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CHARACTERISATION OF DROP AND WEAK DROP PROPERTIES FOR CLOSED BOUNDED CONVEX SETS

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Modifying the concept underlying Daneš' drop theorem, Rolewicz introduced the notion of the drop property of a norm which was later generalised to the weak drop property of a norm. Kutzarova extended the discussion to consider the drop property for closed bounded convex sets. Here we characterise the drop and weak drop properties for such sets by upper semi-continuous and compact valued subdifferential mappings.

Consider a closed bounded convex set A in a Banach space $(X, \|\cdot\|)$. Given $x \notin A$, the set $D(x, A)$ is the convex hull of x and A is called the *drop* generated by x. Daneš [2] proved that for every closed set C at positive distance from the closed unit ball $\overline{B}(X)$ there exists an $x \in C$ such that $D(x, \overline{B}(X)) \cap C = \{x\}$. Rolewicz [11], modifying the Daneš drop theorem assumption said that the *norm* $\|\cdot\|$ *has the drop property* if for every closed set C disjoint from $\overline{B}(X)$ there exists an $x \in C$ such that $D(x, \overline{B}(X)) \cap C = \{x\}$. Rolewicz [11, p.34] proved that if the norm $\|\cdot\|$ has the drop property then X is reflexive. In [6] Giles, Sims and Yorke noted that Rolewicz' drop property of the norm can be characterised by the duality mapping from the dual sphere $S(X^*)$ into subsets of the second dual sphere $S(X^{**})$, being upper semi-continuous and compact valued. This characterisation led them to introduce a drop property weaker than that of Rolewicz. They said that the *norm* $\|\cdot\|$ *has the weak drop property* if for every weakly sequentially closed set C disjoint from $\overline{B}(X)$ there exists an $x \in C$ such that $D(x, \overline{B}(X)) \cap C = \{x\}$ and they showed that this property is equivalent to X being reflexive and able to be characterised by the duality mapping from the dual sphere, being upper semi-continuous from the norm to the weak topology and weak compact valued. Kutzarova [8] generalised Rolewicz' drop property of the norm in a different direction. Instead of drops formed from the closed unit ball she considered drops formed from any closed bounded convex set. She said that such a set A has the *drop property* if for every closed set C disjoint from A there exists an $x \in C$ such that $D(x, A) \cap C = \{x\}$. In papers [8] and [9] such sets with the drop property were studied.

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In this paper we carry the discussion to its next logical stage. We take Kutzarova's idea and consider a weaker drop property for closed bounded convex sets. We say that such a set A has the *weak drop property* if for every weakly sequentially closed set C disjoint from A there exists an $x \in C$ such that $D(x, A) \cap C = \{x\}$. Closed bounded convex sets with the weak drop property have a very satisfying characterisation, more so than those with the drop property, for such sets with the weak drop property are precisely those which are weakly compact. We show that closed bounded convex sets with the weak drop property and closed bounded convex sets with the drop property can be characterised by the subdifferential mapping $f \mapsto \partial p(f)$ of the gauge p of the polar A^0 mapping from X^* into subsets of X^{**}, being upper semi-continuous and compact valued on $X^* \backslash \{0\}$ under appropriate topologies.

We need a special case of Daneš' generalised drop theorem for closed bounded convex sets, [**3**, p.448]. We include a simple proof derived from Ekeland's variational principle, [**4**, p.169].

Theorem 1. *Given a closed bounded convex set A in a Banach space X, for every closed set C at positive distance from A, there exists an $x_0 \in C$ such that $D(x_0, A) \cap C = \{x_0\}$.*

Proof: We may suppose that $0 \in A$. Since A is bounded there exists a $\rho > 0$ such that $B[0; \rho] \supseteq A$ and $B[0; \rho] \cap C \neq \emptyset$. Now consider the complete metric space $Y \equiv B[0; \rho] \cap C$ with metric induced by the norm. Writing $d \equiv d(A, C) > 0$, consider the continuous function $\psi\colon Y \to \mathbb{R}$ defined by $\psi(x) = (2\rho/d)\|x\|$. By Ekeland's variational principle there exists an $x_0 \in Y$ such that $\psi(x_0) < \psi(x) + \|x - x_0\|$ for all $x \in Y$, $x \neq x_0$. Then $x_0 \in D(x_0, A) \cap C$. Suppose that there exists another point $x' \in D(x_0, A) \cap C$. Then $x' = (1 - \lambda)x_0 + \lambda v$ for some $0 < \lambda < 1$ and $v \in A$. So $\|x'\| \leqslant (1 - \lambda)\|x_0\| + \lambda\|v\|$ and $\lambda d \leqslant \lambda(\|x_0\| - \|v\|) \leqslant \|x_0\| - \|x'\|$. But then

$$\left(\frac{2\rho}{d}\right)\|x_0\| < \left(\frac{2\rho}{d}\right)\|x'\| + \|x' - x_0\| = \left(\frac{2\rho}{d}\right)\|x'\| + \lambda\|x_0 - v\|.$$

Now $\|x_0 - v\| \leqslant 2\rho$ so $\|x_0\| < \|x'\| + (\|x_0\| - \|x'\|)$ which is not possible. So we conclude that $D(x_0, A) \cap C = \{x_0\}$. $\square$

Associated with the drop property Rolewicz introduced a useful sequential concept. Given a closed bounded convex set A, a sequence $\{x_n\}$ in $X \setminus A$ such that $x_{n+1} \in D(x_n, A)$ for all $n \in \mathbb{N}$, is called a *stream*. Rolewicz [**11**, p.29] proved that the norm $\|\cdot\|$ has the drop property if and only if each stream in $X \setminus \overline{B}(X)$ contains a convergent subsequence and this was generalised in [**6**, p.506] to the norm $\|\cdot\|$ has the weak drop property if and only if each stream in $X \setminus \overline{B}(X)$ contains a weakly convergent subsequence.

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Theorem 1 enables us to give a first characterisation of the drop and weak drop properties for closed bounded convex sets by streaming sequences.

Theorem 2. *A closed bounded convex set A in a Banach space X has the drop (weak drop) property if and only if stream in $X \setminus A$ has a norm (weak) convergent subsequence.*

Proof: Suppose that there exists a stream $\{x_n\}$ in $X \setminus A$ which does not contain a norm (weak) convergent subsequence. Then $C \equiv \{x_n : n \in \mathbb{N}\}$ is a norm (weakly sequentially) closed set. Now $x_{n+1} \in D(x_n, A)$ for all $n \in \mathbb{N}$ and we see that there is no $n \in \mathbb{N}$ such that $D(x_n, A) \cap C = \{x_n\}$, so A does not have the drop (weak drop) property.

Conversely, suppose that A does not have the drop (weak drop) property. Then there exists a norm (weakly sequentially) closed set C disjoint from A such that for each $z \in C$, $\inf\{d(x, A) : x \in C \cap D(z, A)\} = 0$, otherwise we would contradict Theorem 1. So there exists a sequence $\{x_n\}$ in C such that $x_{n+1} \in D(x_n, A)$ and there exists a sequence $\{y_n\}$ in A such that $\|x_n - y_n\| \to 0$. As a stream, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ norm (weak) convergent to some x_0. Since C is norm (weakly sequentially) closed, $x_0 \in C$. But $\left\|x_{n_k} - y_{n_k}\right\| \to 0$ and so $\{y_{n_k}\}$ is norm (weak) convergent to x_0. As A is closed and convex, in both cases $x_0 \in A$. But this contradicts the fact that C and A are disjoint. $\square$

We recall some basic properties of polars and subdifferentials of gauge functionals. Given a convex set A in X with $0 \in \operatorname{int} A$, the *gauge* p of A is defined by $p(x) = \inf\{\lambda > 0 : x \in \lambda A\}$ and p is a continuous sublinear functional on X. A *subgradient* of p at $x_0 \in X$ is a continuous linear functional f on X such that $f(x - x_0) \leqslant p(x) - p(x_0)$ for all $x \in X$. The *subdifferential* of p at x_0, denoted by $\partial p(x_0)$, is the set of all subgradients of p at x_0. The Hahn-Banach Theorem guarantees that $\partial p(x) \neq \emptyset$ for each $x \in X$, [4, p.27]. Further, for each $x \in X$, $\partial p(x)$ is a weak* compact convex subset of X^*, and the *subdifferential mapping* $x \mapsto \partial p(x)$ is a set valued mapping from X into subsets of X^* and is *weak* upper semi-continuous*; that is, given $x \in X$ and a weak* neighbourhood W of 0 in X^* there exists a $\delta > 0$ such that $\partial p(y) \subseteq \partial p(x) + W$ when $\|x - y\| < \delta$, [4, p.132].

Given a closed bounded convex set A in X with $0 \in A$, recall that A^0 the *polar* of A is defined by $A^0 \equiv \{f \in X^* : f(x) \leqslant 1 \text{ for all } x \in A\}$ and A^0 is closed convex and $0 \in \operatorname{int} A^0$. The Banach-Alaoglu Theorem tells us that if $0 \in \operatorname{int} A$ then A^0 is weak* compact. A point $a_0 \in A$ is called a *support point* of A if there exists a continuous linear functional f on X, $f \neq 0$ such that $f(a_0) = \sup\{f(x) : x \in A\}$. If $\operatorname{int} A \neq \emptyset$ then the set of support points of A, denoted by $\operatorname{supp} A$, is the set $A \setminus \operatorname{int} A$, [4, p.67].

We note some basic relations between polars and subdifferentials. For a closed bounded convex set A with $0 \in \operatorname{int} A$, its gauge p is a continuous sublinear functional on X. Given $x_0 \in X$, for any $f_0 \in \partial p(x_0)$ we have $f_0(x_0) = p(x_0)$ and $f_0(x) \leqslant p(x)$ for all $x \in X$, so $f_0(x) \leqslant 1$ for all $x \in A$ and $f_0 \in A^0$. Using $\widehat{}$ to denote natural embedding elements, we have $f_0(x_0) = \sup \widehat{x}_0(A^0) \geqslant f(x_0)$ for all $f \in A^0$ and so $f_0 \neq 0$ then $f_0 \in$ supp A^0.

To develop our characterisation theorems we need the following lemma. We will denote by τ the weak (w) or norm (n) topologies and given a closed bounded convex set A with $0 \in$ int A we will say that the subdifferential mapping $x \mapsto \partial p(x)$ where p is the gauge of A, is $(n-\tau)$ *upper semi-continuous* at x if given a τ-open neighbourhood W of 0 in X^* there exists a $\delta > 0$ such that $\partial p(y) \subseteq \partial p(x) + W$ when $\|x - y\| < \delta$.

LEMMA 1. *Consider a closed bounded convex set A with $0 \in A$ in a Banach space X and the gauge p of the polar A^0 on X^*. If A is τ-compact then for each $f \in X^*$, $\partial p(f) \subseteq \widehat{A}$ and the subdifferential mapping $f \mapsto \partial p(f)$ from X^* into subsets of X^{**} is $(n-\tau)$ upper semi-continuous and τ-compact valued on X^*.*

PROOF: Note that the polar A^0 in X^* is closed convex and $0 \in$ int A^0, so its gauge p is a continuous sublinear functional on X^*. Consider the set-valued mapping $f \mapsto \partial p(f) \cap \widehat{A}$ from X^* into subsets of X^{**}. As $\partial p(f)$ is weak* compact and A is τ-compact so $\partial p(f) \cap \widehat{A}$ is τ-compact. We show that the mapping $f \mapsto \partial p(f) \cap \widehat{A}$ is $(n-\tau)$ upper semi-continuous. Suppose that it is not $(n-\tau)$ upper semi-continuous at $f_0 \in X^*$. Then there exists a τ-open neighbourhood W of 0 in X^{**} and for each $n \in \mathbb{N}$ there exists an $f_n \in X^*$ such that $\|f_n - f_0\| < 1/n$ and $\widehat{a}_n \in \partial p(f_n)$ but $\widehat{a}_n \notin \left(\partial p(f_0) \cap \widehat{A}\right) + W$. Now since $\widehat{A}$ is τ-compact, $\{\widehat{a}_n\}$ has τ-cluster point $\widehat{a}_0 \in \widehat{A}$. Given $f \in X^*$, consider

$$\widehat{a}_0(f) - \widehat{a}_0(f_0) = \widehat{a}_0(f) - \widehat{a}_n(f) + \widehat{a}_n(f) - \widehat{a}_n(f_n) + \widehat{a}_n(f_n) - \widehat{a}_n(f_0) + \widehat{a}_n(f_0) - \widehat{a}_0(f_0).$$

Now $\widehat{a}_0(f) - \widehat{a}_n(f)$ and $\widehat{a}_n(f_0) - \widehat{a}_0(f_0)$ can be made arbitrarily small since $\widehat{a}_0$ is a τ-cluster point of $\{\widehat{a}_n\}$. Also $|\widehat{a}_n(f_n) - \widehat{a}_n(f_0)| \leqslant K\|f_n - f_0\| \leqslant K \cdot 1/n$ for some $K > 0$ since $\widehat{A}$ is bounded. Further, $\widehat{a}_n(f) - \widehat{a}_n(f_n) \leqslant p(f) - p(f_n) = p(f) - p(f_0) + p(f_0) - p(f_n)$ and $p(f_n) \to p(f_0)$ since p is continuous. Therefore, $\widehat{a}_0(f) - \widehat{a}_0(f_0) \leqslant p(f) - p(f_0)$ for all $f \in X^*$ and so $\widehat{a}_0 \in \partial p(f_0)$. This contradicts the fact that for each $n \in \mathbb{N}$, $\widehat{a}_n \notin \left(\partial p(f_0) \cap \widehat{A}\right) + W$. Then both $f \mapsto \partial p(f)$ and $f \mapsto \partial p(f) \cap \widehat{A}$ are weak* upper semi-continuous and weak* compact valued mappings on X^*. But since p is convex its subdifferential mapping $f \mapsto \partial p(f)$ is a minimal weak* upper semi-continuous and weak* compact valued mapping on X^*, [**10**, p.100]. Therefore $\partial p(f) \subseteq \widehat{A}$ for all $f \in X^*$. We conclude that the subdifferential mapping $f \mapsto \partial p(f)$ is $(n-\tau)$ upper semi-continuous and τ-compact valued. $\square$

We now present our main characterisation theorem for closed bounded convex sets with the weak drop property. The theorem is a generalisation of the characterisation theorem for the weak drop property for the norm, [**6**, p.506] and that theorem follows as a particular case. Since the weak drop property is translation invariant, we may assume that the closed bounded convex sets we are considering contain 0.

Theorem 3. *For a closed bounded convex set A with $0 \in A$ in a Banach space X, the following statements are equivalent.*

(i) *A has the weak drop property;*
(ii) *A is weakly compact;*
(iii) *the subdifferential mapping $f \mapsto \partial p(f)$ for the gauge p of the polar A^0 mapping from X^* into subsets of X^{**} is $(n-w)$ upper semi-continuous and weak compact valued on $X^* \setminus \{0\}$.*

Proof: (i) $\Rightarrow$ (ii) Consider $f \in X^*$, $\|f\| = 1$. Write $M \equiv \sup\{f(x) : x \in A\}$; since A is bounded $M < \infty$. Consider $x_1 \notin A$ with $f(x_1) > M(2 - 1/4)$ and a sequence $\{y_n\}$ in A such that $f(y_n) > M(1 - 1/4^n)$. We define a sequence $\{x_n\}$ in $X \setminus A$ recursively by $x_{n+1} = (x_n + y_n)/2$.

Then $x_{n+1} \in D(x_n, A)$ and $f(x_n) \geqslant M(1 + 3/4^n)$. As a stream, we have from Theorem 2 that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ weakly convergent to some x_0 and it follows that $f(x_0) \geqslant M$. Write $z_{n+1} \equiv y_1/2^n + y_2/2^{n-1} + \ldots + y_n/2$.

Now z_{n+1} is a convex combination of $\{0, y_1, \ldots, y_n\}$ in A so $z_{n+1} \in A$. Also $x_{n+1} = x_1/2^n + z_{n+1}$ so $\|x_{n+1} - z_{n+1}\| = \|x_1\|/2^n \to 0$ and $n \to \infty$. Therefore $\{z_{n_k}\}$ is weakly convergent to x_0. But A is weakly closed so $x_0 \in A$. Therefore $f(x_0) = M$ and f attains its supremum on A so from James' characterisation [7], A is weakly compact.

(ii) $\Rightarrow$ (i) Consider $x_1 \notin A$ and a stream $\{x_n\}$. Then $x_{n+1} = \alpha_0^{n+1} x_1 + \alpha_1^{n+1} y_1 + \ldots + \alpha_n^{n+1} y_n$, a convex combination of x_1 and $\{y_1, y_2, \ldots, y_n\}$ in A. Write $z_{n+1} \equiv \alpha_1^{n+1} y_1 + \alpha_2^{n+1} y_2 + \ldots + \alpha_n^{n+1} y_n$. Then z_{n+1} is a convex combination of $\{0, y_1, \ldots, y_n\}$ in A so $z_{n+1} \in A$. Passing to a subsequence, we may assume that $\alpha_0^{n+1} \to$ to some α_0 as $n \to \infty$. As A is weakly compact there exists a subsequence $\{z_{n_k}\}$ weakly convergent to some $z_0 \in A$. Then $\{x_{n_k}\}$ is weakly convergent to $\alpha_0 x_1 + z_0$ and our result follows from Theorem 2.

(ii) $\Rightarrow$ (iii) It follows as a direct application of Lemma 1 that the subdifferential mapping $f \mapsto \partial p(f)$ is a $(n-w)$ upper semi-continuous mapping of X^* into subsets of X^{**} and is weak compact valued.

(iii) $\Rightarrow$ (ii) Suppose that (iii) holds but that A is not weakly compact. From James' characterisation there exists an $f_0 \in X^* \setminus \{0\}$ which does not attain its supremum on A. Then $\partial p(f_0)$ is a weakly compact convex set disjoint from the weakly closed convex set $\widehat{A}$. So $\partial p(f_0)$ and $\widehat{A}$ can be strongly separated by a weakly continuous linear functional on X^{**}. By the Bishop-Pehlps Theorem for convex sets [**1**, p.30], for each

$n \in \mathbb{N}$ there exists an $a_n \in A$ and an $f_n \in X^*$ such that $f_n(a_n) = \sup f_n(A)$ and $\|f_0 - f_n\| < 1/n$.

Then $p(f_n) = f_n(a_n)$ and $p(f) \geqslant f(a_n)$ for all $f \in X^*$ so $\widehat{a}_n(f) - \widehat{a}_n(f_n) \leqslant p(f) - p(f_n)$ for all $f \in X^*$; that is, $\widehat{a}_n \in \partial p(f_n)$.

Therefore, the subdifferential mapping $f \mapsto \partial p(f)$ being $(n-w)$ upper semi-continuous at f_0 implies that for every weak open neighbourhood W of 0 in X^{**}, $\partial p(f_0) + W$ contains elements from $\widehat{A}$. Since $\partial p(f_0)$ is weakly compact, this contradicts the separation property we establish previously. ∎

A similar characterisation theorem for bounded closed convex sets with the drop property is not nearly so straightforward. We draw on results established in [**8**, **9**]. To develop the proof we need to explore characterisations of the upper semi-continuity and compact valued property of subdifferential mappings which we do in the following two lemmas. These generalise results proved for duality mappings in [**5**, p.102 and p.104]

Given a closed bounded convex set A in X and $f \in X^* \setminus \{0\}$ we write $M(f, A) \equiv \sup\{f(x) : x \in A\}$ and for $\delta > 0$, the *slice* of A is the subset $S(A, f, \delta) \equiv \{x \in A : f(x) > M(f, A) - \delta\}$.

LEMMA 2. *Consider a closed bounded convex set A with $0 \in$ int A in a Banach space X and p the gauge of A on X. If the subdifferential mapping $x \mapsto \partial p(x)$ from X into subsets of X^* is $(n-\tau)$ upper semi-continuous at $x \in X \setminus \{0\}$ then for each τ-neighbourhood W of 0 in X^*, $\partial p(x) + W$ contains a slice of A^0 determined by x.*

PROOF: Suppose for some $\delta > 0$, we have $\partial p(y) \subseteq \partial p(x) + W/2$ when $\|y - x\| < \delta$. Choose $0 < \varepsilon < \delta$ such that $\varepsilon B(X^*) \subseteq W/2$. Consider $f \in S(A^0, x, \varepsilon^2)$, then $f(x) > \sup \widehat{x}(A^0) - \varepsilon^2$. By the Brøndsted-Rockafeller Theorem [**4**, p.173], there exists an $f_0 \in X^*$ and an $x_0 \in X$ such that $f_0(x_0) = \sup \widehat{x}_0(A^0) = p(x_0)$, so $f_0 \in \partial p(x_0)$, and $\|x - x_0\| < \varepsilon$ and $\|f - f_0\| < \varepsilon$. But since $\partial p(y) \subseteq \partial p(x) + W/2$ when $\|y - x\| < \delta$ we have $f \in f_0 + \varepsilon B(X^*) \subseteq f_0 + W/2 \subseteq \partial p(x) + W$. ∎

LEMMA 3. *Consider a closed bounded convex set A with $0 \in$ int A in a Banach space X and p the gauge of A on X. The subdifferential mapping $x \mapsto \partial p(x)$ from X into subsets of X^* is $(n-\tau)$ upper semi-continuous at $x \in X \setminus \{0\}$ and $\partial p(x)$ is τ-compact if and only if the weak* and τ topologies on A^0 agree at points of $\partial p(x)$.*

PROOF: Given that the subdifferential mapping $x \mapsto \partial p(x)$ is $(n-\tau)$ upper semi-continuous at $x \in X \setminus \{0\}$ and $\partial p(x)$ is τ-compact, suppose that there exists a net $\{f_\alpha\}$ in A^0 weak* convergent to $f_0 \in \partial p(x)$ but $\{f_\alpha\}$ is not τ-convergent to f_0. Then $\{f_\alpha\}$ is not τ-convergent to any element of $\partial p(x)$. Since $\partial p(x)$ is τ-compact there exists a τ-open set G such that $\partial p(x) \subseteq G$ and a subnet $\{f_{\alpha_\beta}\}$ such that for all β, $f_{\alpha_\beta} \notin G$. Again since $\partial p(x)$ is τ-compact there exists a τ-neighbourhood W of 0 in X^* such that $\partial p(x) + W \subseteq G$ so for all β, $f_{\alpha_\beta} \notin \partial p(x) + W$. However, $f_{\alpha_\beta}(x) \to f_0(x) = \sup \widehat{x}(A^0)$

so $\{f_{\alpha_\beta}\}$ is eventually in any given slice of A^0 determined by x. We see from Lemma 2 that we have contradicted the $(n-\tau)$ upper semi-continuity of the subdifferential mapping $x \mapsto \partial p(x)$ at x.

Conversely, given that for $x \in X \setminus \{0\}$, the weak* and τ-topologies on A^0 agree at points of $\partial p(x)$, it follows that $\partial p(x)$ is τ-compact. Suppose that the subdifferential mapping $x \mapsto \partial p(x)$ is not $(n-\tau)$ upper semi-continuous at x. Then there exists a sequence $\{x_n\}$ in X such that $x_n \to x$ and a τ-neighbourhood W of 0 in X^* and a sequence $\{f_n\}$ where $f_n \in \partial p(x_n)$ such that for all $n \in \mathbb{N}$, $f_n \notin \partial p(x) + W$. Since A^0 is weak* compact there exists a subnet $\{f_{n_\beta}\}$ which is weak* convergent to some $f_0 \in A^0$ and since the subdifferential mapping $x \mapsto \partial p(x)$ is $(n-w^*)$ upper semi-continuous then $f_0 \in \partial p(x)$. However, $\{f_{n_\beta}\}$ is not τ-convergent to f_0. ∎

Given a closed bounded convex set A, a support point x_0 of A is said to be a *point of continuity* if, whenever a sequence $\{x_n\}$ in A is weakly convergent to x_0, then it is norm convergent to x_0. Kutzarova proved that if A has the drop property then every support point of A is a point of continuity [**8**, p.284] and conversely if int $A \neq \emptyset$ and A is weakly compact and every support point is a point of continuity then A has the drop property, [**8**, p.284].

Given that a closed bounded convex set A has the drop property then by [**9**, Theorem 3] we have that A is norm compact or int $A \neq \emptyset$. If A is norm compact then from Lemma 1 we have that the subdifferential mapping $f \mapsto \partial p(f)$ for the gauge p of the polar A° mapping from X^* into subsets of X^{**} is $(n-n)$ upper semi-continuous and norm compact valued on X^*. We will characterise the more interesting case when int $A \neq 0$. Since the drop property is translation invariant, we may assume that $0 \in$ int A.

Theorem 4. *A closed bounded convex set A with $0 \in$ int A in a Banach space X has the drop property if and only if the subdifferential mapping $f \mapsto \partial p(f)$ for the gauge p of the polar A^0 mapping from X^* into subsets of X^{**} is $(n-n)$ upper semicontinuous and norm compact valued on $X^* \setminus \{0\}$.*

Proof: If a has the drop property then by Proposition 2.3, [**8**, p.284] we have that A is weakly compact. But then since int $\neq 0$, X is reflexive. Now the subdifferential mapping $f \mapsto \partial p(f)$ is $(n-w)$ upper semi-continuous and weak compact valued on X^*. Suppose that $f \mapsto \partial p(f)$ is not $(n-n)$ upper semi-continuous at $f_0 \in X^* \setminus \{0\}$. Then there exists a norm open neighbourhood W of 0 in X^{**} and for each $n \in \mathbb{N}$, an $f_n \in X^*$ such that $\|f_n - f_0\| < 1/n$ and $a_n \in \partial p(f_n)$ but $a_n \notin \partial p(f_0) + W$. Since A is weakly compact, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ weakly convergent to a_0 and, as in the proof of Lemma 1, $a_0 \in \partial p(f_0)$. But then a_0 is a support point of A. However, by Proposition 2.2, [**8**, p.284], we have that every support point is a point of continuity so $\{a_{n_k}\}$ is norm convergent to a_0. But this contradicts $a_n \notin \partial p(f_0) + W$.

So we conclude that $f \mapsto \partial p(f)$ is $(n-n)$ upper semi-continuous on $X^* \setminus \{0\}$. Given $f \in X^* \setminus \{0\}$, consider a sequence $\{a_n\}$ in $\partial p(f)$. Since $\partial p(f)$ is weakly compact there exists a subsequence $\{a_{n_k}\}$ weakly convergent to $a \in \partial p(f)$. But since every support point of A is a point of continuity, $\{a_{n_k}\}$ is norm convergent to a. So $\partial p(f)$ is norm compact.

Conversely, if the subdifferential mapping $f \mapsto \partial p(f)$ is $(n-n)$ upper semi-continuous and norm compact valued on $X^* \setminus \{0\}$ then it is $(n-w)$ upper semi-continuous and weak compact valued $X^* \setminus \{0\}$ so from Theorem 3, A is weakly compact and so X is reflexive. Since $0 \in$ int A applying Lemma 3 to A^0 we see that for each $f \in X^* \setminus \{0\}$ the weak and norm topologies on $A^{\circ\circ} = A$ agree at points of $\partial p(f)$. So every support point of A is a point of continuity and it follows from Theorem 2.1, [**8**, p.284] that A has the drop property. ∎

The following is an example of a closed bounded convex set which does not have the drop property but where the subdifferential condition holds. In Hilbert sequence space $(\ell_2, \|\cdot\|_2)$, consider A as the closed unit ball of the linear subspace ℓ_p for a given $1 < p < 2$. Now A is bounded closed and convex and int $A = \emptyset$ in $(\ell_2, \|\cdot\|_2)$. Also A^0 is the intersection of ℓ_2 with the closed unit ball of ℓ_q where $1/p + 1/q = 1$ and int $A^0 \neq \emptyset$. But the norm of ℓ_q is Fréchet differentiable away from 0, which will give us that the subdifferential mapping $f \mapsto \partial p(f)$ of the gauge p of A^0 is $(n-n)$ upper semi-continuous and norm compact valued on $X^* \setminus \{0\}$ in $(\ell_2, \|\cdot\|_2)$, [4, p.147]. However, as A is not norm compact and int $A = \emptyset$ we have from [**8**, p.285] that A does not have the drop property.

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