NONEXPONENTIAL ASYMPTOTICS FOR THE SOLUTIONS OF RENEWAL EQUATIONS, WITH APPLICATIONS

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Abstract

Nonexponential asymptotics for solutions of two specific defective renewal equations are obtained. These include the special cases of asymptotics for a compound geometric distribution and the convolution of a compound geometric distribution with a distribution function. As applications of these results, we study the asymptotic behavior of the demographic birth rate of females, the perpetual put option in mathematics of finance, and the renewal function for terminating renewal processes.

Keywords: Asymptotics; defective renewal equation; deterministic population model; compound geometric distribution; heavy-tailed distribution; perpetual put option; renewal function

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1. Introduction

Throughout this paper, all distributions are supported on $[0, \infty)$. We denote the tail of distribution F by $\overline{F} = 1 - F$ and the moment generating function of F by $m_F(s) = \int_0^\infty e^{sx} \, dF(x)$. For two functions f_1 and f_2, $f_1(x) \sim f_2(x)$ means that $\lim_{x \to \infty}(f_1(x)/f_2(x)) = 1$ and $f_1(x) = o(f_2(x))$ means that $\lim_{x \to \infty}(f_1(x)/f_2(x)) = 0$.

We consider the renewal equation

$$Z(x) = g(x) + \alpha \int_0^x Z(x - y) \, dF(y), \qquad x \geq 0, \tag{1.1}$$

where $\alpha > 0$, $F(y) = 1 - \overline{F}(y)$ is a proper probability distribution function with $F(0) = 0$, and $g(x) \geq 0$ is locally bounded. The renewal equation is called proper, defective, or excessive, according to whether α is equal to one, is less than one, or is bigger than one, respectively. In applications, the typical situation is that an unknown function Z satisfies a renewal equation of the form (1.1), where g, α, and F are known. Based on this, we want to draw conclusions about the function Z.

The only solution $Z(x)$ to (1.1) which is vanishing for $x < 0$ and bounded on finite intervals is given by

$$Z(x) = g(x) + \sum_{k=1}^\infty \alpha^k \int_0^x g(x - y) \, dF^{*k}(y), \qquad x \geq 0, \tag{1.2}$$

(see, e.g. Rolski *et al.* (1999, Lemma 6.1.2)), where F^{*k} is the distribution function of the k-fold convolution of F with itself.

The renewal equation of the form (1.1) arises in many applied probability models, such as branching processes, queueing theory, insurance ruin theory, reliability theory, demography, and so on. However, it is rarely soluble in closed form. Hence, there has been great interest in estimating the solution. The main purpose of this paper is to study the asymptotic behavior of the solution $Z(x)$ of (1.2) for large x. Set $K(\lambda) = \alpha \int_0^\infty e^{\lambda x} \, dF(x)$ and $l = \sup\{\lambda \geq 0 : K(\lambda) \leq 1\}$. Since $F(0) = 0$, we always have $l < \infty$. Only the following three cases can occur:

(i) $l > 0$ and $K(l) = 1$, the 'Cramér case',

(ii) $l = 0$, the 'heavy-tailed case',

(iii) $l > 0$ and $K(l) < 1$, the 'intermediate case'.

It turns out that the asymptotic behavior of $Z(x)$ heavily depends on the case which applies. For case (i), if F is a nonlattice distribution and $e^{lx} g(x)$ is directly Riemann integrable, then the famous Cramér–Lundberg estimate states that

$$Z(x) \sim \frac{\int_0^\infty e^{ly} g(y) \, dy}{\alpha \int_0^\infty y e^{ly} \, dF(y)} e^{-lx},$$

(see, e.g. Feller (1971)). Obviously, if $\alpha \geq 1$ then we can always find l satisfying condition (i), but if $\alpha < 1$ then there may be no such number and, thus, there is no exponential Cramér–Lundberg estimate for $Z(x)$. Asmussen (1998) investigated the asymptotic behavior of $Z(x)$ under the conditions that g is integrable, $g(x) \sim b(x)$ for some subexponential density b, and F has a subexponential density. Sharp upper and lower bounds on the solution $Z(x)$ to (1.1) have been derived in various situations; see Willmot *et al.* (2001) and references therein for details. In particular, if $g(x) = F(x)$ then (1.2) is a compound geometric distribution which has been studied by many authors (see, e.g. Embrechts *et al.* (1979), Embrechts and Goldie (1982), and Cline (1987)). If $g(x) = G(x)$, a distribution function supported on $[0, \infty)$, then (1.2) is the convolution of a compound geometric distribution with G. For this case, the asymptotic behavior and bounds of $\overline{Z}(x)$ have been studied by Cai and Garrido (2002) and Cai and Tang (2004).

In this paper, we investigate the nonexponential estimates of $Z(x)$ for some functions $g(x)$ in the cases (ii) and (iii), and then discuss applications of these results. In the sequel, we always assume that $\alpha < 1$.

2. Preliminaries

A random variable X (or its distribution function F) is said to be heavy-tailed on the right if $E\, e^{rX} = \infty$ for any $r > 0$. Heavy-tailed distributions have been the focus of study of many researchers in finance and insurance in recent years; see Embrechts *et al.* (1997) for a review.

A distribution function F is said to belong to the class $\mathcal{L}$ of long-tailed distributions if $\lim_{x \to \infty} (\overline{F}(x - y)/\overline{F}(x)) = 1$ for every $y > 0$ (or, equivalently, for some $y > 0$); F is said to belong to the subexponential class $\mathcal{S}$ if $\lim_{x \to \infty} (\overline{F^{*n}}(x)/\overline{F}(x)) = n$ for any $n \geq 2$ (or, equivalently, for $n = 2$); and F is said to belong to the class $\mathcal{D}$ of distribution functions with dominatedly varying tails if $\limsup_{x \to \infty} (\overline{F}(xy)/\overline{F}(x)) < \infty$ for every $y \in (0, 1)$ (or, equivalently, for some $y \in (0, 1)$). It is well known that $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ (see, e.g. Embrechts *et al.* (1997)).

We say that $F \in \mathcal{S}(\gamma)$, $\gamma \geq 0$, if and only if

$$\lim_{x \to \infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2m_F(\gamma) < \infty,$$

$$\lim_{x \to \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\gamma y}, \tag{2.1}$$

for all real y. We say that $F \in \mathcal{L}(\gamma)$ if and only if F satisfies (2.1). Note that $\mathcal{S}(0) = \mathcal{S}$, the class of subexponential distribution functions, and $\mathcal{L}(0) = \mathcal{L}$, the class of long-tailed distribution functions. Distributions in $\mathcal{S}(\gamma)$, $\gamma > 0$, are dominated by an exponential, i.e. $\lim_{x \to \infty} e^{\gamma x} \overline{F}(x) = 0$. One way to construct densities whose distributions are in $\mathcal{S}(\gamma)$ for $\gamma > 0$ is to multiply densities whose distributions are in $\mathcal{S}$ by negative exponentials. A typical example in $\mathcal{S}(\gamma)$ is F with $F'(x) \sim x^{-b} e^{-\gamma x}$ as $x \to \infty$, with $b > 1$. We refer the reader to Chover *et al.* (1973a), Embrechts and Veraverbeke (1982), Embrechts and Goldie (1982), and Klüppelberg (1989) for nice reviews of the classes $\mathcal{S}(\gamma)$ and $\mathcal{L}(\gamma)$, among others.

We now summarize some of the results about the classes of distributions above which will be used in the sequel.

Lemma 2.1. *For $\gamma \geq 0$, if $F_1 \in \mathcal{L}(\gamma)$, $F_2 \in \mathcal{S}(\gamma)$, and $\sup_{x \geq 0}(\overline{F_1}(x)/\overline{F_2}(x)) < \infty$, then*

$$\overline{F_1 * F_2}(x) \sim m_{F_1}(\gamma)\overline{F_2}(x) + m_{F_2}(\gamma)\overline{F_1}(x).$$

Proof. For the case $\gamma = 0$, see Embrechts and Goldie (1980); for the case $\gamma > 0$, see Cline (1986).

Remark 2.1. For the case $\gamma = 0$, if the boundedness of $\overline{F_1}(x)/\overline{F_2}(x)$ in Lemma 2.1, is strengthened to the requirement $\overline{F_1}(x) = o(\overline{F_2}(x))$, then the condition $F_1 \in \mathcal{L}$ can be omitted (see Embrechts *et al.* (1979, Proposition 1)).

Lemma 2.2. *For $\gamma \geq 0$, if $F \in \mathcal{S}(\gamma)$ then,*

  (i) *for all integers $n \geq 1$, $\lim_{x \to \infty}(\overline{F^{*n}}(x)/\overline{F}(x)) = n(m_F(\gamma))^{n-1}$,*

  (ii) *for any $\varepsilon > 0$, there exists a finite constant K_ε such that*

$$\overline{F^{*n}}(x) \leq K_\varepsilon (m_F(\gamma) + \varepsilon)^{n-1} \overline{F}(x)$$

   for all $n \geq 2$ and all $x \geq 0$.

Proof. For the proof of the case $\gamma = 0$, see Embrechts *et al.* (1997); for the case $\gamma > 0$, see Chover *et al.* (1973a), (1973b).

Lemma 2.3. (i) (Cline (1987, Theorem 2.13).) *Assume that $F \in \mathcal{L}(\gamma)$, $\gamma > 0$. Let $\{\lambda_n\}$ be a sequence of nonnegative coefficients such that $\lambda_j > 0$ for some $j > 1$ and $\sum_{n=0}^{\infty} \lambda_n (m_F(\gamma) + \varepsilon)^n < \infty$ for some $\varepsilon > 0$. Let $H = \sum_{n=0}^{\infty} \lambda_n F^{*n}$. The following statements are equivalent:*

  (a) $F \in \mathcal{S}(\gamma)$,

  (b) $\overline{H} \sim c\overline{F}$ for $c = \sum_{n=1}^{\infty} n\lambda_n (m_F(\gamma))^{n-1}$.

(ii) (Embrechts *et al.* (1979, Lemma 3).) *Suppose that* $\alpha \in (0, 1)$ *and G is a proper distribution function on* $[0, \infty)$. *If* $F(x) = (1 - \alpha) \sum_{n=0}^{\infty} \alpha^n G^{*n}(x)$ *then the following assertions are equivalent:*

   (c) $F \in \mathcal{S}$,

   (d) $G \in \mathcal{S}$,

   (e) $\overline{F} \sim (\alpha/(1 - \alpha))\overline{G}$.

Lemma 2.4. (Cai and Tang (2004).) *If* $F_1, F_2 \in \mathcal{D} \cap \mathcal{L}$ *then*

$$F_1 * F_2 \in \mathcal{D} \cap \mathcal{L} \quad and \quad \overline{F_1 * F_2}(x) \sim \overline{F_1}(x) + \overline{F_2}(x).$$

3. Asymptotic behavior of the solutions

Theorem 3.1. *Consider the renewal equation (1.1). Assume that $g(x) \geq 0$ is right-continuous and decreasing to* 0. *Let* $G(x) = (g(0) - g(x))/g(0)$, $x \geq 0$.

   (i) *If* $F \in \mathcal{S}$, $G \in \mathcal{S}$, *and* $\sup_x (\overline{F}(x)/\overline{G}(x)) < \infty$, *then*

$$Z(x) \sim \frac{g(x)}{1 - \alpha}. \tag{3.1}$$

   (ii) *If* $F \in \mathcal{S}$ *and* $\overline{G}(x) = o(\overline{F}(x))$, *then*

$$Z(x) = o(\overline{F}(x)). \tag{3.2}$$

   (iii) *If* $\gamma > 0$, $\alpha m_F(\gamma) < 1$, $F \in \mathcal{S}(\gamma)$, $G \in \mathcal{S}(\gamma)$, *and* $\sup_x (\overline{F}(x)/\overline{G}(x)) < \infty$, *then*

$$Z(x) \sim \frac{\alpha g(0)(m_G(\gamma) - 1)}{(1 - \alpha m_F(\gamma))^2}\overline{F}(x) + \frac{g(x)}{1 - \alpha m_F(\gamma)}. \tag{3.3}$$

   (iv) *If* $\gamma > 0$, $\alpha m_F(\gamma) < 1$, $F \in \mathcal{S}(\gamma)$, $G \in \mathcal{S}(\gamma)$, *and* $\overline{G}(x) = o(\overline{F}(x))$, *then*

$$Z(x) \sim \frac{\alpha g(0)(m_G(\gamma) - 1)}{(1 - \alpha m_F(\gamma))^2}\overline{F}(x). \tag{3.4}$$

Proof. (i) Observe that G is a proper distribution function on $[0, \infty)$ satisfying $G(0) = 0$ with right-tail $\overline{G}(x) = g(x)/g(0)$, $x \geq 0$. It follows from (1.2) that $Z(x)$ can be expressed as

$$\begin{aligned} Z(x) &= \frac{g(0)}{1 - \alpha} \int_0^x \overline{G}(x - y)\, dU_0(y) \\ &= \frac{g(0)}{1 - \alpha}(\overline{G * U_0}(x) - \overline{U_0}(x)), \end{aligned} \tag{3.5}$$

where

$$U_0(x) = (1 - \alpha) \sum_{n=0}^{\infty} \alpha^n F^{*n}(x).$$

By Lemma 2.3(ii) or Cai and Tang (2004, Proposition 1.1(c)),

$$\overline{U_0}(x) \sim \frac{\alpha}{1 - \alpha}\overline{F}(x),$$

which implies that $U_0 \in \mathscr{S}$, since $\mathscr{S}$ is closed with respect to tail equivalence (see, e.g. Teugels (1975)). Using Lemma 2.2(ii) and the condition $\sup_x (\overline{F}(x)/\overline{G}(x)) < \infty$, we find that $\sup_x (\overline{U_0}(x)/\overline{G}(x)) < \infty$. Thus, it follows from Lemma 2.1 that $\overline{G * U_0}(x) \sim \overline{G}(x) + \overline{U_0}(x)$, which, together with (3.5), gives

$$
\begin{aligned}
\frac{\overline{Z}(x)}{\overline{G}(x)} &= \frac{g(0)}{1 - \alpha} \left(\frac{\overline{G}(x) + \overline{U_0}(x) + o(\overline{G}(x) + \overline{U_0}(x))}{\overline{G}(x)} - \frac{\overline{U_0}(x)}{\overline{G}(x)} \right) \\
&= \frac{g(0)}{1 - \alpha} \left(1 + \frac{o(\overline{G}(x) + \overline{U_0}(x))}{\overline{G}(x) + \overline{U_0}(x)} \left(1 + \frac{\overline{U_0}(x)}{\overline{G}(x)} \right) \right) \\
&\to \frac{g(0)}{1 - \alpha}.
\end{aligned}
$$

Since $g(x) = g(0)\overline{G}(x)$, (3.1) follows.

(ii) The condition $\overline{G}(x) = o(\overline{F}(x))$ implies that $\overline{G}(x) = o(\overline{U_0}(x))$ since $\overline{U_0}(x) \sim (\alpha/(1 - \alpha))\overline{F}(x)$. If $F \in \mathscr{S}$ then, by Lemma 2.1 and Remark 2.1 or using Cai and Garrido (2002, Theorem 3.3), we have $\overline{G * U_0}(x) \sim \overline{U_0}(x)$, which gives (3.2) by using (3.5).

Similarly, (3.3) and (3.4) follow from Lemma 2.1 and the fact that (by Lemma 2.3(i))

$$
\overline{U_0}(x) \sim \frac{\alpha(1 - \alpha)}{(1 - \alpha m_F(\gamma))^2} \overline{F}(x),
$$

which implies that $U_0 \in \mathscr{S}(\gamma)$ since $\mathscr{S}(\gamma)$ is closed with respect to tail equivalence (see Embrechts and Goldie (1982, Theorem 2.7)). This ends the proof of Theorem 3.1.

Theorem 3.2. *Consider the renewal equation (1.1). Suppose that $g(x) \geq 0$ with $g(0) = 0$ is a right-continuous and nondecreasing function such that $\lim_{x \to \infty} g(x) = g(\infty) < \infty$. Let $P(x) = g(x)/g(\infty)$, $x \geq 0$.*

(i) *If $F \in \mathscr{S}$, $P \in \mathscr{S}$, and $\sup_x (\overline{F}(x)/\overline{P}(x)) < \infty$, then*

$$
Z(\infty) - Z(x) \sim \frac{g(\infty) - g(x)}{1 - \alpha} + \frac{\alpha g(\infty)}{(1 - \alpha)^2} \overline{F}(x). \tag{3.6}
$$

In particular, if $g(\infty) = 1$ then $\overline{Z}(x) \sim \overline{g}(x)/(1 - \alpha) + (\alpha/(1 - \alpha)^2)\overline{F}(x)$.

(ii) *If $F \in \mathscr{S}$ and $\overline{P}(x) = o(\overline{F}(x))$, then*

$$
Z(\infty) - Z(x) \sim \frac{\alpha g(\infty)}{(1 - \alpha)^2} \overline{F}(x).
$$

In particular, if $g(\infty) = 1$ then $\overline{Z}(x) \sim (\alpha/(1 - \alpha)^2)\overline{F}(x)$.

(iii) *If $\gamma > 0$, $\alpha m_F(\gamma) < 1$, $F \in \mathscr{S}(\gamma)$, $P \in \mathscr{S}(\gamma)$, and $\sup_x (\overline{F}(x)/\overline{P}(x)) < \infty$, then*

$$
Z(\infty) - Z(x) \sim \frac{g(\infty) - g(x)}{1 - \alpha m_F(\gamma)} + \frac{\alpha g(\infty) m_P(\gamma)}{(1 - \alpha m_F(\gamma))^2} \overline{F}(x).
$$

In particular, if $g(\infty) = 1$ then

$$
\overline{Z}(x) \sim \frac{1}{1 - \alpha m_F(\gamma)} \overline{g}(x) + \frac{\alpha m_g(\gamma)}{(1 - \alpha m_F(\gamma))^2} \overline{F}(x).
$$

(iv) *If $\gamma > 0$, $\alpha m_F(\gamma) < 1$, $F \in \mathcal{S}(\gamma)$, $P \in \mathcal{S}(\gamma)$, and $\overline{P}(x) = o(\overline{F}(x))$, then*

$$Z(\infty) - Z(x) \sim \frac{\alpha g(\infty) m_P(\gamma)}{(1 - \alpha m_F(\gamma))^2} \overline{F}(x).$$

In particular, if $g(\infty) = 1$ then $\overline{Z}(x) \sim (\alpha m_g(\gamma)/(1 - \alpha m_F(\gamma))^2)\overline{F}(x)$.

Proof. (i) Clearly, P is a proper distribution function on $[0, \infty)$ satisfying $P(0) = 0$ with right-tail

$$\overline{P}(x) = \frac{g(\infty) - g(x)}{g(\infty)}.$$

It follows from (1.2) that the solution of (1.1) can be written as

$$Z(x) = (1 - \alpha)^{-1} g(\infty)(P * U_0)(x), \qquad x \geq 0, \tag{3.7}$$

where, as before, $U_0(x) = (1 - \alpha) \sum_{n=0}^{\infty} \alpha^n F^{*n}(x)$. Thus,

$$\frac{Z(x)}{Z(\infty)} = (P * U_0)(x), \qquad x \geq 0, \tag{3.8}$$

since $Z(\infty) = g(\infty)(1 - \alpha)^{-1}$ by (3.7). As in the proof of Theorem 3.1(i), $U_0 \in \mathcal{S}$, $\overline{U_0}(x) \sim (\alpha/(1 - \alpha))\overline{F}(x)$, and $\sup_x (\overline{U_0}(x)/\overline{P}(x)) < \infty$. By Lemma 2.1 we obtain

$$\overline{P * U_0}(x) \sim \overline{P}(x) + \overline{U_0}(x) \sim \overline{P}(x) + \frac{\alpha}{1 - \alpha} \overline{F}(x),$$

which, together with (3.8), gives (3.6).

The proof of part (ii) is virtually the same as that of Theorem 3.1(ii). We can prove parts (iii) and (iv) in a similar way to parts (i) and (ii). This ends the proof of Theorem 3.2.

Remark 3.1. We remark that the condition $\sup_x (\overline{F}(x)/\overline{P}(x)) < \infty$ in Theorem 3.2 is weaker than the conditions used by Cai and Garrido (2002, Theorems 3.4, 3.5, and 4.1), and the asymptotic results of the above paper can be deduced from Theorem 3.2.

The following theorem does not assume a domination relationship between F and P.

Theorem 3.3. *Suppose that $g(x)$ satisfies the same conditions as in Theorem 3.2. If $F, P \in \mathcal{D} \cap \mathcal{L}$, where $P(x) = g(x)/g(\infty)$, then*

$$Z(\infty) - Z(x) \sim \frac{g(\infty) - g(x)}{1 - \alpha} + \frac{\alpha g(\infty)}{(1 - \alpha)^2} \overline{F}(x).$$

In particular, if $g(\infty) = 1$ then $\overline{Z}(x) \sim \overline{g}(x)/(1 - \alpha) + (\alpha/(1 - \alpha)^2)\overline{F}(x)$.

Proof. The proof is similar to that of Theorem 3.2, but using Lemma 2.4 instead of Lemma 2.1.

4. Applications

In this section, we study the asymptotic behavior of various renewal equations arising in applied probability models. Several examples from insurance ruin theory, branching processes, and queueing theory can be found in Willmot *et al.* (2001). We first consider an example from demography.

Example 4.1. (*Deterministic population model.*) Let $B(t)$ denote the rate at which females are born at time t, i.e. $B(t)\,dt$ is the number of female babies born between times t and $t + dt$. Given $B(t)$ for $t \leq 0$, project $B(t)$ for $t > 0$. For this purpose, assume a survival function $s(x)$ and an age-specific force of birth $\beta(x)$, $x > 0$. Thus, $s(x)$ is the probability that a newly born female survives to age x, and $\beta(x)\,dt$ is the number of female babies borne by a parent of age x in a time interval of length dt. Then $B(t)$ satisfies the following renewal equation (see Gerber (1979)):

$$B(t) = \int_t^\infty B(t - x)s(x)\beta(x)\,dx + \int_0^t B(t - x)s(x)\beta(x)\,dx. \tag{4.1}$$

Clearly, (4.1) is of the form (1.1) with $g(x) = \int_x^\infty B(x - y)s(y)\beta(y)\,dy$, $\alpha = \int_0^\infty s(x)\beta(x)\,dx$, and

$$F(x) = \frac{\int_0^x s(y)\beta(y)\,dy}{\int_0^\infty s(y)\beta(y)\,dy}.$$

If there exist positive constants c_i such that $c_1 \leq \beta(x) \leq c_2$ and $c_3 \leq B(-x) \leq c_4$, for $x \geq 0$, then

$$\frac{c_1}{c_2} \leq \frac{\overline{F}(x)}{\overline{S}(x)} \leq \frac{c_2}{c_1}$$

for all $x \in [0, \infty)$ and $\sup_{x \geq 0}(\overline{F}(x)/\overline{G}(x)) < \infty$, where

$$\overline{S}(x) = \frac{\int_x^\infty s(y)\,dy}{\int_0^\infty s(y)\,dy}, \qquad \overline{G}(x) = \frac{\int_x^\infty B(x - y)s(y)\beta(y)\,dy}{\int_0^\infty B(x - y)s(y)\beta(y)\,dy}.$$

Thus, if $S \in \mathcal{S}(\gamma)$ then $F \in \mathcal{S}(\gamma)$ (see, e.g. Klüppelberg (1988, Theorem 2.1)). If, in addition, $\alpha m_F(\gamma) < 1$, then the conditions of Theorem 3.1 are fulfilled. Consequently, Theorem 3.1 yields

$$B(x) \sim \begin{cases} \dfrac{\int_x^\infty B(x - y)s(y)\beta(y)\,dy}{1 - \int_0^\infty s(y)\beta(y)\,dy} & \text{if } \gamma = 0, \\[2em] \dfrac{(m_G(\gamma) - 1)\int_x^\infty s(y)\beta(y)\,dy \int_0^\infty B(-y)s(y)\beta(y)\,dy}{(1 - \alpha m_F(\gamma))^2} \\[1.5em] \quad + \dfrac{\int_x^\infty B(x - y)s(y)\beta(y)\,dy}{1 - \alpha m_F(\gamma)} & \text{if } \gamma > 0. \end{cases}$$

The second example is from mathematics of finance.

Example 4.2. (*Pricing a perpetual put option.*) Let $S(t)$ denote the price at time $t \geq 0$ of a non-dividend-paying stock. We assume that the logarithm of the stock price process follows a process of the form $U(t) = u + ct - Z(t)$, where $c > 0$ is constant and $\{Z(t)\}$ is a compound Poisson process specified by the Poisson parameter $\lambda > 0$ and the distribution of the jump amounts. To simplify notation, we suppose that the latter is continuous with probability density $p(x)$, $x \geq 0$. We further assume that the market is frictionless and risk-neutral, and that there is a constant risk-free force of interest, $r > 0$.

We consider a perpetual put option with payoff function $\Pi(s) = \max(K - s, 0)$, where K is the exercise price. We restrict ourselves to options where it is clear a priori that the optimal exercise strategy is a stopping time of the form $T_L = \inf\{t : S(t) < L\}$. Let

$$V(s; L) = \mathrm{E}[\mathrm{e}^{-rT_L}\Pi(S(T_L)) \mid S(0) = s], \qquad s \geq L,$$

denote the expected discounted value of the payoff if the initial stock price is s and the strategy T_L is applied. Then $V(s; L)$ satisfies the equation (see Gerber and Shiu (1998))

$$V(s; L) = \int_0^{\ln(s/L)} V(se^{-y}; L)h(y)\,dy + \int_{\ln(s/L)}^{\infty} \Pi(se^{-y})h(y)\,dy, \qquad s \geq L,$$

where

$$h(y) = \frac{\lambda}{c}e^y \int_y^{\infty} e^{-x}p(x)\,dx.$$

Replace s by $x = \ln(s/L)$ and consider the function $W(x; L)$, defined by $W(x; L) = V(s; L)$. Then W satisfies the following defective renewal equation:

$$W(x; L) = \int_0^x W(x - y; L)h(y)\,dy + \int_x^{\infty} \Pi(Le^{x-y})h(y)\,dy, \qquad x \geq 0. \tag{4.2}$$

Clearly, (4.2) is of the form (1.1) with $g(x) = \int_x^{\infty} \Pi(Le^{x-y})h(y)\,dy$, $\alpha = \int_0^{\infty} h(x)\,dx$, and $F(x) = (\int_0^{\infty} h(x)\,dx)^{-1} \int_0^x h(x)\,dx$. Let $\overline{G}(x) = g(x)/g(0)$. If $F \in \mathcal{S}(\gamma)$, $\gamma \geq 0$, then it is not difficult to verify that

$$\lim_{x \to \infty} \frac{\overline{F}(x)}{\overline{G}(x)} = \begin{cases} \dfrac{\int_0^{\infty} \Pi(Le^{-y})h(y)\,dy}{\int_0^{\infty} h(y)\,dy(K - (\gamma L \int_0^{\infty} h(y)\,dy)/(1 + \gamma \int_0^{\infty} h(y)\,dy))} & \text{if } \gamma > 0, \\[4mm] \dfrac{\int_0^{\infty} \Pi(Le^{-y})h(y)\,dy}{K \int_0^{\infty} h(y)\,dy} & \text{if } \gamma = 0. \end{cases}$$

Thus, $G \in \mathcal{S}(\gamma)$ and $\sup_{u \geq 0}(\overline{F}(u)/\overline{G}(u)) < \infty$, since $\mathcal{S}(\gamma)$ is closed with respect to tail equivalence. Applying Theorem 3.1 yields the following results.

(i) For $\gamma = 0$,

$$W(x; L) \sim \frac{\int_x^{\infty} \Pi(Le^{x-y})h(y)\,dy}{1 - \int_0^{\infty} h(y)\,dy}, \qquad \text{as } x \to \infty,$$

or

$$V(s; L) \sim \frac{\int_{\ln(s/L)}^{\infty} \Pi(se^{-y})h(y)\,dy}{1 - \int_0^{\infty} h(y)\,dy}, \qquad \text{as } s \to \infty.$$

(ii) For $\gamma > 0$,

$$W(x; L) \sim \frac{(m_G(\gamma) - 1)\int_0^{\infty} \Pi(Le^{-y})h(y)\,dy \int_x^{\infty} h(y)\,dy}{(1 - m_F(\gamma)\int_0^{\infty} h(y)\,dy)^2}$$
$$+ \frac{\int_x^{\infty} \Pi(Le^{x-y})h(y)\,dy}{1 - m_F(\gamma)\int_0^{\infty} h(y)\,dy}, \qquad \text{as } x \to \infty,$$

or

$$V(s; L) \sim \frac{(m_G(\gamma) - 1)\int_0^{\infty} \Pi(Le^{-y})h(y)\,dy \int_{\ln(s/L)}^{\infty} h(y)\,dy}{(1 - m_F(\gamma)\int_0^{\infty} h(y)\,dy)^2}$$
$$+ \frac{\int_{\ln(s/L)}^{\infty} \Pi(se^{-y})h(y)\,dy}{1 - m_F(\gamma)\int_0^{\infty} h(y)\,dy}, \qquad \text{as } s \to \infty.$$

The last example is from renewal theory.

Example 4.3. (*The renewal function for terminating renewal processes.*) Let $T_1, T_2, \dots$ be a sequence of nonnegative, independent, and identically distributed random variables with common distribution function F; we define the renewal counting process

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}\{T_1 + \cdots + T_n \le t\}, \qquad t \ge 0,$$

where $\mathbf{1}$ is the indicator function. The function $M(t) = \mathrm{E}\, N(t)$ is called the renewal function. It gives the expected number of renewals in the interval $[0, t]$. To avoid trivialities we assume that the interrenewal distances $T_1, T_2, \dots$ are not concentrated at zero, that is, $F(0) < 1$. If $F(\infty) < 1$ then the resulting renewal process is called terminating or transient. Then M satisfies (see, e.g. Feller (1971)) the following defective renewal equation:

$$M(t) = F(t) + \int_0^t M(t - x)\, \mathrm{d}F(x), \qquad x \ge 0. \tag{4.3}$$

Clearly, (4.3) is of the form (1.1) with $\alpha = F(\infty)$ and $g(t) = F(t)$. Assume that $F \in \mathscr{S}(\gamma)$, $\gamma \ge 0$. Then Theorem 3.2 yields

$$M(\infty) - M(t) \sim \begin{cases} \dfrac{F(\infty) - F(t)}{(1 - F(\infty))^2} & \text{if } \gamma = 0, \\[2ex] \dfrac{F(\infty) - F(t)}{(1 - m_F(\gamma))^2} & \text{if } \gamma > 0. \end{cases}$$

The result for the case $\gamma = 0$ can be found in Teugels (1975, Theorem 4); here we have provided a different proof.

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