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ON THE EXISTENCE OF UNI-INSTANTANEOUS Q-PROCESSES WITH A GIVEN FINITE μ-INVARIANT MEASURE

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Abstract

Let S be a countable set and let $Q = (q_{ij},\ i, j \in S)$ be a conservative q-matrix over S with a single instantaneous state b. Suppose that we are given a real number $\mu \geq 0$ and a strictly positive probability measure $m = (m_j,\ j \in S)$ such that $\sum_{i \in S} m_i q_{ij} = -\mu m_j$, $j \neq b$. We prove that there exists a Q-process $P(t) = (p_{ij}(t),\ i, j \in S)$ for which m is a μ-invariant measure, that is $\sum_{i \in S} m_i p_{ij}(t) = \mathrm{e}^{-\mu t} m_j$, $j \in S$. We illustrate our results with reference to the Kolmogorov 'K1' chain and a birth–death process with catastrophes and instantaneous resurrection.

Keywords: Markov chain; q-matrix; birth–death process; construction theory

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1. Introduction

We begin with a conservative q-matrix over a countable set S; that is, a collection $Q = (q_{ij},\ i, j \in S)$ of real numbers that satisfy $0 \leq q_{ij} < \infty$, $i, j \in S$, $j \neq i$; $q_i := -q_{ii} \leq \infty$, $i \in S$; and $\sum_{j \neq i} q_{ij} = q_i$, $i \in S$.

We shall assume that Q has a single instantaneous state; that is, a state $b \in S$ such that $q_b = \infty$ and $q_i < \infty$ for $i \neq b$. A set of real-valued functions $P(t) = (p_{ij}(t),\ i, j \in S)$ defined on $(0, \infty)$ is called a *standard transition function* or *process* if

$$p_{ij}(t) \geq 0, \qquad i, j \in S,\ t > 0, \tag{1}$$

$$\sum_{j \in S} p_{ij}(t) \leq 1, \qquad i \in S,\ t > 0, \tag{2}$$

$$p_{ij}(s+t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \qquad i, j \in S,\ s, t > 0, \tag{3}$$

$$\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij}, \qquad i, j \in S, \tag{4}$$

where δ_{ij} is the Kroneker delta. The process P is then *honest* if equality holds in (2) for some (and, thus, all) $t > 0$, and it is called a Q-*transition function* (or Q-*process*) if $p'_{ij}(0+) = q_{ij}$ for each $i, j \in S$.

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If μ is some fixed nonnegative real number, a collection of strictly positive numbers $m = (m_j,\ j \in S)$ is called a *μ-subinvariant measure (on S) for Q* if $\sum_{i\in S} m_i q_{ij} \le -\mu m_j$, $j \in S$, and is called *μ-invariant* if

$$\sum_{i\in S} m_i q_{ij} = -\mu m_j, \qquad j \in S. \tag{5}$$

Here, we shall suppose that m is a finite measure (i.e. $\sum_{i\in S} m_i < \infty$) which is *almost* μ-invariant for Q, that is

$$\sum_{i\in S} m_i q_{ij} = -\mu m_j, \qquad j \ne b, \tag{6}$$

and we will show that there always exists a Q-process P such that m is a *μ-invariant measure (on S) for P*, that is

$$\sum_{i\in S} m_i p_{ij}(t) = \mathrm{e}^{-\mu t} m_j, \qquad j \in S,\ t > 0. \tag{7}$$

(When $\mu = 0$, all of the above notions reduce to the more common ones of invariance and subinvariance.) Note that if we were given a μ-invariant measure m for a particular Q-process P, then, since (7) may be rewritten as

$$\sum_{i\ne j} m_i p_{ij}(t) + (1 - \mathrm{e}^{-\mu t}) m_j = (1 - p_{jj}(t)) m_j,$$

Fatou's lemma would give

$$\sum_{i\ne j} m_i q_{ij} + \mu m_j \le q_j m_j$$

for all $j \in S$, meaning that m would be μ-subinvariant for Q. However, under what conditions is m μ-invariant for Q? In Section 2, we provide necessary and sufficient conditions for m to be almost invariant for Q and delay addressing the interesting question of whether or not $\sum_{i\ne b} m_i q_{ib} = \infty$, which would be the remaining requirement for (5) to hold; this question will be considered in Section 6.

Here, we are assuming that Q is uni-instantaneous. When Q is *totally stable*, that is $q_i < \infty$ for all $i \in S$, the relationship between (5) and (7) is well understood, and has been divined completely for the minimal Q-process F. It was shown by Tweedie [14] that if m is a μ-invariant measure for F, then it is also μ-invariant for Q. Conversely [8], [9], if m is μ-invariant for Q, then it is μ-subinvariant for F and μ-invariant for F if and only if the equations

$$\sum_{i\in S} y_i q_{ij} = -\nu y_j, \qquad 0 \le y_j \le m_j,\ j \in S,$$

have only the trivial solution for some (and, thus, all) $\nu < \mu$. This result holds whether or not S is irreducible and does not require m to be finite. If, as we are assuming here, m *is* finite, then, for μ to be strictly positive, it is necessary that F be dishonest. Furthermore, if F is the unique Q-process satisfying the forward equations, then m is μ-invariant for F.

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Recently, Zhang, Lin and Hou solved the existence problem for the case $\mu = 0$ in the totally stable case [17] and the uni-instantaneous case [18]. They proved that if m is a strictly positive, (almost-)invariant probability measure for Q, then there exists a Q-process P for which m is an invariant measure (and, hence, a stationary distribution). We will extend their results to the case $\mu > 0$.

The structure of the paper is as follows. We begin, in Section 2, by examining the relationship between (6) and (7). Next, we recall the *resolvent decomposition theorem* of [2], which is the major tool for constructing uni-instantaneous Q-processes. This, and some other preliminary results, are presented in Section 3. Our main result on the existence of a Q-process with a given finite, almost-μ-invariant measure for Q is proved in Section 4. In Section 5, we discuss two examples illustrative of our results and, finally, in Section 6, we provide some necessary conditions for μ-invariance. The terminology and notation used will follow that established by Anderson [1] and Yang [16].

2. Almost μ-invariance

Our aim here is to provide necessary and sufficient conditions for a measure m that satisfies (7) (but is not necessarily finite) to be almost μ-invariant for Q. To do so, we recall the notions of an almost-B-type and an almost-F-type Q-process.

Definition 1. (*Chen and Renshaw [3].*) A uni-instantaneous Q-process P with instantaneous state b is called *almost B-type* if it satisfies the Kolmogorov backward equations over the noninstantaneous states, that is if

$$p'_{ij}(t) = \sum_{k\in S} q_{ik}p_{kj}(t), \qquad i \neq b,\ j \in S. \tag{8}$$

The process P is called *almost F-type* if it satisfies the Kolmogorov forward equations over the noninstantaneous states, that is if

$$p'_{ij}(t) = \sum_{k\in S} p_{ik}(t)q_{kj}, \qquad i \neq b,\ j \in S.$$

By adapting the proof of Theorem 1 of [11], we can establish the following result.

Theorem 1. *If m is a μ-invariant measure for P, then m is almost μ-invariant for Q if and only if P is almost F-type.*

Proof. Since (7) holds, we may define an honest standard transition function $P^*(t) = (p^*_{ij}(t),\ i, j \in S)$ over S by

$$p^*_{ij}(t) = \mathrm{e}^{\mu t}\frac{m_j p_{ji}(t)}{m_i}, \qquad i, j \in S,\ t > 0.$$

Indeed, P^* is a Q^*-transition function, where $Q^* = (q^*_{ij},\ i, j \in S)$ is the q-matrix with entries

$$q^*_{ij} = \frac{m_j q_{ji}}{m_i} + \mu\delta_{ij}, \qquad i, j \in S.$$

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(P^* is called the μ-reverse of P with respect to m and Q^* the μ-reverse of Q with respect to m; see [9].) It is easy to see that Q^* is uni-instantaneous with instantaneous state b and, for $i \neq b$, that

$$m_i \sum_{j \in S} q^*_{ij} = \sum_{j \neq i} m_j q_{ji} + \mu m_i - m_i q_i \leq 0.$$

Moreover, all of the states $i \neq b$ are conservative states for Q^* if and only if (6) holds. It is easy to verify that P^* is almost B-type if and only if P is almost F-type. Thus, if (6) holds then Q^* is conservative for the states $i \neq b$. Hence, the backward equations (8) hold for P^* over the states $i \neq b$, implying that P is almost F-type. Conversely, if P is almost F-type then P^* is almost B-type; however, P^* is honest, implying that the states $i \neq b$ are conservative states for Q^* and, hence, (6) holds.

3. The resolvent decomposition theorem

Henceforth, we will find it convenient to specify transition functions through their Laplace transforms. If P is a specified transition function, then the function $\Psi(\alpha) = (\psi_{ij}(\alpha),\ i, j \in S)$ given by

$$\psi_{ij}(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha t} p_{ij}(t)\,\mathrm{d}t, \qquad i, j \in S,\ \alpha > 0, \tag{9}$$

is called the *resolvent* of P. Indeed, if $i, j \in C$, where C is any irreducible class, then the integral in (9) converges for all $\alpha > -\lambda_P(C)$, where $\lambda_P(C)$ is the *decay parameter* of C (for P); see [6]. In analogy to properties (1)–(4) of P, the resolvent satisfies

$$\psi_{ij}(\alpha) \geq 0, \qquad i, j \in S,\ \alpha > 0, \tag{10}$$

$$\sum_{j \in S} \alpha \psi_{ij}(\alpha) \leq 1, \qquad i \in S,\ \alpha > 0, \tag{11}$$

$$\psi_{ij}(\alpha) - \psi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in S} \psi_{ik}(\alpha)\psi_{kj}(\beta) = 0, \qquad i, j \in S,\ \alpha, \beta > 0, \tag{12}$$

$$\lim_{\alpha \to \infty} \alpha \psi_{ij}(\alpha) = \delta_{ij}, \qquad i, j \in S. \tag{13}$$

(Note that (12) is called the *resolvent equation*.) Indeed, any Ψ that satisfies (10)–(13) is the resolvent of a standard transition function P; see Lemma 1.1 of [12]. Furthermore, (11) is satisfied with equality if and only if P is honest, in which case the resolvent is said to be honest. Also, the q-matrix of P can be recovered from Ψ using the following identity:

$$q_{ij} = \lim_{\alpha \to \infty} \alpha(\alpha \psi_{ij}(\alpha) - \delta_{ij}). \tag{14}$$

Finally, a resolvent Ψ that satisfies (14) is called a Q-resolvent.

We can identify μ-invariant measures using resolvents. If P is a Q-process with resolvent Ψ and $m = (m_j,\ j \in S)$ is a μ-invariant measure for P, then $\mu \leq \lambda_P(S)$, where $\lambda_P(S) = \inf_C \lambda_P(C)$ (the infimum being taken over all the irreducible classes comprising S); see Lemma 4.1 of [15]. Furthermore, since the integral in (9) converges for all $\alpha > -\lambda_P(S)$, we have

$$\sum_{i \in S} m_i \alpha \psi_{ij}(\alpha - \mu) = m_j \tag{15}$$

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for all $j \in S$ and $\alpha > 0$. We refer to m as μ-invariant for Ψ if (15) is satisfied. Finally, a simple extension of Lemma 1 of [10] establishes both that m is μ-invariant for Ψ if it is μ-invariant for P, and that if $\mu \le \lambda_P(S)$, then m is μ-invariant for P if it is μ-invariant for Ψ.

We are assuming that Q is a uni-instantaneous q-matrix with instantaneous state b, so let us write $N = S \setminus \{b\}$ and denote by $Q_N = (q_{ij},\ i, j \in N)$ the restriction of Q to N. If $m = (m_i,\ i \in S)$ is a measure on S, then $m_N = (m_i,\ i \in N)$ will be the restriction of m to N.

The following important result combines Theorems 7.7 and 7.8 of [2]. It characterizes Q-processes with a single instantaneous state. In preparation, define families H_Ψ and K_Ψ, for a given Q_N-resolvent $\Psi(\alpha) = (\psi_{ij}(\alpha),\ i, j \in N)$, as follows: H_Ψ is the set of all nonnegative row vectors $\eta(\alpha) = (\eta_i(\alpha),\ i \in N)$, $\alpha > 0$, satisfying $\sum_{j\in N} \eta_j(\alpha) < \infty$ and

$$\eta_j(\alpha) - \eta_j(\beta) + (\alpha - \beta) \sum_{k\in N} \eta_k(\alpha)\psi_{kj}(\beta) = 0, \qquad j \in N, \tag{16}$$

and K_Ψ is the set of all column vectors $\xi(\alpha) = (\xi_i(\alpha),\ i \in N)$, $\alpha > 0$, satisfying $0 \le \xi_i(\alpha) \le 1$, $i \in N$, and

$$\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k\in N} \psi_{ik}(\alpha)\xi_k(\beta) = 0, \qquad i \in N.$$

Theorem 2. (Resolvent decomposition theorem.) *For the uni-instantaneous q-matrix Q, every Q-resolvent $R(\alpha) = (r_{ij}(\alpha),\ i, j \in S)$ can be decomposed uniquely as*

$$R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \psi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha)\eta(\alpha) \end{pmatrix}, \tag{17}$$

where $\Psi(\alpha) = (\psi_{ij}(\alpha),\ i, j \in N)$ is a Q_N-resolvent and $\eta(\alpha) = (\eta_i(\alpha),\ i \in N)$ and $\xi(\alpha) = (\xi_i(\alpha),\ i \in N)$ satisfy the following conditions:

(i) *$\eta(\alpha) \in H_\Psi$ and $\xi(\alpha) \in K_\Psi$,*

(ii) *$\xi_i(\alpha) \le 1 - \sum_{j\in N} \alpha\psi_{ij}(\alpha)$, $i \in N$,*

(iii) *$\lim_{\alpha\to\infty} \alpha\eta_j(\alpha) = q_{bj}$, $j \in N$,*

(iv) *$\lim_{\alpha\to\infty} \alpha\xi_i(\alpha) = q_{ib}$, $i \in N$, and*

(v) *$r_{bb}(\alpha) = (C + \alpha + \alpha \sum_{j\in N} \eta_j(\alpha)\xi_j)^{-1}$, where $\xi_j := \lim_{\alpha\to 0} \xi_j(\alpha)$ and $C < \infty$ satisfy*

$$C \ge \lim_{\alpha\to\infty} \alpha \sum_{j\in N} \eta_j(\alpha)(1 - \xi_j), \tag{18}$$

$$\lim_{\alpha\to\infty} \alpha \sum_{j\in N} \eta_j(\alpha)\xi_j = \infty \quad \left(\text{or, equivalently,} \quad \lim_{\alpha\to\infty} \alpha \sum_{j\in N} \eta_j(\alpha) = \infty\right).$$

Conversely, if there exists a Q_N-resolvent Ψ, and vectors $\eta(\alpha)$ and $\xi(\alpha)$ satisfying the above conditions, then R, defined by (17), is a Q-resolvent.

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Our main result rests on the following three lemmas.

Lemma 1. *Suppose that the uni-instantaneous q-matrix Q admits an almost-μ-invariant measure $m = (m_i,\ i \in S)$. Then $d_i(\alpha) = (d_i(\alpha),\ i \in N)$, defined by*

$$d_i(\alpha) = m_i - (\alpha + \mu) \sum_{k \in N} m_k \phi_{ki}(\alpha), \qquad i \in N,\ \alpha > 0, \tag{19}$$

where $\Phi_N(\alpha) = (\phi_{ij}(\alpha),\ i, j \in N)$ is the minimal Q_N-resolvent, satisfies

$$\lim_{\alpha \to \infty} \alpha d_i(\alpha) = m_b q_{bi}, \qquad i \in N.$$

Proof. Since m is almost μ-invariant for Q, it is clear that the restriction $m_N = (m_i,\ i \in N)$ is a μ-subinvariant measure for Q_N. Therefore, because m_N is then μ-subinvariant for Φ_N, we find that $d_i(\alpha) \geq 0$, $i \in N$, $\alpha > 0$. Also, since Φ_N is the minimal Q_N-resolvent, it satisfies the resolvent equation

$$\phi_{ij}(\alpha) - \phi_{ij}(\beta) + (\alpha - \beta) \sum_{k \in N} \phi_{ik}(\alpha)\phi_{kj}(\beta) = 0, \qquad i, j \in N,\ \alpha, \beta > 0,$$

and, therefore,

$$\bar{d}_i(\alpha) - \bar{d}_i(\beta) + (\alpha - \beta) \sum_{k \in N} \bar{d}_k(\alpha)\phi_{kj}(\beta) = 0, \qquad i \in N,\ \alpha, \beta > 0, \tag{20}$$

where

$$\bar{d}_i(\alpha) = m_i - \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha), \qquad i \in N,\ \alpha > 0.$$

Since $d_i(\alpha) \geq 0$, $i \in N$, $\alpha > 0$, we have $\bar{d}_i(\alpha) \geq 0$, $i \in N$, $\alpha > 0$. Using (20) we see that, for each $i \in N$, $\bar{d}_i(\alpha)$ is nonincreasing in α and, hence, $\alpha \sum_{k \in N} m_k \phi_{ki}(\alpha)$ is nondecreasing in α. Therefore, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha)$ exists. However, by Fatou's lemma, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha) \geq m_i$, and, hence, $\lim_{\alpha \to \infty} \alpha \sum_{k \in N} m_k \phi_{ki}(\alpha) = m_i$ because $\bar{d}_i(\alpha) \geq 0$. Since Φ_N satisfies the forward equation

$$\alpha \phi_{ij}(\alpha) = \delta_{ij} + \sum_{k \in N} \phi_{ik}(\alpha) q_{kj}, \qquad i, j \in N,\ \alpha > 0,$$

and (19) can be rewritten as

$$d_i(\alpha) = \sum_{k \in N} m_k (\delta_{ki} - (\alpha + \mu)\phi_{ki}(\alpha)), \qquad i \in N,\ \alpha > 0,$$

we deduce that

$$\begin{aligned} \alpha d_i(\alpha) &= -\alpha \sum_{k \in N} m_k \sum_{j \in N} \phi_{kj}(\alpha) q_{ji} - \alpha\mu \sum_{k \in N} m_k \phi_{ki}(\alpha), \\ &= -\sum_{j \in N} q_{ji} \alpha \sum_{k \in N} m_k \phi_{kj}(\alpha) - \mu\alpha \sum_{k \in N} m_k \phi_{ki}(\alpha), \end{aligned}$$

which leads to

$$\lim_{\alpha \to \infty} \alpha d_i(\alpha) = -\sum_{j \in N} m_j q_{ji} - \mu m_i = m_b q_{bi}, \qquad i \in N.$$

This completes the proof.

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Lemma 2. *Let $\Psi(\alpha) = (\psi_{ij}(\alpha),\ i, j \in N)$ be a Q_N-resolvent and let $\xi_i = \lim_{\alpha\to 0}\xi_i(\alpha)$, where $\xi_i(\alpha) = 1 - \alpha\sum_{j\in N}\psi_{ij}(\alpha)$, $i \in N$. If $\eta(\alpha) \in H_\Psi$ then $\alpha\sum_{i\in N}\eta_i(\alpha)(1-\xi_i)$ is finite and does not depend on α.*

Proof. By the dominated convergence theorem,

$$\begin{aligned}\lim_{\beta\to 0}\alpha\beta\sum_{i\in N}\sum_{j\in N}\eta_i(\alpha)\psi_{ij}(\beta) &= \alpha\sum_{i\in N}\eta_i(\alpha)\lim_{\beta\to 0}\beta\sum_{j\in N}\psi_{ij}(\beta)\\ &= \alpha\sum_{i\in N}\eta_i(\alpha)\lim_{\beta\to 0}(1-\xi_i(\beta))\\ &= \alpha\sum_{i\in N}\eta_i(\alpha)(1-\xi_i).\end{aligned}$$

On the other hand, using (16), we obtain

$$\begin{aligned}\lim_{\beta\to 0}\alpha\beta\sum_{i\in N}\sum_{j\in N}\eta_i(\alpha)\psi_{ij}(\beta) &= \lim_{\beta\to 0}\alpha\beta\sum_{j\in N}\sum_{i\in N}\eta_i(\alpha)\psi_{ij}(\beta)\\ &= \lim_{\beta\to 0}\frac{\alpha\beta}{\beta-\alpha}\sum_{j\in N}(\eta_j(\alpha)-\eta_j(\beta))\\ &= \lim_{\beta\to 0}\frac{\alpha\beta}{\beta-\alpha}\sum_{j\in N}\eta_j(\alpha) + \lim_{\beta\to 0}\frac{\alpha\beta}{\alpha-\beta}\sum_{j\in N}\eta_j(\beta).\end{aligned}$$

The first term vanishes because $\sum_{j\in N}\eta_j(\alpha) < \infty$. The second term equals

$$\lim_{\beta\to 0}\beta\sum_{j\in N}\eta_j(\beta),$$

which exists, because it is easy to deduce, from (16), that $\beta\sum_{j\in N}\eta_j(\beta)$ is nondecreasing in β. Since this limit does not depend on α, the proof is complete.

Lemma 3. *Suppose that $m = (m_i,\ i \in S)$ is a strictly positive probability measure. If m is μ-invariant for the Q-resolvent R defined in (17), then*

(i) *$m_N = (m_i,\ i \in N)$ is a μ-subinvariant measure for Ψ, and*

(ii) *$\eta_i(\alpha) = d_i(\alpha)/m_b$, where $d_i(\alpha) = m_i - (\alpha+\mu)\sum_{k\in N}m_k\psi_{ki}(\alpha)$, $i \in N$, $\alpha > 0$.*

Conversely, if (i) and (ii) hold, then, on setting $\xi_i(\alpha) = 1 - \alpha\sum_{j\in N}\psi_{ij}(\alpha)$, $i \in N$, and $C = \mu/m_b + \alpha\sum_{i\in N}\eta_i(\alpha)(1-\xi_i)$, where $\xi_i = \lim_{\alpha\to 0}\xi_i(\alpha)$, (17) determines a Q-resolvent R for which m is a μ-invariant measure.

Proof. If m is μ-invariant for R, that is

$$(\alpha+\mu)\sum_{i\in S}m_i r_{ij}(\alpha) = m_j, \qquad j \in S,\ \alpha > 0, \tag{21}$$

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then $(\alpha+\mu)\sum_{i\in N} m_i\psi_{ij}(\alpha) \le m_j$, $j\in N$, since, from (17), we have $\psi_{ij}(\alpha)\le r_{ij}(\alpha)$, $i,j\in N$. This proves part (i). Next, from (17) and (21), we have

$$(\alpha+\mu)r_{bb}(\alpha)m_b + (\alpha+\mu)\sum_{k\in N} m_k\xi_k(\alpha)r_{bb}(\alpha) = m_b \tag{22}$$

and, for all $i\in N$ and $\alpha>0$,

$$(\alpha+\mu)\eta_i(\alpha)r_{bb}(\alpha)m_b+(\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha)+(\alpha+\mu)\sum_{k\in N} m_k\xi_k(\alpha)r_{bb}(\alpha)\eta_i(\alpha) = m_i. \tag{23}$$

These equations combine to give $m_b\eta_i(\alpha)+(\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha) = m_i$, $i\in N$, and, hence, part (ii) holds.

To prove the converse, set $\xi_i(\alpha) = 1-\alpha\sum_{j\in N}\psi_{ij}(\alpha)$ in (17) and take $\eta(\alpha)$ to satisfy (16). Then, by Lemma 2, $\alpha\sum_{i\in N}\eta_i(\alpha)(1-\xi_i)$ is finite and independent of α, and, so, the given C satisfies (18). It follows that

$$r_{bb}(\alpha) = \left(\frac{\mu}{m_b}+\alpha+\alpha\sum_{i\in N}\eta_i(\alpha)\right)^{-1}.$$

Since parts (i) and (ii) hold and $\sum_{i\in S} m_i = 1$, we have

$$\begin{aligned}
&(\alpha+\mu)r_{bb}(\alpha)m_b + (\alpha+\mu)\sum_{i\in N} m_i\xi_i(\alpha)r_{bb}(\alpha)\\
&\quad = r_{bb}(\alpha)\left((\alpha+\mu)m_b+(\alpha+\mu)(1-m_b)-\alpha\sum_{j\in N}(\alpha+\mu)\sum_{i\in N} m_i\psi_{ij}(\alpha)\right)\\
&\quad = r_{bb}(\alpha)\left(\mu+\alpha m_b+\alpha m_b\sum_{j\in N}\eta_j(\alpha)\right)\\
&\quad = m_b
\end{aligned}$$

and, for $i\in N$,

$$\begin{aligned}
&(\alpha+\mu)\eta_i(\alpha)r_{bb}(\alpha)m_b + (\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha) + (\alpha+\mu)\sum_{k\in N} m_k\xi_k(\alpha)r_{bb}(\alpha)\eta_i(\alpha)\\
&\quad = (\alpha+\mu)r_{bb}(\alpha)d_i(\alpha) + (\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha) + (\alpha+\mu)r_{bb}(\alpha)\frac{d_i(\alpha)}{m_b}\sum_{k\in N} m_k\xi_k(\alpha)\\
&\quad = (\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha) + \frac{d_i(\alpha)}{m_b}\left((\alpha+\mu)r_{bb}(\alpha)m_b+(\alpha+\mu)\sum_{i\in N} m_i\xi_i(\alpha)r_{bb}(\alpha)\right)\\
&\quad = (\alpha+\mu)\sum_{k\in N} m_k\psi_{ki}(\alpha) + d_i(\alpha)\\
&\quad = m_i.
\end{aligned}$$

Thus, (22) and (23) hold. These in turn imply that (21) holds, meaning that m is a μ-invariant measure for R.

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4. Existence

We are now ready to state our main result.

Theorem 3. *Let $\mu \geq 0$ and suppose that the uni-instantaneous q-matrix Q admits a finite, almost-μ-invariant measure $m = (m_i,\ i \in S)$. Then there exists a Q-process for which m is a μ-invariant measure.*

Proof. Without loss of generality, we may assume that $\sum_{i \in S} m_i = 1$. Let $\Phi(\alpha) = (\phi_{ij}(\alpha),\ i, j \in N)$ be the minimal Q_N-resolvent. Since m is almost μ-invariant for Q, the restriction $m_N = (m_i,\ i \in N)$ is a μ-subinvariant measure for Q_N and, hence, is μ-subinvariant for Φ. Set

$$d_i(\alpha) = m_i - (\alpha + \mu) \sum_{k \in N} m_k \phi_{ki}(\alpha), \qquad i \in N,\ \alpha > 0, \tag{24}$$

$$\eta_i(\alpha) = \frac{d_i(\alpha)}{m_b}, \qquad i \in N,\ \alpha > 0, \tag{25}$$

$$\xi_i(\alpha) = 1 - \alpha \sum_{j \in N} \phi_{ij}(\alpha), \qquad i \in N,\ \alpha > 0, \tag{26}$$

and

$$r_{bb}(\alpha) = \left(\frac{\mu}{m_b} + \alpha + \alpha \sum_{i \in N} \eta_i(\alpha) \right)^{-1}. \tag{27}$$

Since Φ satisfies the resolvent equation, $\eta(\alpha)$ and $\xi(\alpha)$ given in (25) and (26) satisfy

$$\eta_i(\alpha) - \eta_i(\beta) + (\alpha - \beta) \sum_{k \in N} \eta_k(\alpha) \phi_{ki}(\beta) = 0, \qquad i \in N, \tag{28}$$

and

$$\xi_i(\alpha) - \xi_i(\beta) + (\alpha - \beta) \sum_{k \in N} \phi_{ik}(\alpha) \xi_k(\beta) = 0, \qquad i \in N.$$

Using Lemma 1, we see that

$$\lim_{\alpha \to \infty} \alpha \eta_j(\alpha) = \lim_{\alpha \to \infty} \alpha \frac{d_j(\alpha)}{m_b} = q_{bj}, \qquad j \in N,$$

and

$$\lim_{\alpha \to \infty} \alpha \sum_{j \in N} \eta_j(\alpha) = \lim_{\alpha \to \infty} \frac{1}{m_b} \sum_{j \in N} \alpha d_j(\alpha) = \sum_{j \in N} q_{bj} = \infty.$$

Also,

$$\lim_{\alpha \to \infty} \alpha \xi_i(\alpha) = \lim_{\alpha \to \infty} \sum_{k \in N} \alpha(\delta_{ik} - \alpha \phi_{ik}(\alpha)) = -\sum_{k \in N} q_{ik} = q_{ib}, \qquad i \in N.$$

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Therefore, using (26), (28), and Lemma 2, we deduce that $\alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i)$ is finite and independent of α. Now set

$$C = \frac{\mu}{m_b} + \alpha \sum_{i \in N} \eta_i(\alpha)(1 - \xi_i),$$

where $\xi = \lim_{\alpha \to 0} \xi(\alpha)$, and observe that C satisfies (18). Hence, in view of Theorem 2, we may use (24)–(27) to construct a Q-resolvent R by setting

$$R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha)\eta(\alpha) \end{pmatrix},$$

and then use the second part of Lemma 3 to deduce that m is a μ-invariant measure for R. This completes the proof.

Remark 1. When $\mu = 0$, Theorem 3 reduces to the result of [18].

5. Examples

Example 1. We will begin with an example, generally known as the 'K1' chain, described by Kolmogorov [7] and analysed by Kendall and Reuter [5] and Reuter [13] (see also the discussions in [4] and [1]). The chain has a q-matrix over the nonnegative integers given by

$$Q = \begin{pmatrix} -\infty & 1 & 1 & 1 & \cdots \\ q_1 & -q_1 & 0 & 0 & \cdots \\ q_2 & 0 & -q_2 & 0 & \cdots \\ q_3 & 0 & 0 & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{29}$$

where $q_i > 0$, $i \geq 1$. If a μ-subinvariant measure exists for Q then $\mu \leq \inf_i q_i$; see Corollary 1 of [6]. We will assume that $\mu < q_i$ for all $i \geq 1$. Then, for any such μ, Q admits a μ-invariant measure $m = (m_i,\ i \geq 0)$ given by $m_i = m_0/(q_i - \mu)$, $i \geq 1$, with m_0 arbitrary. This is finite if and only if

$$\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty, \tag{30}$$

in which case Q has the unique μ-invariant probability measure

$$m_0 = \frac{1}{A}, \qquad m_i = \frac{m_0}{q_i - \mu}, \quad i \geq 1, \tag{31}$$

where $A = 1 + \sum_{i=1}^{\infty} 1/(q_i - \mu)$. Therefore, an immediate consequence of Theorems 2 and 3 and Lemma 3 is the following simple result.

Proposition 1. *If Q defined in (29) satisfies (30), then there exists a Q-process for which m, defined by (31), is a μ-invariant probability measure. The resolvent of one such process is given by*

$$R(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(\alpha) \end{pmatrix} + r_{bb}(\alpha) \begin{pmatrix} 1 & \eta(\alpha) \\ \xi(\alpha) & \xi(\alpha)\eta(\alpha) \end{pmatrix},$$

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where

$$\phi_{ij}(\alpha) = \frac{\delta_{ij}}{\alpha + q_i}, \qquad i, j \geq 1,\ \alpha > 0,$$
$$\xi_i(\alpha) = \frac{q_i}{\alpha + q_i}, \qquad i \geq 1,\ \alpha > 0,$$
$$\eta_j(\alpha) = \frac{1}{\alpha + q_j}, \qquad j \geq 1,\ \alpha > 0,$$

and

$$r_{bb}(\alpha) = \left(\frac{\mu}{m_0} + \alpha + \alpha \sum_{i=1}^{\infty} \eta_i(\alpha)\right)^{-1}.$$

Example 2. Next we consider the following q-matrix, describing a birth–death process incorporating catastrophes to state 0 and instantaneous resurrection from state 0:

$$Q = \begin{pmatrix} -\infty & h_1 & h_2 & h_3 & \cdots \\ d_1 & -(d_1 + b_1) & b_1 & 0 & \cdots \\ d_2 & a_2 & -(a_2 + b_2 + d_2) & b_2 & \cdots \\ d_3 & 0 & a_3 & -(a_3 + b_3 + d_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{32}$$

Here, $d_i > 0$, $b_i > 0$, $i \geq 0$, $a_i > 0$, $i \geq 1$, $h_j \geq 0$, $j \geq 1$, and $\sum_{j=1}^{\infty} h_j = \infty$. Define $\pi = (\pi_i,\ i \geq 1)$ by $\pi_1 = 1$ and

$$\pi_i = \prod_{j=2}^{i} \frac{b_{j-1}}{a_j}, \qquad i \geq 2.$$

It is easy to show that if μ satisfies $0 \leq \mu \leq \inf_{i \geq 1} d_i$ and if $h_i = c\pi_i(d_i - \mu)$, $i \geq 1$, where c is a positive constant, then $m = (m_i,\ i \geq 0)$ given by

$$m_0 = 1, \qquad m_i = c\pi_i, \quad i \geq 1, \tag{33}$$

is a μ-invariant measure for Q.

Proposition 2. *If μ satisfies $0 \leq \mu \leq \inf_{i \geq 1} d_i$ and Q defined in (32) satisfies $\sum_{i=1}^{\infty} \pi_i < \infty$ and $\sum_{i=1}^{\infty} \pi_i d_i = \infty$, then there exists a Q-process for which m, defined by (33), is a μ-invariant probability measure.*

Proof. The condition $\sum_{i=1}^{\infty} \pi_i < \infty$ implies that m is a finite measure, and the facts that $\sum_{i=1}^{\infty} \pi_i d_i = \infty$ and $\sum_{i=1}^{\infty} \pi_i < \infty$ together imply that $\sum_{j=1}^{\infty} h_j = \infty$. Hence, the result follows from Theorem 3.

6. Necessary conditions

In both of the examples above, our finite measure m satisfied

$$\sum_{i \neq b} m_i q_{ib} = \infty \tag{34}$$

and, hence, was invariant for Q (that is, (5) holds for *all* $j \in S$). We have established that only *almost* μ-invariance is needed for the existence of a Q-process for which the given (finite)

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measure is μ-invariant. It would therefore be of interest to know whether (34) is actually necessary for a (finite or infinite) measure m to be μ-invariant for P. We shall content ourselves with the following result, which shows that (34) is necessary in the $\mu = 0$ case under the condition that P is reversible.

Theorem 4. *Let Q be a uni-instantaneous q-matrix with instantaneous state b and let P be a Q-process with invariant measure m. If P is reversible with respect to m, that is if*

$$m_i p_{ij}(t) = m_j p_{ji}(t), \qquad i, j \in S, \tag{35}$$

then (34) holds.

Proof. On dividing (35) by t and letting $t \downarrow 0$, we obtain $m_i q_{ij} = m_j q_{ji}$, $j \neq i$. Hence,

$$\sum_{i \neq j} m_i q_{ij} = m_j \sum_{i \neq j} q_{ji}, \qquad j \in S,$$

meaning that, in particular,

$$\sum_{i \neq b} m_i q_{ib} = m_b \sum_{i \neq b} q_{bi} = \infty,$$

since Q is conservative.

We gain some insight into the general case from the following simple result, which follows directly from the proof of Theorem 1.

Theorem 5. *Let Q be a uni-instantaneous q-matrix with instantaneous state b and let P be a Q-process with μ-invariant measure m. Let P^* and Q^* be, respectively, the μ-reverse of P with respect to m and the μ-reverse of Q with respect to m. Then P^* is honest. In particular, b is an honest state for P^*, while being instantaneous for Q^*. Moreover,*

$$m_b \sum_{j \neq b} q^*_{bj} = \sum_{j \neq b} m_j q_{jb},$$

meaning that, in particular, b is a conservative state for Q^* if and only if (34) holds.

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References

[1] Anderson, W. J. (1991). *Continuous-Time Markov Chains. An Applications-Oriented Approach*. Springer, New York.

[2] Chen, A. Y. and Renshaw, E. (1990). Markov branching processes with instantaneous immigration. *Prob. Theory Relat. Fields* **87,** 209–240.

[3] Chen, A. Y. and Renshaw, E. (1993). Existence and uniqueness criteria for conservative uni-instantaneous denumerable Markov processes. *Prob. Theory Relat. Fields* **94,** 427–456.

https://doi.org/10.1239/jap/1127322022 Published online by Cambridge University Press

[4] CHUNG, K. (1967). *Markov Chains with Stationary Transition Probabilities*, 2nd edn. Springer, New York.
[5] KENDALL, D. G. AND REUTER, G. E. H. (1956). Some pathological Markov processes with a denumerable infinity of states and the associated semigroups of operators on l. In *Proc. Internat. Congr. Math. 1954, Amsterdam*, Vol. 3, Noordhoff, Groningen, pp. 377–415.
[6] KINGMAN, J. F. C. (1963). The exponential decay of Markov transition probabilities. *Proc. London Math. Soc.* **13,** 337–358.
[7] KOLMOGOROV, A. N. (1951). On the differentiability of the transition probabilities in stationary Markov processes with a denumerable number of states. *Moskov. Gos. Univ. Učenye Zapiski Mate.* **148,** 53–59 (in Russian).
[8] POLLETT, P. K. (1986). On the equivalence of μ-invariant measures for the minimal process and its q-matrix. *Stoch. Process. Appl.* **22,** 203–221.
[9] POLLETT, P. K. (1988). Reversibility, invariance and μ-invariance. *Adv. Appl. Prob.* **20,** 600–621.
[10] POLLETT, P. K. (1991). On the construction problem for single-exit Markov chains. *Bull. Austral. Math. Soc.* **43,** 439–450.
[11] POLLETT, P. K. AND VERE-JONES, D. (1992). A note on evanescent processes. *Austral. J. Statist.* **34,** 531–536.
[12] REUTER, G. E. H. (1959). Denumerable Markov processes. II. *J. London Math. Soc.* **34,** 81–91.
[13] REUTER, G. E. H. (1969). Remarks on a Markov chain example of Kolmogorov. *Z. Wahrscheinlichkeitsth.* **13,** 315–320.
[14] TWEEDIE, R. L. (1974). Some ergodic properties of the Feller minimal process. *Quart. J. Math. Oxford Ser.* **25,** 485–495.
[15] VERE-JONES, D. (1967). Ergodic properties of nonnegative matrices. I. *Pacific J. Math.* **22,** 361–386.
[16] YANG, X. Q. (1990). *The Construction Theory of Denumerable Markov Processes*. John Wiley, Chichester.
[17] ZHANG, H. J., LIN, X. AND HOU, Z. T. (2001). Invariant distributions of Q-processes. I. *Chinese Ann. Math. Ser. A* **22,** 323–330.
[18] ZHANG, H. J., LIN, X. AND HOU, Z. T. (2002). Invariant distributions of Q-processes. II. *Chinese Ann. Math. Ser. A* **23,** 361–370.

https://doi.org/10.1239/jap/1127322022 Published online by Cambridge University Press