J. Appl. Prob. **56,** 1269–1270 (2019)
*doi:*10.1017/jpr.2019.64

LETTER TO THE EDITOR

Dear Editor,

Kemeny's constant for infinite DTMCs is infinite

Consider a positive recurrent discrete-time Markov chain $(X_n)_{n\geq 0}$ with (countable) state space $\mathcal{S}$. For $x \in \mathcal{S}$, define the positive hitting time $T_x = \inf\{n \geq 1 \colon X_n = x\}$ and the hitting time $\theta_x = \inf\{n \geq 0 \colon X_n = x\}$. Let $\mathbb{P}_x$ denote the law of the process started from state x, and let $\mathbb{E}_x$ denote the corresponding expectation. It was observed by Kemeny and Snell [3] that, when $\mathcal{S}$ is finite, the expected hitting time of a random stationary target, i.e. the quantity

$$\kappa_x = \sum_{y\in\mathcal{S}} \pi_y \mathbb{E}_x[T_y], \tag{1}$$

does not depend on x. (Here $\boldsymbol{\pi} = (\pi_y)_{y\in\mathcal{S}}$ is the stationary distribution for the chain.) Thus, the quantity $\kappa = \kappa_x$ in (1) is called Kemeny's constant. Considerable effort has been devoted to giving an 'intuitive' proof of this result. In [1] it was argued that it is more natural to consider the quantity

$$\omega_x = \sum_{y\in\mathcal{S}} \pi_y \mathbb{E}_x[\theta_y].$$

Note that $\mathbb{E}_x[\theta_y] = \mathbf{1}_{\{y\neq x\}}\mathbb{E}_x[T_y]$, from which it follows that $\kappa_x = 1 + \omega_x$ (since $\pi_x\mathbb{E}_x[T_x] = 1$). For finite $\mathcal{S}$, Hunter [2] established the sharp bound $\kappa \geq (|\mathcal{S}| + 1)/2$ (the bound is achieved by the directed non-random walk on the cycle). It was conjectured in [1, p. 1031] that κ is infinite for any infinite state chain. In this note we verify this conjecture.

Theorem 1. *For an irreducible positive recurrent, discrete-time Markov chain with infinite state space and for any $x \in \mathcal{S}$, we have $\kappa_x = \sum_{y\in\mathcal{S}} \pi_y \mathbb{E}_x[T_y] = \infty$.*

This theorem is an immediate consequence of the following result.

Lemma 1. *Let $\mathcal{S}$ be finite or infinite. Then, for every $x, y \in \mathcal{S}$, $\mathbb{E}_x[T_y] \geq \pi_x/(2\pi_y)$.*

Proof. We first prove by induction on $n \geq 0$ that $\mathbb{P}_x(X_n = y) \leq \pi_y/\pi_x$ for every x, y. The case $n = 0$ is trivial (for both $x = y$ and $x \neq y$). For $n \geq 1$, we have

$$\mathbb{P}_x(X_n = y) = \sum_{u\in\mathcal{S}} \mathbb{P}_x(X_{n-1} = u)p_{u,y} \leq \sum_{u\in\mathcal{S}} \frac{\pi_u}{\pi_x} p_{u,y} = \frac{\pi_y}{\pi_x}, \tag{2}$$

where $(p_{w,z})_{w,z\in\mathcal{S}}$ are the one-step transition probabilities, and we have used the induction hypothesis and the full balance equations. Using (2), we have

$$\mathbb{P}_x(T_y \leq n) = \mathbb{P}_x\left(\bigcup_{j=1}^{n} \{X_j = y\}\right) \leq \sum_{j=1}^{n} \mathbb{P}_x(X_j = y) \leq \frac{n\pi_y}{\pi_x}.$$

Therefore, $\mathbb{P}_x(T_y > n) \geq 1 - n\pi_y/\pi_x$, and

$$\mathbb{E}_x[T_y] = \sum_{n=0}^{\infty} \mathbb{P}_x(T_y > n) \geq \sum_{n=0}^{\lfloor \pi_x/\pi_y \rfloor} \left(1 - \frac{n\pi_y}{\pi_x}\right) \geq \frac{\pi_x}{2\pi_y}.$$

Received 07 June 2019; revision received 18 June 2019.

The last step uses the fact that, for $a \geq 0$,

$$\sum_{n=0}^{\lfloor a \rfloor} \left(1 - \frac{n}{a}\right) = \frac{(2a - \lfloor a \rfloor)(\lfloor a \rfloor + 1)}{2a} \geq \frac{a}{2}. \qquad \square$$

Acknowledgements

OA was supported in part by NSERC. MH was supported by a Future Fellowship grant (no. FT160100166) from the Australian Research Council.

References

[1] Bini, D., Hunter, J. J., Latouche, G., Meini, B. and Taylor, P. G. (2018). Why is Kemeny's constant a constant? *J. Appl. Prob.* **55**, 1025–1036.

[2] Hunter, J. J. (2006). Mixing times with applications to perturbed Markov chains. *Linear Algebra Appl.* **417**, 108–123.

[3] Kemeny, J. G. and Snell, J. L. (1960). *Finite Markov Chains*. Van Nostrand, Princeton, NJ.

Yours sincerely,

Omer Angel* and Mark Holmes**

* Department of Mathematics,
The University of British Columbia,
1984 Mathematics Road,
Vancouver, BC V6T1Z2,
Canada.

** School of Mathematics and Statistics,
The University of Melbourne,
Parkville,
VIC 3010,
Australia.