\infty} c_{2k+1} \ne 0$.

Again, following Chen *et al.* [4], we define the generating functions of the two known sequences $\{c_k; k \geq 0\}$ and $\{b_k; \geq 0\}$, respectively, as

$$C(s) = \sum_{k=0}^{\infty} c_k s^k \quad \text{and} \quad B(s) = \sum_{k=0}^{\infty} b_k s^k.$$

By Chen *et al.* [4, Theorem 3.2] we know that, for any given IBC q-matrix $\boldsymbol{Q}$, there exists only one IBCP. Now, let $\{X(t); t \geq 0\}$ denote this unique IBCP with the given IBC q-matrix $\boldsymbol{Q}$ as defined in (2.1)–(2.2) and let $\boldsymbol{P}(t) = (p_{ij}(t); i, j \in \mathbb{Z}_+)$ be the $\boldsymbol{Q}$-function of this unique IBCP. Also, let

$$\tau_0 = \inf\{t > 0; X(t) = 0\}$$

and

$$a_i = \mathrm{P}(\tau_0 < \infty \mid X(0) = i), \qquad i \geq 1,$$

be the extinction time and extinction probability, respectively.

The following conclusions were obtained by Chen *et al.* [4].

Proposition 2.1. (i) *The equation $C(s) = 0$ has either two roots or three roots in the complex disk $\{z; |z| \leq 1\}$ and all these roots are real. More specifically, if $C'(1) \leq 0$ then $C(s) > 0$ for all $s \in [0, 1)$ and 1 is the only root of the equation $C(s) = 0$ in $[0, 1]$, which is simple or with multiplicity 2 according to $C'(1) < 0$ or $C'(1) = 0$, respectively, while if $0 < C'(1) \leq \infty$ then $C(s) = 0$ has an additional simple root ρ_c satisfying $0 < \rho_c < 1$ such that $C(s) > 0$ for $s \in (0, \rho_c)$ and $C(s) < 0$ for $s \in (\rho_c, 1)$. In addition, $C(s) = 0$ has exactly one root, denoted by ξ_c, in $[-1, 0]$ such that $C(s) > 0$ for all $s \in (\xi_c, 0]$ and $|\xi_c| < \rho_c$. Moreover, $C(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.*

(ii) *The equation $B(s) = 0$ has either one root or two roots in the complex disk $\{z; |z| \leq 1\}$ and all these roots are positive. More specifically, if $B'(1) \leq 0$ then $B(s) > 0$ for all $s \in [-1, 1)$ and 1 is the only root of $B(s) = 0$ in $[0, 1]$. If $0 < B'(1) \leq +\infty$ then $B(s) = 0$ has an additional root in $[0, 1)$, denoted by ρ_b, such that $B(s) > 0$ for all $s \in [-1, \rho_b)$ and $B(s) < 0$ for $s \in (\rho_b, 1)$. Moreover, $B(z) = 0$ has no other root in the complex disk $\{z; |z| \leq 1\}$.*

Throughout this paper, we shall let ρ_c and ρ_b denote the smallest nonnegative root of $C(s) = 0$ and $B(s) = 0$, respectively.

Proposition 2.2. *Suppose that $\boldsymbol{Q}$ is an IBC q-matrix as defined in (2.1)–(2.2) and let $\boldsymbol{P}(t) = (p_{ij}(t); i, j \geq 0)$ and $\boldsymbol{\Phi}(\lambda) = (\phi_{ij}(\lambda); i, j \geq 0)$ be the (in fact, unique, see Proposition 2.3, below) $\boldsymbol{Q}$-function and its $\boldsymbol{Q}$-resolvent that satisfy the Kolmogorov forward equations, respectively. Then, for any $i \geq 0$, $t \geq 0$, $\lambda > 0$ and $|s| < 1$, we have*

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{C(s)}{2} \frac{\partial^2 F_i(t,s)}{\partial s^2} + B(s) \frac{\partial F_i(t,s)}{\partial s}$$

and

$$\Phi_i(\lambda, s) - s^i = \frac{C(s)}{2} \frac{\partial^2 \Phi_i(\lambda, s)}{\partial s^2} + B(s) \frac{\partial \Phi_i(\lambda, s)}{\partial s},$$

where $F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t) s^j$ and $\Phi_i(\lambda, s) = \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j$.

Proposition 2.3. (i) *For any IBC q-matrix $\boldsymbol{Q}$, there exists only one IBCP which is the Feller minimal $\boldsymbol{Q}$-process. Moreover, this unique IBCP is honest (i.e. $\boldsymbol{Q}$ is regular) if and only if $C'(1) \le 0$.*

(ii) *The extinction probability of this unique IBCP, starting from state $i \ge 1$, is 1 if and only if either*

(a) $C'(1) \le 0$ *and* $B'(1) \le 0$ *or*

(b) $C'(1) \le 0$, $0 < B'(1) \le \infty$, *and* $J = \int_{\xi_c}^1 (A(y)/C(y))\,\mathrm{d}y = +\infty$ *(equivalently,* $J_0 = \int_0^1 (A(y)/C(y))\,\mathrm{d}y = +\infty$*), where*

$$A(s) = \exp\biggl\{\int_0^s \frac{2B(x)}{C(x)}\,\mathrm{d}x\biggr\}. \tag{2.3}$$

It should be noticed that, throughout this paper, we shall use (a_n) to denote the extinction probability of IBCP when the process starts at state n which has no relationship with $A(s)$ defined in (2.3). In particular, (a_n) is not the nth coefficient of the Taylor series expansion of the function $A(s)$.

Based on the above propositions, Chen *et al.* [4] further deeply investigated the extinction probability. In particular, they proved that if $C'(1) < 0$ and $0 < B'(1) < \infty$ (and thus $\boldsymbol{Q}$ is regular) then the extinction probabilities $a_i = 1$ is true. They further showed that if $C'(1) = 0$ and $0 < B'(1) < \infty$ (and thus $\boldsymbol{Q}$ is still regular) then both the cases of $a_i < 1$ (for all $i \ge 1$) and $a_i \equiv 1$ (for all $i \ge 1$) may occur and when $a_i < 1$, the exact expressions for a_i $(i \ge 1)$ are given. If $C'(1) > 0$ (and thus $\boldsymbol{Q}$ is irregular by Proposition 2.3), then $a_i < 1$ (for all $i \ge 1$) is certain and all kinds of expressions for the extinction probabilities a_i $(i \ge 1)$ are given in [4].

However, although the explicit expressions for extinction probabilities for IBCP are given in [4], these expressions are sometimes extremely complicated; see [4, Theorem 5.8], for example. It seems very hard to draw useful information from these very complicated expressions. For example, we could know little about the asymptotic behaviour of these extinction probabilities to which we are particularly interested in. Another disadvantage of these expressions is that their forms look extremely different for different cases; thus, the deep relationships among these expressions seem very vague.

The main aim of this paper is therefore to investigate the simple forms of the asymptotic behaviour of these extinction probabilities in order to overcome the above mentioned shortcomings. Of course, in investigating asymptotic behaviour, we are only interested in the case of $a_i < 1$ since otherwise the question will be trivial. Hence, in discussing asymptotic behaviour, we are only interested in, by Proposition 2.3, two cases: either $0 < C'(1) \le \infty$ or $C'(1) = 0$ and $J_0 = \int_0^1 (A(y)/C(y))\,\mathrm{d}y < \infty$. For the latter case, we shall further assume that $B'(1) < \infty$ since for this latter case we are less interested in the uninformative situation of $B'(1) = \infty$.

Surprisingly, we shall show that, as the asymptotic behaviour is concerned, the extinction probabilities of the IBCP display an extremely simple and harmonic feature. Indeed, the asymptotic behaviour of the extinction probabilities just follows a simple power law and, moreover, different situations, to which the exact expressions for extinction probabilities are very complicated and extremely different as mentioned above, are just referring to a constant value, see Remark 3.3 together with Remark 3.4, below.

Our main results obtained in this paper are the following two conclusions which deal with two different cases: the q-matrix $\boldsymbol{Q}$ is regular or irregular.

Theorem 2.1. *Suppose that $C'(1) = 0$, $C''(1) < \infty$, $B'(1) < \infty$, and*

$$J_0 = \int_0^1 \frac{A(y)}{C(y)}\,\mathrm{d}y < \infty;$$

thus, $\rho_c = 1$ (and hence the q-matrix $\boldsymbol{Q}$ is regular). Then, as $n \to \infty$, we have

$$a_n \sim kn^{-\alpha}, \qquad n \to \infty, \tag{2.4}$$

where $\alpha = 4B'(1)/C''(1) - 1 > 0$ and k is a constant which is independent of n.

Theorem 2.2. *Suppose that $0 < C'(1) \le +\infty$; thus, $0 < \rho_c < 1$ (and hence the q-matrix $\boldsymbol{Q}$ is irregular). Further assume that $\rho_b \neq \rho_c$. Then, as $n \to \infty$,*

$$a_n \sim kn^{-\alpha}\rho_c^n, \qquad n \to \infty, \tag{2.5}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c)$ and k is a constant which is independent of n.

We see that the form of (2.5) is extremely simple and harmonic. If one compares (2.5) with the complicated expressions given in [4, Theorems 5.6–5.10], one would feel that they have addressed totally different problems. Note that, in Theorem 2.2, we have assumed that $\rho_b \neq \rho_c$. This is because if $\rho_b = \rho_c$, then $a_n = \rho_c^n$, see [4, Theorem 5.1]; thus, the asymptotic behaviour for the extinction probabilities $\{a_n\}$ is trivial. However, it is easily seen that, even for $\rho_c = \rho_b$, (2.5) is still true, since for this case we have $\alpha = 0$ and $k = 1$. For more detailed explanation, see Remark 3.3, below.

3. Proofs of the main results

As a preparation, we first provide the following simple lemma which describes some simple but useful properties of the function $A(s)$ which is defined in (2.3).

Recall that, by Proposition 2.1, we know that the generating function $C(s) = \sum_{j=0}^{\infty} c_j s^i$ has a negative zero $-1 < \xi_c < 0$ and a smallest positive zero $0 < \rho_c \le 1$ and, furthermore, $\rho_c < 1$ if and only if $0 < C'(1) \le +\infty$.

Lemma 3.1. *The function $A(s)$ defined in (2.3) possesses the following properties.*

(i) *$A(y) \sim l(y - \xi_c)^{\beta}$ as $y \to \xi_c^+$, where $0 < l < \infty$ is a constant and $\beta = 2B(\xi_c)/C'(\xi_c)$.*

(ii) *Suppose that $0 < C'(1) \le \infty$; thus, $\rho_c < 1$. Then*

$$A(y) \sim l(\rho_c - y)^{\alpha} \quad \text{as } y \to \rho_c^-,$$

where $0 < l < \infty$ is a constant (i.e. independent of y) and $\alpha = 2B(\rho_c)/C'(\rho_c)$.

(iii) *Suppose that $C'(1) = 0$ and $C''(1) < 4B'(1) < \infty$. Then*

$$A(y) \sim l(1 - y)^{\gamma} \quad \text{as } y \to 1^-,$$

where $0 < l < \infty$ is a constant (i.e. independent of y) and $\gamma = 4B'(1)/C''(1) > 1$.

Proof. We first prove (ii). By Proposition 2.1, we know that the condition $0 < C'(1) \le \infty$ implies that $\rho_c < 1$ is a single zero of $C(s)$; thus, if we let

$$g(x) = \frac{2B(x)(\rho_c - x)}{C(x)}, \tag{3.1}$$

then $g(x)$, as a complex function of x, has only one negative zero ξ_c in the open unit disk $\{z; |z| < 1\}$. In particular, $g(x)$ is analytic on the open disk $\{z; |z| < |\xi_c|\}$; thus, it could be expanded as a power series of x on the interval $[0, \rho_c)$. Note that in the latter we have viewed $g(x)$ as a real valued function of x. Suppose that the expansion takes the form of

$$g(x) = \sum_{k=0}^{\infty} g_k x^k, \tag{3.2}$$

where $g_k = g^{(k)}(0)/k!$. By (3.1) and (3.2), we have, for $0 < y < \rho_c$,

$$\begin{aligned}\int_0^y \frac{2B(x)}{C(x)}\,\mathrm{d}x &= \int_0^y \frac{g(x)}{\rho_c - x}\,\mathrm{d}x \\ &= \sum_{k=0}^{\infty} g_k \int_0^y \frac{x^k}{\rho_c - x}\,\mathrm{d}x \\ &= \bigg(\sum_{k=0}^{\infty} g_k \rho_c^k\bigg) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} + \sum_{k=1}^{\infty} g_k \sum_{m=1}^{k} (-1)^m \binom{k}{m} \rho_c^{k-m} \int_0^y (\rho_c - x)^{m-1}\,\mathrm{d}x \\ &= J_1 + J_2, \end{aligned}\tag{3.3}$$

where the meaning of J_1 and J_2 should be self-explanatory.

By noting (3.2), we see that J_1 in (3.3) is just

$$J_1 = \bigg(\sum_{k=0}^{\infty} g_k \rho_c^k\bigg) \int_0^y \frac{\mathrm{d}x}{\rho_c - x} = g(\rho_c) \int_0^y \frac{\mathrm{d}x}{\rho_c - x},$$

where $g(\rho_c) = \lim_{x\to\rho_c^+} 2B(x)(\rho_c - x)/C(x) = -2B(\rho_c)/C'(\rho_c)$, which is finite.

Similarly, after some trivial algebra, J_2 in (3.3) can be written as

$$J_2 = \sum_{k=1}^{\infty} g_k \rho_c^k \sum_{m=1}^{k} (-1)^m \frac{\binom{k}{m}}{m} - \sum_{k=1}^{\infty} g_k \rho_c^k \sum_{m=1}^{k} \frac{\binom{k}{m}}{m} \bigg(\frac{y}{\rho_c} - 1\bigg)^m. \tag{3.4}$$

We recognize that the first term on the right-hand side of (3.4) is just a constant that is independent of y and the second term on the right-hand side of (3.4) is just a rational function of y; thus, it is a bounded function of y on $[0, \rho_c]$. It follows from the mean-value theorem that J_2 can be written as a constant k_1, say, as $y \to \rho_c^-$. Therefore, we obtain that there exists a constant k such that

$$A(y) = \exp\bigg\{\int_0^y \frac{2B(x)}{C(x)}\,\mathrm{d}x\bigg\} \sim k(\rho_c - y)^\alpha \quad \text{as } y \to \rho_c^-,$$

where $\alpha = 2B(\rho_c)/C'(\rho_c)$. The proof of (ii) is completed.

The proof of (i) is similar to the proof of (ii); we just note that ξ_c is also a single zero of $C(s)$ and $0 < B(\xi_c) < \infty$.

We now prove (iii). Since $C'(1) = 0$ we know from Proposition 2.1 that $C(s)$ has no zero on $[0, 1)$ and 1 is the zero of $C(s)$ with multiplicity 2. Also, since $0 < B'(1) < \infty$, we know that 1 is also a single zero of $B(s)$. It follows that if we let

$$g_1(x) = \frac{2B(x)(1 - x)}{C(x)}$$

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then, by the same reasoning as we used in the proof of (ii), we get that $g_1(x)$ can also be expanded as a power series of x on $[0, 1)$. Now, using similar arguments as we used in the proof of (ii) and noting the fact that

$$\lim_{s \to 1^+} g_1(x) = \frac{-4B'(1)}{C''(1)},$$

it is then easily shown that (iii) is true.

Remark 3.1. Lemma 3.1(iii) was essentially proved in [4]; see the proof of [4, Corollary 4.3] under the further assumption that $B''(1) < +\infty$. However, it is easily seen that the proof in [4] does not depend on this latter assumption; thus, this assumption can be removed.

In the rest of the paper we shall constantly use the following simple and well-known analytic lemma whose proof can be found in any standard textbook of analysis.

Lemma 3.2. *For any complex number a we have*

$$\lim_{z \to \infty} \frac{\Gamma(z + a)}{\Gamma(z)} z^{-a} = 1$$

so long as $\mathcal{R}(a) > 0$, where $\mathcal{R}(a)$ denotes the real part of the complex number a and $\Gamma(\cdot)$ is the gamma function.

In our later application of Lemma 3.2 we actually will only meet the case that the complex number a is just a real (and thus must be a positive) number.

We are now ready to prove the main results of this paper which were stated in Section 2. For the purpose of giving some further useful information regarding the asymptotic behaviour of the extinction probability than that stated in Theorems 2.1 and 2.2, we shall state and prove the following main conclusions, separately.

Theorem 3.1. *If $C'(1) = 0$ and $B'(1) < \infty$, then the extinction probability $\{a_n\}$, starting from state $n \geq 1$, is less than 1 (for all $n \geq 1$) if and only if $C''(1) < 4B'(1)$. Moreover, if $C''(1) < 4B''(1)$ is satisfied, then*

$$a_n \sim k_1 n^{-\alpha} + k_2 n^{-\beta} \xi_c^n \quad \text{as } n \to \infty, \tag{3.5}$$

where k_1 and k_2 are two constants, $\alpha = 4B'(1)/C''(1) - 1 > 0$, and $\beta = 2B(\xi_c)/C'(\xi_c) > 0$.

Furthermore, we have

$$a_n \sim k n^{-\alpha} \quad \text{as } n \to \infty, \tag{3.6}$$

where k is a constant and $\alpha = 4B'(1)/C''(1) - 1 > 0$.

Proof. The first part of the conclusion has been proved in [4]; thus, we prove (3.5) first. Now, suppose that $C'(1) = 0$ and $C''(1) < 4B'(1) < \infty$, then by [4, Theorem 4.2 and Corollary 4.3], we know that the extinction probability $\{a_n\}$, starting from $n \geq 1$, is given by

$$a_n = \frac{1}{J} \int_{\xi_c}^{1} \frac{y^n A(y)}{C(y)} \, dy, \qquad n \geq 1,$$

where $J = \int_{\xi_c}^{1} (A(y)/C(y)) \, dy$ is a finite constant which is independent of n. In order to obtain (3.5), we consider the two integrals

$$I_1^{(n)} = \int_0^1 \frac{y^n A(y)}{C(y)} \, dy \quad \text{and} \quad I_2^{(n)} = \int_{\xi_c}^{0} \frac{y^n A(y)}{C(y)} \, dy.$$

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Note that under the condition stated in this theorem, we know that, for any $n \geq 1$ and $0 < \varepsilon < 1$, the function $y^n A(y)/C(y)$ is bounded on $[0, \varepsilon]$; thus, we only need to consider the behaviour of $I_1^{(n)}$ when $y \to 1^-$. Noting that $C'(1) = 0$ and $C''(1) < \infty$, we know that the function $C(y)/(1-y)^2$ is bounded on $[0, 1]$. This fact together with Lemma 3.1(iii), implies that we may write

$$I_1^{(n)} = c\int_0^1 \frac{y^n(1-y)^\gamma}{(1-y)^2}\,\mathrm{d}y = c\int_0^1 y^n(1-y)^{\gamma-2}\,\mathrm{d}y,$$

where $0 < c < \infty$ is a constant and $\gamma = 4B'(1)/C''(1) > 1$ since we have $0 < C''(1) < 4B'(1) < \infty$. However,

$$\int_0^1 y^n(1-y)^{\gamma-2}\,\mathrm{d}y = \frac{\Gamma(n+1)\Gamma(\gamma-1)}{\Gamma(n+\gamma)}.$$

Now, noting the fact that $\gamma - 1 > 0$ and applying Lemma 3.2, we get that there exists a constant k such that

$$I_1^{(n)} \sim kn^{1-\gamma}, \qquad n \to \infty. \tag{3.7}$$

We now consider $I_2^{(n)}$. We treat this in a similar way to $I_1^{(n)}$ but noting the difference that ξ_c is a single zero of $C(s)$, we can get that, by also using Lemma 3.1(i), there exists a constant $\tilde{c}$ such that

$$I_2^{(n)} = \tilde{c}\int_{\xi_c}^0 y^n(y-\xi_c)^{\beta-1}\,\mathrm{d}y,$$

where $\beta = 2B(\xi_c)/C'(\xi_c) > 0$. After performing a similar transformation to that above, we obtain that there exists a constant c such that

$$I_2^{(n)} = c\xi_c^n \frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(n+1+\beta)}.$$

Now, applying Lemma 3.2 once again and noting that $\beta > 0$, we get that there exists a constant $\tilde{k}$ such that

$$I_2^{(n)} \sim \tilde{k}n^{-\beta}\xi_c^n, \qquad n \to \infty. \tag{3.8}$$

Combining (3.7) and (3.8) and noting that there exists another constant $1/J$ in the form of $\{a_n\}$, we see that (3.5) is true by simply letting $\alpha = \gamma - 1$ (and hence $\alpha > 0$). Finally, considering $|\xi_c| < 1$, we immediately get (3.6) by using the already proven (3.5).

Remark 3.2. We see that (3.6) is the same as (2.4), which we claimed in Theorem 2.1. However, we can see that (3.5) is a finer result than (3.6). Indeed, (3.5) provides some further information than that given in (3.6).

We now turn to consider the more interesting and challenging irregular case, i.e. $0 < C'(1) \leq \infty$. Although our initial aim is to prove Theorem 2.2, we shall discuss this case more extensively. The reward is that we can get much more information than that stated in Theorem 2.2. Note that, in discussing this irregular case, neither $C'(1) < \infty$ nor $B'(1) < \infty$ is assumed. In other words, we shall cover all possible cases, even if both $C'(1)$ and $B'(1)$ are infinite.

By Proposition 2.3, we know that the condition $0 < C'(1) \leq \infty$ implies that $\rho_c < 1$, where ρ_c is the smallest positive zero of $C(s)$. We also know that the generating function $B(s)$ has the smallest positive zero ρ_b. Hence, three relationships between them may occur, i.e. $\rho_b < \rho_c < 1$, $\rho_b = \rho_c < 1$, and $\rho_c < \rho_b \le 1$. However, as the asymptotic property of the extinction probability is concerned, the case of $\rho_b = \rho_c < 1$ is trivial since in this case we have $a_n = \rho_c^n$ $(n \ge 1)$. We shall therefore only consider the other two cases. We first investigate the case of $\rho_b < \rho_c < 1$.

Theorem 3.2. *If $\rho_b < \rho_c < 1$, then the extinction probability of the IBCP, starting from $n \ge 1$, denoted by $\{a_n\}$, possesses the following asymptotic behaviour:*

$$a_n \sim k_1 n^{-\alpha} \rho_c^n + k_2 n^{-\beta} \xi_c^n \quad \text{as } n \to \infty, \tag{3.9}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c) > 0$, $\beta = 2B(\xi_c)/C'(\xi_c) > 0$, and k_1 and k_2 are constants which are independent of n. Furthermore, we have

$$a_n \sim k n^{-\alpha} \rho_c^n \quad \text{as } n \to \infty, \tag{3.10}$$

where k is a constant and $\alpha = 2B(\rho_c)/C'(\rho_c) > 0$.

Proof. By [4, Theorem 5.3], we know that the extinction probability $\{a_n\}$, starting from $n \ge 1$, is given by

$$a_n = \frac{\int_{\xi_c}^{\rho_c} (y^n A(y)/C(y))\,\mathrm{d}y}{\int_{\xi_c}^{\rho_c} (A(y)/C(y))\,\mathrm{d}y}. \tag{3.11}$$

Since the denominator of the right-hand side of (3.11) is just a constant which is independent of n, we only need to consider the two integrals $I_1^{(n)} = \int_0^{\rho_c} (y^n A(y)/C(y))\,\mathrm{d}y$ and $I_2^{(n)} = \int_{\xi_c}^0 (y^n A(y)/C(y))\,\mathrm{d}y$.

However, the latter is already analyzed in Theorem 3.1, i.e. (3.8) is still true for our current situation; thus, we shall only consider the former. But this is simpler than the case considered in Theorem 3.1 and also very similar to the case of $I_1^{(n)}$ in Theorem 3.1. Indeed, considering $\rho_c < 1$ is the single zero of $C(s)$, and by applying Theorem 2.1(ii), we know that there exists a constant k such that

$$I_1^{(n)} = \int_0^{\rho_c} \frac{y^n A(y)}{C(y)}\,\mathrm{d}y = k \int_0^{\rho_c} y^n (\rho_c - y)^{\alpha - 1}\,\mathrm{d}y,$$

where $\alpha = 2B(\rho_c)/C'(\rho_c) > 0$, since both $B(\rho_c)$ and $C'(\rho_c)$ are negative due to the assumption that $\rho_b < \rho_c < 1$.

Now, since $\alpha > 0$, we have

$$\int_0^{\rho_c} y^n (\rho_c - y)^{\alpha-1}\,\mathrm{d}y = \rho_c^{n+\alpha} \int_0^1 x^n (1-x)^{\alpha-1}\,\mathrm{d}x = \rho_c^{n+\alpha} \frac{\Gamma(n+1)\Gamma(\alpha)}{\Gamma(n+\alpha+1)};$$

thus, by applying Lemma 3.2 once again (since $\alpha > 0$), we obtain that there exists a constant, again denoted by k, such that

$$I_1^{(n)} \sim k n^{-\alpha} \rho_c^n.$$

This, together with (3.8), shows that (3.9) is true. Finally, (3.10) follows from (3.11) by noting the fact that $|\xi_c| < \rho_c$. This completes the proof.

Now we turn to consider the more subtle case of $\rho_c < \rho_b \le 1$. By Proposition 2.1 we know that for this case we have $C'(\rho_c) < 0$ and $B'(\rho_c) > 0$. Following these facts we may face the following three subcases: $C'(\rho_c) + 2B(\rho_c) < 0$, $C'(\rho_c) + 2B(\rho_c) = 0$, and $C'(\rho_c) + 2B(\rho_c) > 0$. We shall discuss these three subcases separately. We first consider the subcase $C'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.3. *If $\rho_c < \rho_b \le 1$ and $C'(\rho_c) + 2B(\rho_c) = 0$, then the extinction probability $\{a_n\}$, starting from $n \ge 1$, is given by*

$$a_n = \rho_c^n + \sigma n \rho_c^{n-1}, \tag{3.12}$$

where $\sigma = -B(\rho_c)/B'(\rho_c)$. Furthermore,

$$a_n \sim k n^{-\alpha} \rho_c^n, \qquad n \to \infty, \tag{3.13}$$

where $k = \sigma/\rho_c$ is a constant and $\alpha = 2B(\rho_c)/C'(\rho_c) = -1$.

Proof. Equation (3.12) is proved in [4, Theorem 5.5], and then (3.13) follows from (3.12) directly. Also, it is easily seen that the condition $C'(\rho_c)+2B(\rho_c) = 0$ is equivalent to $\alpha = -1$.

Secondly, we consider the subcase $C'(\rho_c) + 2B(\rho_c) < 0$.

Theorem 3.4. *Suppose that $\rho_c < \rho_b \le 1$ and $C'(\rho_c) + 2B(\rho_c) < 0$. Then the extinction probability $\{a_n\}$ of the IBCP, starting from $n \ge 1$, possesses the following asymptotic behaviour:*

$$a_n \sim k_1 n^{-\alpha} \rho_c^n + k_2 n^{-\beta} \xi_c^n, \qquad n \to \infty, \tag{3.14}$$

where k_1 and k_2 are constants and $\alpha = 2B(\rho_c)/C'(\rho_c) < 0$ and $\beta = 2B(\xi_c)/C'(\xi_c) > 0$. Furthermore, we have

$$a_n \sim k \rho_c^n n^{-\alpha}, \qquad n \to \infty, \tag{3.15}$$

where $-1 < \alpha = 2B(\rho_c)/C'(\rho_c) < 0$ and k is a constant.

Proof. By [4, Theorem 5.6], we know that if $\rho_c < \rho_b \le 1$ and $C'(\rho_c) + 2B(\rho_c) < 0$, then the extinction probability $\{a_n\}$ is given by

$$\begin{aligned} a_n = {} & \int_{\xi_c}^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} \exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y \\ & \times \biggl(\int_{\xi_c}^{\rho_c} \frac{B'(y)}{A_1(y)} \exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y\biggr)^{-1}, \end{aligned} \tag{3.16}$$

where

$$A_1(s) = \frac{C(s)B(s)}{2} \quad \text{and} \quad B_1(s) = \frac{B(s)(2B(s) + C'(s) - C(s)B'(s))}{2}. \tag{3.17}$$

It follows from (3.16) that there exists a constant k which is independent of n such that

$$a_n = k \int_{\xi_c}^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} \exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y. \tag{3.18}$$

In order to understand the asymptotic property of $\{a_n\}$ in (3.18), we first carefully consider the property of the function $\exp(\int_0^y (B_1(x)/A_1(x))\,\mathrm{d}x)$ which is the key term in the expression (3.18). Let

$$\begin{aligned} a_n^+ &= k \int_0^{\rho_c} \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} \exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y, \\ a_n^- &= k \int_{\xi_c}^0 \frac{y^n B'(y) - n y^{n-1} B(y)}{A_1(y)} \exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y. \end{aligned} \tag{3.19}$$

Following [4], we denote $A_0(s) = C(s)/2$ and $B_0(s) = B(s)$, then $B_1(s)$ and $A_1(s)$ given in (3.17) can be rewritten as

$$A_1(s) = A_0(s)B_0(s)$$

and

$$B_1(s) = B_0(s)[B_0(s) + A_0'(s)] - A_0(s)B_0'(s).$$

Hence,

$$\begin{aligned}\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x &= \int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x + \int_0^y \frac{A_0'(x)}{A_0(x)}\,\mathrm{d}x - \int_0^y \frac{B_0'(x)}{B_0(x)}\,\mathrm{d}x \\ &= \int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x + \ln\frac{A_0(y)}{B_0(y)} + \ln\frac{B_0(0)}{A_0(0)},\end{aligned}$$

where $B_0(0) = b_0 > 0$ and $A_0(0) = c_0/2 > 0$.

It follows that

$$\exp\biggl(\int_0^y \frac{B_1(x)}{A_1(x)}\,\mathrm{d}x\biggr) = k_1\frac{A_0(y)}{B_0(y)}\exp\biggl(\int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x\biggr), \tag{3.20}$$

where k_1 is a constant which is independent of y. Substituting (3.20) into (3.19) shows that there exists a constant, denoted by k again, which is independent of both y and n, such that

$$a_n^+ = k\int_0^{\rho_c} \frac{ny^{n-1}B_0(y) - y^nB_0'(y)}{(B_0(y))^2}\exp\biggl(\int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y.$$

Since $\rho_c < \rho_b \le 1$ we know that $B_0(s) \equiv B(s)$ has no zero on $[0, \rho_c]$; thus, $1/B_0(s)$ is bounded on $[0, \rho_c]$. It follows from this crucial fact and the mean-value theorem, together with the simple facts that both $B_0(s)$ and $B_0'(s)$ are bounded functions of $s \in [0, \rho_c]$, we know that there exist two constants k_1 and k_2 which are both independent of y and n such that

$$a_n^+ = k_1n\int_0^{\rho_c} y^{n-1}\exp\biggl(\int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y + k_2\int_0^{\rho_c} y^n\exp\biggl(\int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y.$$

However, the function $\exp(\int_0^y (B_0(x)/A_0(x))\,\mathrm{d}x)$ is just $A(y)$ defined in (2.3); thus, by using Lemma 3.1(ii) once again we know that there exist two constants, again denoted by k_1 and k_2, such that

$$a_n^+ = k_1n\int_0^{\rho_c} y^{n-1}(\rho_c - y)^\alpha\,\mathrm{d}y + k_2\int_0^{\rho_c} y^n(\rho_c - y)^\alpha\,\mathrm{d}y, \tag{3.21}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c) < 0$.

Noting that $C'(\rho_c) + 2B(\rho_c) < 0$ and $C'(\rho_c) < 0$, we know that $1 + 2B(\rho_c)/C'(\rho_c) > 0$; thus,

$$-1 < \alpha = \frac{2B(\rho_c)}{C'(\rho_c)} < 0.$$

Therefore, we can obtain that

$$\begin{aligned}\int_0^{\rho_c} y^{n-1}(\rho_c - y)^\alpha\,\mathrm{d}y &= \rho_c^n\rho_c^\alpha\int_0^1 x^{n-1}(1-x)^\alpha\,\mathrm{d}x \\ &= \rho_c^{n+\alpha}\int_0^1 x^{n-1}(1-x)^{1+\alpha-1}\,\mathrm{d}x \\ &= \rho_c^{n+\alpha}\frac{\Gamma(n)\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)}.\end{aligned} \tag{3.22}$$

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Similarly, we have

$$\int_0^{\rho_c} y^n(\rho_c - y)^\alpha \, \mathrm{d}y = \rho_c^{n+1+\alpha} \frac{\Gamma(n+1)\Gamma(1+\alpha)}{\Gamma(n+2+\alpha)}. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21), using the fact that $1+\alpha > 0$, and applying Lemma 3.2 together with some trivial algebra, we then can write that there exists a constant k such that

$$a_n^+ \sim k_1 n^{-\alpha} \rho_c^n, \qquad n \to \infty.$$

Similarly, we obtain

$$a_n^- \sim k_2 n^{-\beta} \xi_c^n, \qquad n \to \infty,$$

with $\beta = 2B(\xi_c)/C'(\xi_c) > 0$. Then (3.14) follows. Again, (3.15) follows from (3.14) directly.

Finally we consider the subcase $C'(\rho_c)+2B(\rho_c) > 0$. Since now $C'(\rho_c)+2B(\rho_c) > 0$, but $C'(\rho_c) < 0$ and $B(\rho_c) > 0$, we can certainly find the smallest positive integer $m \geq 2$ such that $mC'(\rho_c) + 2B(\rho_c) \leq 0$, but for all $1 \leq n < m$ we have $nC'(\rho_c) + 2B(\rho_c) > 0$. Equivalently, if we let $\alpha = 2B(\rho_c)/C'(\rho_c)$, then $0 < m-1 < -\alpha \leq m$. We first consider the easy subcase of $mC'(\rho_c) + 2B(\rho_c) = 0$.

Theorem 3.5. *Suppose that $\rho_c < \rho_b \leq 1$ and $C'(\rho_c) + 2B(\rho_c) > 0$. If there exists a positive integer m such that $mC'(\rho_c)+2B(\rho_c) = 0$, then there exist $(m+1)$ constants $\{k_0, k_1, \ldots, k_m\}$ with $k_0 = 1$ such that the extinction probability $\{a_n\}$, starting from $n \geq 1$, can be written as*

$$a_n = \sum_{l=0}^{m} k_l n^l \rho_c^{n-l}. \tag{3.24}$$

In particular, there exists a constant k such that

$$a_n \sim k n^{-\alpha} \rho_c^n, \qquad n \to \infty, \tag{3.25}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c) = -m$.

Proof. Equation (3.24) follows directly from [4, Equation (5.23)]. Then (3.25) is an easy consequence of (3.24) by noting that we have denoted m as $-\alpha$.

Note that Theorem 3.3 can be viewed as a special case of Theorem 3.5 when $m = 1$.

We now turn to the final subcase, $mC'(\rho_c) + 2B(\rho_c) < 0$, for some $m \geq 2$, where m is the smallest positive integer such that $mC'(\rho_c) + 2B(\rho_c) < 0$ holds. Then, as detailed in [4], in addition to defining $A_0(s) = C(s)/2$ and $B_0(s) = B(s)$, we have to define $A_n(s)$ and $B_n(s)$ $(n \geq 1)$ sequentially, until we get $A_m(s)$ and $B_m(s)$, as follows:

$$A_n(s) = A_{n-1}(s)B_{n-1}(s), \tag{3.26}$$

$$B_n(s) = B_{n-1}(s)[B_{n-1}(s) + A'_{n-1}(s)] - A_{n-1}(s)B'_{n-1}(s). \tag{3.27}$$

As also detailed in [4], without loss of generality, we may assume that $A_m(s) > 0$ for all $s \in (\xi_c, \rho_c)$.

Theorem 3.6. *Suppose that $\rho_c < \rho_b \leq 1$, $C'(\rho_c) + 2B(\rho_c) > 0$, and that $-2B(\rho_c)/C'(\rho_c)$ is not an integer. Let m be the smallest positive integer such that $m = \min\{k \geq 1, kC'(\rho_c) + 2B(\rho_c) < 0\}$; thus, $-m < \alpha = 2B(\rho_c)/C'(\rho_c) < -(m-1)$. Further assume that $A_m(s) > 0$*

for all $s \in (\xi_c, \rho_c)$, where $A_m(s)$ is defined sequentially as in (3.26) and (3.27). Then the extinction probability $\{a_n\}$ of the IBCP ($n \geq 1$), starting from $n \geq 1$, possesses the asymptotic behaviour that there exist $m+1$ constants $\{k_0, k_1, \dots, k_{m-1}\}$ such that

$$a_n \sim \sum_{l=0}^{m} k_l \frac{n!}{(n-l)!} \rho_c^{n-l} n^{-\alpha}, \qquad n \to \infty, \tag{3.28}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c)$. Furthermore, we have

$$a_n \sim k\rho_c^n n^{-\alpha}, \qquad n \to \infty, \tag{3.29}$$

where $-m < \alpha = 2B(\rho_c)/C'(\rho_c) < -(m-1)$.

Proof. By [4, Theorem 5.8], we know that, for a sufficiently large n, the extinction probability $\{a_n\}$ is given by

$$a_n = k \sum_{l=0}^{m} \frac{n!}{(n-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{n-l} D_{m,l}(y)}{A_m(y)} \exp\biggl(\int_0^y \frac{B_m(x)}{A_m(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y, \tag{3.30}$$

for some constant k that is independent of n, where $A_m(s)$ and $B_m(s)$ are defined in (3.26) and (3.27) and the functions $D_{m,l}(s)$ are given recursively as

$$D_{1,0}(s) = -B'(s), \qquad D_{1,1}(s) = B(s), \tag{3.31}$$

$$D_{n,k}(s) = D_{n-1,k-1}(s)B_{n-1}(s) - D_{n-1,k}(s)B'_{n-1}(s) + D'_{n-1,k}(s)B_{n-1}(s), \qquad k \leq n-1, \tag{3.32}$$

$$D_{n,n}(s) = \prod_{m=0}^{n-1} B_m(s). \tag{3.33}$$

By (3.31)–(3.33), it is easily seen that all $D_{m,l}(s)$ are analytic functions of s, since they are all power series of s. Hence, they are all bounded on the finite interval $[\xi_c, \rho_c]$. It follows that the $\{a_n\}$ in (3.30) can be written as

$$a_n = \sum_{l=0}^{m} k_l \frac{n!}{(n-l)!} \int_{\xi_c}^{\rho_c} \frac{y^{n-l}}{A_m(y)} \exp\biggl(\int_0^y \frac{B_m(x)}{A_m(x)}\,\mathrm{d}x\biggr)\,\mathrm{d}y, \tag{3.34}$$

where $\{k_0, k_1, \dots, k_m\}$ are $m+1$ constants.

Again, let $\{a_n^+\}$ be the part of $\{a_n\}$ regarding the integral of '$\int_0^{\rho_c}$' and $\{a_n^-\}$ be the part regarding the integral of '$\int_{\xi_c}^{0}$'; thus, $a_n = a_n^+ + a_n^-$.

Firstly, in a similar way as we obtained (3.20), we can get, by using (3.26) and (3.27), that

$$\frac{B_m(s)}{A_m(s)} = \frac{B_{m-1}(s)}{A_{m-1}(s)} + \frac{A'_{m-1}(s)}{A_{m-1}(s)} - \frac{B'_{m-1}(s)}{B_{m-1}(s)}; \tag{3.35}$$

thus,

$$\exp\biggl(\int_0^y \frac{B_m(x)}{A_m(x)}\,\mathrm{d}x\biggr) = \exp\biggl(\int_0^y \frac{B_{m-1}(x)}{A_{m-1}(x)}\,\mathrm{d}x\biggr) \frac{A_{m-1}(y)}{B_{m-1}(y)} \frac{B_{m-1}(0)}{A_{m-1}(0)}. \tag{3.36}$$

By repeatedly using (3.35) and (3.36) and noting that $B_{m-1}(0)/A_{m-1}(0)$ is just a constant, we obtain

$$\exp\left(\int_0^y \frac{B_m(x)}{A_m(x)}\,\mathrm{d}x\right) = k\exp\left(\int_0^y \frac{B_0(x)}{A_0(x)}\,\mathrm{d}x\right)\frac{\prod_{l=0}^{m-1} A_l(y)}{\prod_{l=0}^{m-1} B_l(y)}, \tag{3.37}$$

where k is a constant.

By using (3.26) we may easily see that, for any $n \geq 1$, $A_n(s) = A_0(s)\prod_{k=0}^{n-1} B_k(s)$. Substituting this latter expression into (3.37) and then substituting the result into (3.34), we obtain that there exists $m+1$ constants, again denoted by $\{k_0, k_1, \ldots, k_m\}$, such that

$$a_n^+ = \sum_{l=0}^m k_l \frac{n!}{(n-l)!}\int_0^{\rho_c} y^{n-l}\frac{A_0(y)\prod_{k=0}^{m-1} A_k(y)}{(A_m(y))^2}\exp\left(\int_0^y \frac{2B(x)}{C(x)}\,\mathrm{d}x\right)\mathrm{d}y. \tag{3.38}$$

Now since m is the minimal value of k such that $kC'(\rho_c) + 2B(\rho_c) < 0$, we obtain that ρ_c is not a zero of the function $A_0(y)\prod_{k=0}^{m-1} A_k(y)/(A_m(y))^2$. Thus, by applying the mean-value theorem together with Lemma 3.1(ii), we see that $\{a_n^+\}$ in (3.38) can be written as

$$a_n^+ = \sum_{l=0}^m k_l \frac{n!}{(n-l)!}\int_0^{\rho_c} y^{n-l}(\rho_c - y)^{-\alpha}\,\mathrm{d}y,$$

where $\alpha = 2B(\rho_c)/C'(\rho_c) < 0$.

Similarly, we have

$$a_n^- = \sum_{l=0}^m \tilde{k}_l \frac{n!}{(n-l)!}\int_{\xi_c}^0 y^{n-l}(y - \xi_c)^{-\beta}\,\mathrm{d}y.$$

Using the same transformation as we did before together with applying Lemma 3.2 and using the fact that $|\xi_c| < \rho_c < 1$, we can similarly prove (3.28). Then (3.29) follows directly from (3.28).

Remark 3.3. If we carefully check the results obtained in Theorems 3.2–3.6, particularly (3.10), (3.13), (3.15), (3.25), and (3.29), we may see that if the IBC q-matrix $\boldsymbol{Q}$ is irregular, then the extinction probabilities $\{a_n\}$ always satisfy uniformly the asymptotic behaviour

$$a_n \sim kn^{-\alpha}\rho_c^n, \qquad n \to \infty, \tag{3.39}$$

where $\alpha = 2B(\rho_c)/C'(\rho_c)$ and k is a constant which is independent of n. Hence, Theorem 2.2 is fully proved. We also note that the basic conclusions in Theorems 3.2–3.6 are nothing but special cases of (3.39) with the value of α being $\alpha > 0$ (Theorem 3.2), $\alpha = -1$ (Theorem 3.3), $-1 < \alpha < 0$ (Theorem 3.4), $\alpha = -m$ for some positive integer $m \geq 2$ (Theorem 3.5), and $-m < \alpha < -(m-1)$ for some positive integer $m \geq 2$ (Theorem 3.6).

Remark 3.4. By checking Remark 3.3, it seems that the case of $\alpha = 0$ is missing! Note that, however, in discussing irregular cases we have omitted the case $\rho_b = \rho_c < 1$ for its triviality. Now, we can see that, even for $\rho_b = \rho_c < 1$, (3.39) is still true in the sense of $k = 1$ and $\alpha = 0$. Indeed, in this case we have $a_n = \rho_c^n$; thus, (3.39) takes the form of $\alpha = 0$. Thus, this case fills the gap of the 'spectrum' from $-\infty$ to $+\infty$ well distributed among Theorems 3.2–3.6.

Finally, we use a simple example that was discussed in both [4] and [10] to end this paper. In this simple example the birth structure for both the branching and collision components takes a single birth form. More specifically, we assume that

$$b_0 = a > 0, \qquad b_1 = -(a+b), \qquad b_2 = b > 0, \qquad b_j \equiv 0 \quad (\text{for all } j \geq 3) \tag{3.40}$$

and that

$$\begin{gathered} c_0 = d > 0, \qquad c_1 = r \geq 0, \qquad c_2 = -(d+r+c), \qquad c_3 = c > 0, \\ c_j \equiv 0 \quad (\text{for all } j \geq 4). \end{gathered} \tag{3.41}$$

Using the above quantities, we can easily construct an IBC q-matrix $\boldsymbol{Q}$. It is clear that, for this IBC q-matrix $\boldsymbol{Q}$, we have

$$B(s) = a - (a+b)s + bs^2 = a(1-s)\left(1 - \frac{bs}{a}\right)$$

and

$$C(s) = d + rs - (d+r+c)s^2 + cs^3 = c(s-1)(s-\rho_c)(s-\xi_c), \tag{3.42}$$

where $\rho_c = ((d+r) + \sqrt{(d+r)^2 + 4dc})/2c$ and $\xi_c = ((d+r) - \sqrt{(d+r)^2 + 4dc})/2c < 0$. It is easily seen that $C'(1) = c - (2d+r)$ and $B'(1) = b - a$.

By [4, Theorem 6.1], we know that, for this IBC-q-matrix $\boldsymbol{Q}$, the extinction probabilities are less then 1 if and only if either $c > 2d + r$, or $c = (2d+r)$, $b > a$, and $3d + r < 2(b-a)$. Hence, in studying asymptotic behaviour, we only need to consider these latter two cases. Now, combining our Theorems 2.4 and 2.5 with [4, Theorem 6.1], we obtain the following satisfactory conclusion.

Corollary 3.1. *For the IBC q-matrix determined by (3.40) and (3.41) we have the following conclusions.*

(i) *There always exists only one IBCP which is the Feller minimal process and that this Feller minimal process is honest if and only if* $c \leq 2d + r$.

(ii) *The extinction probabilities* $a_n = 1$ *(for all* $n \geq 1$*) if and only if one of the following three conditions holds:*

   (a) $c < (2d+r)$,

   (b) $c = (2d+r)$ *and* $b \leq a$,

   (c) $c = (2d+r)$, $b > a$, *and* $3d + r \geq 2(b-a)$.

(iii) *If* $c = (2d+r)$, $b > a$, *and* $3d + r < 2(b-a)$, *then* $a_n < 1$ *(for all* $n \geq 1$*) and in this case, the asymptotic behaviour of the extinction probability* $\{a_n\}$ *is given by*

$$a_n \sim kn^{-\alpha}, \qquad n \to \infty,$$

where $\alpha = 2(b-a)/(3d+r) - 1 > 0$.

(iv) *If* $c > 2d + r$, *then* $a_n < 1$ *(for all* $n \geq 1$*) and in this case, the asymptotic behaviour of the extinction probabilities follows the power law of* $a_n \sim kn^{-\alpha}\rho_c^n$ $(n \to \infty)$, *where* ρ_c *is given below (3.42),* $\alpha = 2B(\rho_c)/C'(\rho_c)$ *which is easily given, and k is a constant.*

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