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NUMERICAL APPROXIMATION OF STATIONARY DISTRIBUTIONS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper we discuss an exponential integrator scheme, based on spatial discretization and time discretization, for a class of stochastic partial differential equations. We show that the scheme has a unique stationary distribution whenever the step size is sufficiently small, and that the weak limit of the stationary distribution of the scheme as the step size tends to 0 is in fact the stationary distribution of the corresponding stochastic partial differential equations.

Keywords: Stochastic partial differential equation; mild solution; stationary distribution; exponential integrator scheme; numerical approximation

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1. Introduction

The convergence and the stability of numerical schemes for finite-dimensional stochastic differential equations (SDEs) have been extensively investigated; see, e.g. [18] and [22]. Nowadays, numerical approximate schemes for stochastic partial differential equations (SPDEs) are also becoming more and more popular. There is extensive literature on strong/weak convergence of approximate solutions for SPDEs. For instance, under a dissipative condition, Caraballo and Kloeden [3] showed pathwise convergence of finite-dimensional approximations for a class of reaction–diffusion equations. Applying the Malliavin calculus approach, Debussche [7] discussed weak convergence of an implicit Euler scheme for the stochastic heat equation with multiplicative noise. Greksch and Kloeden [8] investigated time-discretised Galerkin approximations of parabolic SPDEs through an eigenfunction argument. Gyöngy [9], [10], Shardlow [23], and Yoo [25] applied finite element methods to approximate solutions of parabolic SPDEs driven by space–time white noise. Hausenblas [12], [13] utilized discretization in time, including implicit Euler, explicit Euler, and Crank–Nicholson schemes, to approximate quasilinear evolution equations. Higher-order pathwise numerical approximations of SPDEs with additive noise was considered in [15]. For the Taylor approximations of SPDEs, we refer the reader to the monograph [16].

There are few results however on the asymptotic behavior of numerical solutions for infinite-dimensional SPDEs although the counterpart for the finite-dimensional case has been extensively studied; see, e.g. [22]. In this work we will investigate the long-term behavior

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of a certain numerical scheme for a class of SPDEs. To begin with, we introduce some notation and, thus, give the framework of our work. Let $(H, \langle\cdot,\cdot\rangle_{\mathrm{H}}, \|\cdot\|_{\mathrm{H}})$ be a real separable Hilbert space. Let $\mathrm{id}_{\mathrm{H}}\colon H \to H$ be the identity operator, and denote by $(\mathcal{L}(H), \|\cdot\|)$ and $(\mathcal{L}_{\mathrm{HS}}(H), \|\cdot\|_{\mathrm{HS}})$ the families of bounded linear operators and Hilbert–Schmidt operators from H into H, respectively. In this paper we consider an SPDE on the real separable Hilbert space $(H, \langle\cdot,\cdot\rangle_{\mathrm{H}}, \|\cdot\|_{\mathrm{H}})$ of the form

$$\mathrm{d}X(t) = \{AX(t) + b(X(t))\}\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t) \tag{1}$$

with initial value $X(0) = x \in H$, where $W(t)$ is an H-valued cylindrical id_{H}-Wiener process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, $b\colon H \to H$ is a Lipschitz continuous mapping, $\sigma(x) := \sigma^0 + \sigma^1(x)$, $x \in H$, such that $\sigma^0 \in \mathcal{L}(H)$, and $\sigma^1\colon H \to \mathcal{L}_{\mathrm{HS}}(H)$. The interested reader is referred to the classical book [4] for further details on SPDEs.

Throughout the paper, we impose the following assumptions.

(H1) $(A, \mathcal{D}(A))$ is a self-adjoint operator on H generating an immediately compact C_0-semigroup $\{\mathrm{e}^{tA}\}_{t\ge 0}$ such that $\|\mathrm{e}^{tA}\| \le \mathrm{e}^{-\alpha t}$ for some $\alpha > 0$. In this case, by [17, Theorem 6.26 and Theorem 6.29], $-A$ has a discrete spectrum $\{\lambda_i\}_{i\ge 1}$ such that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots$ and $\lim_{i\to\infty}\lambda_i = \infty$ with corresponding eigenbasis $\{e_i\}_{i\ge 1}$ of H.

(H2) There exist $\theta_1 \in (0, 1)$ and $\delta_1 \in (0, \infty)$ such that $\int_0^t \|(-A)^{\theta_1}\mathrm{e}^{sA}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \le \delta_1$ for any $t > 0$, where $(-A)^{\theta_1} := \sum_{k\ge 1}\lambda_k^{\theta_1}(e_k \otimes e_k)$ denotes the fractional power of $-A$.

(H3) There exist $L_1, L_2 > 0$ such that

$$\|b(x)-b(y)\|_{\mathrm{H}} \le L_1\|x-y\|_{\mathrm{H}}, \qquad \|\sigma^1(x)-\sigma^1(y)\|_{\mathrm{HS}} \le L_2\|x-y\|_{\mathrm{H}}, \qquad x, y \in H.$$

(H4) There exists $\gamma \in \mathbb{R}$ such that

$$2\langle x-y, b(x)-b(y)\rangle_{\mathrm{H}} + \|\sigma^1(x)-\sigma^1(y)\|_{\mathrm{HS}}^2 \le -\gamma\|x-y\|_{\mathrm{H}}^2, \qquad x, y \in H.$$

By [5, Theorem 5.3.1] we know that (H1)–(H3) imply existence and uniqueness of the mild solution to (1), i.e. there exists a unique H-valued adapted process $X_x(t)$ with the initial value $x \in H$ such that

$$X_x(t) = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A}b(X_x(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\sigma(X_x(s))\,\mathrm{d}W(s). \tag{2}$$

Remark 1. In fact, under (H1), (H3), and $\int_0^t \|\mathrm{e}^{sA}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \le \delta_2$ for any $t > 0$ and some $\delta_2 > 0$, (1) also admits a unique mild solution on H. Condition (H2) is just imposed for the later numerical analysis. Let $\sigma^0 = \mathrm{id}_{\mathrm{H}}$, and let $Ax := \partial_\xi^2 x$ for $x \in \mathcal{D}(A) := H^2(0,\pi) \cap H_0^1(0,\pi)$. Then A is a self-adjoint negative operator and $Ae_k = -k^2 e_k$, $k \in \mathbb{N}$, where $e_k(\xi) := (2/\pi)^{1/2}\sin k\xi$, $\xi \in [0,\pi]$ and $k \in \mathbb{N}$. A simple computation shows that

$$\int_0^t \|(-A)^{\theta_1}\mathrm{e}^{sA}\|_{\mathrm{HS}}^2\,\mathrm{d}s = \sum_{k=1}^{\infty}(k^2)^{2\theta_1}\int_0^t \mathrm{e}^{-2k^2 s}\,\mathrm{d}s \le \frac{1}{2}\sum_{k=1}^{\infty}(k^2)^{2\theta_1-1}.$$

Then (H2) holds with $\delta_1 = \frac{1}{2}\sum_{k=1}^{\infty}(k^2)^{2\theta_1-1}$ for $\theta_1 \in (0, \frac{1}{4})$.

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Remark 2. By (H3), it is readily seen that

$$\|b(x)\|_{\mathrm{H}}^2 + \|\sigma^1(x)\|_{\mathrm{HS}}^2 \le \bar{L}(1 + \|x\|_{\mathrm{H}}^2), \qquad x \in H, \tag{3}$$

where $\bar{L} := 2((L_1^2 + L_2^2) \vee \mu)$ with $\mu := \|b(0)\|_{\mathrm{H}}^2 + \|\sigma^1(0)\|_{\mathrm{HS}}^2$. Moreover, by (H4) we have

$$\begin{aligned} 2\langle x, b(x)\rangle_{\mathrm{H}} &+ \|\sigma^1(x)\|_{\mathrm{HS}}^2 \\ &= 2\langle x, b(x) - b(0)\rangle_{\mathrm{H}} + \|\sigma^1(x) - \sigma^1(0)\|_{\mathrm{HS}}^2 + 2\langle x, b(0)\rangle_{\mathrm{H}} \\ &\quad + 2\langle \sigma^1(x) - \sigma^1(0), \sigma^1(0)\rangle_{\mathrm{HS}} + \|\sigma^1(0)\|_{\mathrm{HS}}^2 \\ &\le -(\gamma - \varepsilon)\|x\|_{\mathrm{H}}^2 + 2(L_2^2 + 1 + \varepsilon)\mu\varepsilon^{-1}, \qquad \varepsilon \in (0,1),\ x \in H, \end{aligned} \tag{4}$$

where $\langle T, S\rangle_{\mathrm{HS}} := \sum_{i=1}^{\infty} \langle Te_i, Se_i\rangle_{\mathrm{H}}$ for $S, T \in \mathcal{L}_{\mathrm{HS}}(H)$.

Before defining the numerical scheme, we need to introduce some further notation. For any $n \in \mathbb{N}$, let $\pi_n \colon H \to H_n := \mathrm{span}\{e_1, \ldots, e_n\}$ be the orthogonal projection, i.e. $\pi_n x = \sum_{i=1}^{n} \langle x, e_i\rangle_{\mathrm{H}} e_i$, $x \in H$, let $A_n := \pi_n A \in \mathcal{L}(H_n)$, let $b_n := \pi_n b \colon H_n \to H_n$, and let $\sigma_n := \pi_n \sigma \colon H_n \to \mathcal{L}_{\mathrm{HS}}(H_n)$. Moreover, throughout the paper, let $x_n := \pi_n x$ for arbitrary $x \in U$, where U is a bounded subset of H.

Consider the finite-dimensional approximation associated with (1) on $H_n \simeq \mathbb{R}^n$

$$\mathrm{d}X^n(t) = \{A_n X^n(t) + b_n(X^n(t))\}\,\mathrm{d}t + \sigma_n(X^n(t))\,\mathrm{d}W(t), \qquad X^n(0) = x_n. \tag{5}$$

The spatial approximation (5) is also called the Galerkin approximation of (1). Owing to

$$\pi_n A x = \pi_n A\biggl(\sum_{i=1}^{n} \langle x, e_i\rangle_{\mathrm{H}} e_i\biggr) = -\sum_{i=1}^{n} \langle x, e_i\rangle_{\mathrm{H}} \lambda_i e_i, \qquad x \in H_n,$$

it follows that

$$A_n x = Ax, \qquad \mathrm{e}^{tA_n} x = \mathrm{e}^{tA} x, \quad \text{and} \quad \langle x, b_n(y)\rangle_{\mathrm{H}} = \langle x, b(y)\rangle_{\mathrm{H}} \tag{6}$$

for all $x, y \in H_n$. By (H3) and the property of π_n, for $x, y \in H_n$, we have

$$\|A_n(x - y) + b_n(x) - b_n(y)\|_{\mathrm{H}}^2 + \|\sigma_n^1(x) - \sigma_n^1(y)\|_{\mathrm{HS}}^2 \le 2(\lambda_n^2 + L_1^2 + L_2^2)\|x - y\|_{\mathrm{H}}^2.$$

Hence, under (H1) and (H3), (5) admits a unique strong solution $\{X_{x_n}^n(t)\}_{t \ge 0}$ with the starting point $x_n \in H_n$.

Next we introduce the time-discretization scheme for (5). For a step size $\triangle \in (0, 1)$ and each integer $k \ge 0$, compute the discrete *exponential integrator* (EI) scheme $\bar{Y}_{x_n}^{n,\triangle}(k\triangle) \approx X_{x_n}^n(k\triangle)$ by setting $\bar{Y}_{x_n}^{n,\triangle}(0) := x_n$ and forming

$$\bar{Y}_{x_n}^{n,\triangle}((k+1)\triangle) := \mathrm{e}^{\triangle A_n}\{\bar{Y}_{x_n}^{n,\triangle}(k\triangle) + b_n(\bar{Y}_{x_n}^{n,\triangle})\triangle + \sigma_n(\bar{Y}_{x_n}^{n,\triangle}(k\triangle))\triangle W_k\}, \tag{7}$$

where $\triangle W_k := W((k+1)\triangle) - W(k\triangle)$, and define the continuous EI scheme by

$$Y_{x_n}^{n,\triangle}(t) := \mathrm{e}^{tA_n} x_n + \int_0^t \mathrm{e}^{(t - s_\triangle)A_n} b_n(Y_{x_n}^{n,\triangle}(s_\triangle))\,\mathrm{d}s + \int_0^t \mathrm{e}^{(t - s_\triangle)A_n} \sigma_n(Y_{x_n}^{n,\triangle}(s_\triangle))\,\mathrm{d}W(s), \tag{8}$$

where $t_\triangle := \lfloor t/\triangle \rfloor \triangle$ with $\lfloor t/\triangle \rfloor$ standing for the integer part of $t/\triangle$. It is easy to see from (6) and (8) that

$$Y_{x_n}^{n,\triangle}(t) = \mathrm{e}^{(t-s)A} Y_{x_n}^{n,\triangle}(s) + \int_s^t \mathrm{e}^{(t-r_\triangle)A} b_n(Y_{x_n}^{n,\triangle}(r_\triangle))\,\mathrm{d}r + \int_s^t \mathrm{e}^{(t-r_\triangle)A} \sigma_n(Y_{x_n}^{n,\triangle}(r_\triangle))\,\mathrm{d}W(r), \qquad 0 \le s \le t. \tag{9}$$

By $Y_{x_n}^{n,\triangle}(0) = \bar{Y}_{x_n}^{n,\triangle}(0)$, we deduce from (7) and (9) that $Y_{x_n}^{n,\triangle}(k\triangle) = \bar{Y}_{x_n}^{n,\triangle}(k\triangle)$, i.e. $Y_{x_n}^{n,\triangle}(t)$ coincides with the discrete EI approximate solution at the grid points.

Remark 3. For finite-dimensional SDEs, the discrete/continuous Euler–Maruyama (EM) scheme is standard; see, e.g. [20, p. 113]. The roots of constructing schemes (8) and (9) go back to, e.g. [6] and [19].

For the discrete EI scheme (7), in this paper we are concerned with the following two questions.

- Given $n \in \mathbb{N}$, for what choices of the step size $\triangle \in (0, 1)$ does the EI scheme have a unique stationary distribution?
- Will the stationary distribution of the EI scheme converge weakly to some probability measure? If so, what is the weak limit probability measure?

In this paper we will give positive answers to these two questions.

It is also worth pointing out that, for the finite-dimensional case, by a Lyapunov–Foster argument, Yuan and Mao [26] studied the asymptotic stability in distribution of EM numerical solutions for a class of SDEs, and, by a global attractor approach, Yevik and Zhao [24] investigated the existence of a stationary distribution of the EM scheme for random dynamical systems associated with a class of SDEs. Comparing the EI scheme (7) with the EM scheme for the finite-dimensional case, e.g. [20, p. 113], we note that the explicit EI scheme (7) is based not only on spatial discretization but also on time discretization. Moreover, in (1), the linear operator A is generally unbounded, and the diffusion coefficient is not Hilbert–Schmidt, so the Itô formula does not apply. Therefore, our approaches are different from those in [24] and [26]. Furthermore, using Malliavin calculus, Bréhier [2] discussed the asymptotic behavior of the invariant measure for an implicit Euler scheme associated with a class of parabolic SPDEs driven by additive noise, where the drift coefficient is bounded.

The organization of this paper is as follows. In Section 2, for a given $n \in \mathbb{N}$ and a sufficiently small step size $\triangle \in (0, 1)$, we show that the EI approximate solution $\{\bar{Y}_{x_n}^{n,\triangle}(k\triangle)\}_{k\ge 0,\, x_n \in H_n}$ admits a unique stationary distribution under properties (P1) and (P2). In Section 3 we provide some sufficient conditions such that (P1) and (P2) hold. In the last section we show that the weak limit of the EI scheme as the step size tends to 0 is in fact the stationary distribution of (1).

2. Stationary distribution for the EI scheme

For fixed integer $n \in \mathbb{N}$, arbitrary integer $k \ge 0$, and $\Gamma \in \mathcal{B}(H_n)$, define the k-step transition probability kernel for the discrete EI approximate solution $\bar{Y}_{x_n}^{n,\triangle}(k\triangle)$ by

$$\mathbb{P}_k^{n,\triangle}(x_n, \Gamma) := \mathbb{P}(\bar{Y}_{x_n}^{n,\triangle}(k\triangle) \in \Gamma).$$

Following the argument of [26, Theorem 1.2], we deduce the following result.

Lemma 1. *It holds that $\{\bar{Y}_{x_n}^{n,\triangle}(k\triangle)\}_{k\ge 0}$ is a homogeneous Markov process.*

We still need to introduce some additional notation and notions. For a real separable Hilbert space $(K, \|\cdot\|_K)$, let $\mathscr{P}(K)$ stand for the collection of all probability measures on K. For $P_1, P_2 \in \mathscr{P}(K)$, define the metric $\mathrm{d}_{\mathbb{L}}$ as

$$\mathrm{d}_{\mathbb{L}}(P_1, P_2) := \sup_{f \in \mathbb{L}} \left| \int_K f(u) P_1(\mathrm{d}u) - \int_K f(u) P_2(\mathrm{d}u) \right|,$$

where $\mathbb{L} := \{f : K \to \mathbb{R} : |f(u) - f(v)| \le \|u - v\|_K \text{ and } |f(\cdot)| \le 1\}$.

Remark 4. It is known that the weak convergence of probability measures is a metric concept; see, e.g. [14, Proposition 2.5]. In other words, a sequence of probability measures $\{P_k\}_{k\ge1} \in \mathscr{P}(K)$ converges weakly to a probability measure $P_0 \in \mathscr{P}(K)$ if and only if $\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(P_k, P_0) = 0$.

Definition 1. For a given $n \in \mathbb{N}$ and a given step size $\triangle$, $\{\bar{Y}^{n,\triangle}_{x_n}(k\triangle)\}_{k\ge0,\, x_n\in H_n}$ is said to have a stationary distribution $\pi^{n,\triangle} \in \mathscr{P}(H_n)$ if $\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(x_n, \cdot), \pi^{n,\triangle}(\cdot)) = 0$ for every $x_n \in H_n$.

Definition 2. For a given $n \in \mathbb{N}$ and a given step size $\triangle$, $\{\bar{Y}^{n,\triangle}_{x_n}(k\triangle)\}_{k\ge0,\, x_n\in H_n}$ is said to have property (P1) if

$$\sup_{k\ge0} \sup_{x_n\in U} \mathbb{E}\|\bar{Y}^{n,\triangle}_{x_n}(k\triangle)\|^2_{\mathrm{H}} < \infty,$$

while it is said to have property (P2) if

$$\lim_{k\to\infty} \sup_{x_n, y_n\in U} \mathbb{E}\|\bar{Y}^{n,\triangle}_{x_n}(k\triangle) - \bar{Y}^{n,\triangle}_{y_n}(k\triangle)\|^2_{\mathrm{H}} = 0,$$

where U is a bounded subset of H_n.

We now state our main result in this section.

Theorem 1. *Assume that (P1) and (P2) hold. Then, for a given $n \in \mathbb{N}$ and a given step size $\triangle$, $\{\bar{Y}^{n,\triangle}_{x_n}(k\triangle)\}_{k\ge0,\, x_n\in H_n}$ has a unique stationary distribution $\pi^{n,\triangle} \in \mathscr{P}(H_n)$.*

Proof. For fixed $n \in \mathbb{N}$, we note that $H_n \simeq \mathbb{R}^n$ is finite dimensional, and choose a bounded subset $U \subseteq H_n$ such that $x_n, y_n \in U$. Following the argument used to derive [26, Lemma 2.4 and Lemma 2.6], we deduce that

$$\lim_{k\to\infty} \sup_{x_n, y_n\in U} \mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(x_n, \cdot), \mathbb{P}^{n,\triangle}_k(y_n, \cdot)) = 0, \tag{10}$$

and that, invoking Lemma 1, there exists $\pi^{n,\triangle} \in \mathscr{P}(H_n)$ such that

$$\lim_{k\to\infty} \mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(0, \cdot), \pi^{n,\triangle}(\cdot)) = 0. \tag{11}$$

Then the desired assertion follows from (10), (11), and the triangle inequality

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(x_n, \cdot), \pi^{n,\triangle}(\cdot)) \le \mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(x_n, \cdot), \mathbb{P}^{n,\triangle}_k(0, \cdot)) + \mathrm{d}_{\mathbb{L}}(\mathbb{P}^{n,\triangle}_k(0, \cdot), \pi^{n,\triangle}(\cdot)).$$

3. Sufficient conditions for properties (P1) and (P2)

To make Theorem 1 more applicable, in this section we give some sufficient conditions for (P1) and (P2) to hold. In what follows, $C > 0$ is a generic constant whose values may change from line to line. For notational simplicity, let

$$Z^{n,\triangle}(t) := \int_0^t \mathrm{e}^{(t-s_\triangle)A}\sigma_n^0\,\mathrm{d}W(s) \quad \text{and} \quad \tilde{Y}_{x_n}^{n,\triangle}(t) := Y_{x_n}^{n,\triangle}(t) - Z^{n,\triangle}(t).$$

Lemma 2. *Under (H1)–(H3),*

$$\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t) - \tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2 \le \beta_1\triangle(1 + \mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2), \qquad t \ge 0, \tag{12}$$

where $\beta_1 := 3\{(\lambda_n^2 + 2\bar{L}) \vee (2\bar{L}(1 + \|(-A)^{-\theta_1}\|^2\delta_1))\}$.

Proof. Observe from (8) that

$$\tilde{Y}_{x_n}^{n,\triangle}(t) = \mathrm{e}^{tA}x_n + \int_0^t \mathrm{e}^{(t-s_\triangle)A}b_n(Y_{x_n}^{n,\triangle}(s_\triangle))\,\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s_\triangle)A}\sigma_n^1(Y_{x_n}^{n,\triangle}(s_\triangle))\,\mathrm{d}W(s). \tag{13}$$

Then, by the Hölder inequality, the Itô isometry, and (H1), we have

$$\begin{aligned}
&\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t) - \tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2 \\
&\quad\le 3\bigg\{\mathbb{E}\|(\mathrm{e}^{(t-t_\triangle)A} - \mathrm{id}_{\mathrm{H}})\tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2 + \mathbb{E}\int_{t_\triangle}^t \|b(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{H}}^2\,\mathrm{d}s \\
&\qquad + \mathbb{E}\int_{t_\triangle}^t \|\sigma^1(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{HS}}^2\,\mathrm{d}s\bigg\} \\
&\quad=: 3\{I_1(t) + I_2(t) + I_3(t)\}.
\end{aligned} \tag{14}$$

Recalling the fundamental inequality $1 - \mathrm{e}^{-y} \le y,\ y > 0$, from (H1) we obtain

$$\|(\mathrm{e}^{(t-t_\triangle)A} - \mathrm{id}_{\mathrm{H}})u\|_{\mathrm{H}}^2 \le (1 - \mathrm{e}^{-\lambda_n(t-t_\triangle)})^2\|u\|_{\mathrm{H}}^2 \le \lambda_n^2\triangle^2\|u\|_{\mathrm{H}}^2, \qquad u \in H_n. \tag{15}$$

Thus, we arrive at

$$I_1(t) \le \lambda_n^2\triangle^2\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2. \tag{16}$$

Note that, from the Itô isometry, (H1), and (H2),

$$\begin{aligned}
\mathbb{E}\|Z^{n,\triangle}(t)\|_{\mathrm{H}}^2 &= \int_0^t \|\mathrm{e}^{(s-s_\triangle)A}\mathrm{e}^{(t-s)A}\sigma_n^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \\
&\le \int_0^t \|(-A)^{-\theta_1}(-A)^{\theta_1}\mathrm{e}^{(t-s)A}\sigma_n^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \\
&\le \|(-A)^{-\theta_1}\|^2\int_0^t \|(-A)^{\theta_1}\mathrm{e}^{(t-s)A}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \\
&\le \|(-A)^{-\theta_1}\|^2\delta_1.
\end{aligned} \tag{17}$$

Thus, by (3) and (17), it follows that

$$\begin{aligned}
I_2(t) + I_3(t) &\le \triangle\mathbb{E}\{\|b(Y_{x_n}^{n,\triangle}(t_\triangle))\|_{\mathrm{H}}^2 + \|\sigma^1(Y_{x_n}^{n,\triangle}(t_\triangle))\|_{\mathrm{HS}}^2\} \\
&\le 2\bar{L}\triangle\{1 + \mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2 + \mathbb{E}\|Z^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2\} \\
&\le 2\bar{L}\triangle\{1 + \|(-A)^{-\theta_1}\|^2\delta_1 + \mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(t_\triangle)\|_{\mathrm{H}}^2\}.
\end{aligned} \tag{18}$$

As a result, (12) follows by substituting (16) and (18) into (14).

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Theorem 2. *Let (H1)–(H4) hold, and assume further that* $2\alpha+\gamma>0$. *If* $\triangle<\min\{1,(2\alpha+\gamma)^2/(4\rho_1^2)\}$ *then*

$$\sup_{t\geq 0}\sup_{x_n\in U}\mathbb{E}\|Y^{n,\triangle}_{x_n}(t)\|^2_{\mathrm{H}}<\infty, \tag{19}$$

where

$$\rho_1:=2+\bigg(\frac{|14\alpha-\gamma|^2}{64}+2\bar{L}+\frac{|14\alpha-\gamma|}{8}\bigg)\beta_1+2(1+\beta_1+\lambda_n^2\bar{L})$$

and U *is a bounded subset of* H_n. *Hence, property (P1) holds whenever the step size* $\triangle$ *is sufficiently small.*

Proof. Note that (13) can be rewritten in the differential form

$$\mathrm{d}\tilde{Y}^{n,\triangle}_{x_n}(t)=\{A\tilde{Y}^{n,\triangle}_{x_n}(t)+\mathrm{e}^{(t-t_\triangle)A}b_n(Y^{n,\triangle}_{x_n}(t_\triangle))\}\,\mathrm{d}t+\mathrm{e}^{(t-t_\triangle)A}\sigma^1_n(Y^{n,\triangle}_{x_n}(t_\triangle))\,\mathrm{d}W(t) \tag{20}$$

with $\tilde{Y}^{n,\triangle}_{x_n}(0)=x_n$. For any $\nu>0$, by Itô's formula, from (20) and (H1), we obtain

$$\begin{aligned}&\mathbb{E}(\mathrm{e}^{\nu t}\|\tilde{Y}^{n,\triangle}_{x_n}(t)\|^2_{\mathrm{H}})\\&\quad\leq\|x\|^2_{\mathrm{H}}+\mathbb{E}\int_0^t\mathrm{e}^{\nu s}\{-(2\alpha-\nu)\|\tilde{Y}^{n,\triangle}_{x_n}(s)\|^2_{\mathrm{H}}+2\langle\tilde{Y}^{n,\triangle}_{x_n}(s),\mathrm{e}^{(s-s_\triangle)A}b_n(Y^{n,\triangle}_{x_n}(s_\triangle))\rangle_{\mathrm{H}}\\&\qquad\qquad+\|\sigma^1(Y^{n,\triangle}_{x_n}(s_\triangle))\|^2_{\mathrm{HS}}\}\,\mathrm{d}s.\end{aligned} \tag{21}$$

Since

$$\begin{aligned}\|\tilde{Y}^{n,\triangle}_{x_n}(t)\|^2_{\mathrm{H}}=&\|\tilde{Y}^{n,\triangle}_{x_n}(t_\triangle)\|^2_{\mathrm{H}}+2\langle\tilde{Y}^{n,\triangle}_{x_n}(t_\triangle),\tilde{Y}^{n,\triangle}_{x_n}(t)-\tilde{Y}^{n,\triangle}_{x_n}(t_\triangle)\rangle_{\mathrm{H}}\\&+\|\tilde{Y}^{n,\triangle}_{x_n}(t)-\tilde{Y}^{n,\triangle}_{x_n}(t_\triangle)\|^2_{\mathrm{H}},\end{aligned} \tag{22}$$

it follows from (21) that

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\nu t}\|\tilde{Y}^{n,\triangle}_{x_n}(t)\|^2_{\mathrm{H}})\leq\|x\|^2_{\mathrm{H}}+\mathbb{E}\int_0^t\mathrm{e}^{\nu s}\{&-(2\alpha-\nu)\|\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle)\|^2_{\mathrm{H}}+\|\sigma^1(Y^{n,\triangle}_{x_n}(s_\triangle))\|^2_{\mathrm{HS}}\\&+2\langle Y^{n,\triangle}_{x_n}(s_\triangle),b(Y^{n,\triangle}_{x_n}(s_\triangle))\rangle_{\mathrm{H}}\\&-2(2\alpha-\nu)\langle\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle),\tilde{Y}^{n,\triangle}_{x_n}(s)-\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle)\rangle_{\mathrm{H}}\\&-(2\alpha-\nu)\|\tilde{Y}^{n,\triangle}_{x_n}(s)-\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle)\|^2_{\mathrm{H}}\\&+2\langle\tilde{Y}^{n,\triangle}_{x_n}(s)-\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle),b(Y^{n,\triangle}_{x_n}(s_\triangle))\rangle_{\mathrm{H}}\\&-2\langle Z^{n,\triangle}(s_\triangle)),b(Y^{n,\triangle}_{x_n}(s_\triangle))\rangle_{\mathrm{H}}\\&+2\langle\tilde{Y}^{n,\triangle}_{x_n}(s),(\mathrm{e}^{(s-s_\triangle)A}-\mathrm{id}_{\mathrm{H}})b_n(Y^{n,\triangle}_{x_n}(s_\triangle))\rangle_{\mathrm{H}}\}\,\mathrm{d}s.\end{aligned}$$

This, together with (4), yields

$$\begin{aligned}&\mathbb{E}(\mathrm{e}^{\nu t}\|\tilde{Y}^{n,\triangle}_{x_n}(t)\|^2_{\mathrm{H}})\\&\quad\leq\|x\|^2_{\mathrm{H}}-(2\alpha+\gamma-\varepsilon-\nu)\mathbb{E}\int_0^t\mathrm{e}^{\nu s}\|\tilde{Y}^{n,\triangle}_{x_n}(s_\triangle)\|^2_{\mathrm{H}}\,\mathrm{d}s\end{aligned}$$

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$$
\begin{aligned}
&+\mathbb{E}\int_0^t \mathrm{e}^{\nu s}\{-2(2\alpha-\nu)\langle \tilde{Y}_{x_n}^{n,\triangle}(s_\triangle), \tilde{Y}_{x_n}^{n,\triangle}(s)-\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\rangle_{\mathrm{H}}\\
&\qquad -(2\alpha-\nu)\|\tilde{Y}_{x_n}^{n,\triangle}(s)-\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\\
&\qquad +2\langle \tilde{Y}_{x_n}^{n,\triangle}(s)-\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle), b(Y_{x_n}^{n,\triangle}(s_\triangle))\rangle_{\mathrm{H}}\}\,\mathrm{d}s\\
&+2\mathbb{E}\int_0^t \mathrm{e}^{\nu s}\langle \tilde{Y}_{x_n}^{n,\triangle}(s), (\mathrm{e}^{(s-s_\triangle)A}-\mathrm{id}_{\mathrm{H}})b_n(Y_{x_n}^{n,\triangle}(s_\triangle))\rangle_{\mathrm{H}}\,\mathrm{d}s\\
&+\mathbb{E}\int_0^t \mathrm{e}^{\nu s}\{2(L_2^2+1+\varepsilon^{-1})\mu\varepsilon^{-1}-2\langle Z^{n,\triangle}(s_\triangle), b(Y_{x_n}^{n,\triangle}(s_\triangle))\rangle_{\mathrm{H}}\\
&\qquad -2(\gamma-\varepsilon)\langle Z^{n,\triangle}(s_\triangle), \tilde{Y}_{x_n}^{n,\triangle}(s_\triangle))\rangle_{\mathrm{H}}-(\gamma-\varepsilon)\|Z^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\}\,\mathrm{d}s\\
=:&\ J_1(t)+J_2(t)+J_3(t)+J_4(t). \qquad (23)
\end{aligned}
$$

By the elementary inequality $2ab\le \kappa a^2+b^2/\kappa,\ a,b\in\mathbb{R},\kappa>0$, and (12), we arrive at

$$
\begin{aligned}
J_2(t)\le \mathbb{E}\int_0^t \mathrm{e}^{\nu s}\{&2\triangle^{1/2}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2+2^{-1}\triangle^{1/2}+\triangle^{1/2}\|Z^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\\
&+\{(|2\alpha-\nu|^2+2\bar{L})\triangle^{-1/2}+|2\alpha-\nu|\}\|\tilde{Y}_{x_n}^{n,\triangle}(s)-\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\}\,\mathrm{d}s,
\end{aligned}
$$

where in the last step we used (3). Combining (12) with (17), we thus obtain

$$
\begin{aligned}
J_2(t)\le \int_0^t \mathrm{e}^{\nu s}\{&\{2+(|2\alpha-\nu|^2+2\bar{L}+|2\alpha-\nu|)\beta_1\}\triangle^{1/2}\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\\
&+\{1+\|(-A)^{-\theta_1}\|^2\delta_1+(|2\alpha-\nu|^2+2\bar{L}+|2\alpha-\nu|)\beta_1\}\triangle^{1/2}\}\,\mathrm{d}s. \qquad (24)
\end{aligned}
$$

On the other hand, we deduce from (3), (15), (17), and (22) that

$$
\begin{aligned}
J_3(t)&\le \mathbb{E}\int_0^t \mathrm{e}^{\nu s}\{2\triangle^{1/2}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2+2\triangle^{1/2}\|\tilde{Y}_{x_n}^{n,\triangle}(s)-\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\\
&\qquad +\triangle^{-1/2}\|(\mathrm{e}^{(s-s_\triangle)A}-\mathrm{id}_{\mathrm{H}})b_n(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{H}}^2\}\,\mathrm{d}s\\
&\le \int_0^t \mathrm{e}^{\nu s}\{2(1+\beta_1+\lambda_n^2\bar{L})\triangle^{1/2}\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\\
&\qquad +2(\beta_1+\lambda_n^2\bar{L}(1+\|(-A)^{-\theta_1}\|^2\delta_1))\triangle^{1/2}\}\,\mathrm{d}s. \qquad (25)
\end{aligned}
$$

Furthermore, due to (3) and (17), for arbitrary $\kappa>0$, we have

$$
\begin{aligned}
J_4(t)\le \mathbb{E}\int_0^t \mathrm{e}^{\nu s}\{&2(L_2^2+1+\varepsilon^{-1})\mu\varepsilon^{-1}+\kappa^{-1}\|Z^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2+\kappa\|b(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{H}}^2\\
&+|\gamma-\varepsilon|^2\kappa^{-1}\|Z^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}+\kappa\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2+|\gamma-\varepsilon|\|Z^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\}\,\mathrm{d}s
\end{aligned}
$$

$$\leq \int_0^t e^{\nu s}\{\kappa \bar{L} + (\kappa^{-1} + 2\kappa\bar{L} + |\gamma-\varepsilon|^2\kappa^{-1} + |\gamma-\varepsilon|)\|(-A)^{-\theta_1}\|^2\delta_1$$
$$+ 2(L_2^2 + 1 + \varepsilon^{-1})\mu\varepsilon^{-1} + (1+2\bar{L})\kappa\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\}\,\mathrm{d}s.$$

In particular, taking $\varepsilon = \nu = (2\alpha+\gamma)/8$ and $\kappa = (2\alpha+\gamma)/(4(1+2\bar{L}))$ yields

$$J_4(t) \leq \int_0^t e^{\nu s}\{4^{-1}(2\alpha+\gamma)\mathbb{E}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2 + C\}\,\mathrm{d}s. \tag{26}$$

Substituting (24)–(26) into (23), we deduce that

$$\mathbb{E}(e^{\nu t}\|\tilde{Y}_{x_n}^{n,\triangle}(t)\|_{\mathrm{H}}^2) \leq \|x\|_{\mathrm{H}}^2 + C\int_0^t e^{\nu s}\,\mathrm{d}s - \frac{2\alpha+\gamma-2\rho_1\triangle^{1/2}}{2}\mathbb{E}\int_0^t e^{\nu s}\|\tilde{Y}_{x_n}^{n,\triangle}(s_\triangle)\|_{\mathrm{H}}^2\,\mathrm{d}s.$$

For $\triangle < (2\alpha+\gamma)^2/(4\rho_1^2)$, it is trivial to see that $2\alpha+\gamma-2\rho_1\triangle^{1/2} > 0$. Thus, we have

$$\sup_{t\geq 0}\sup_{x_n\in U}\mathbb{E}(\|\tilde{Y}_{x_n}^{n,\triangle}(t)\|_{\mathrm{H}}^2) < \infty.$$

Finally, recalling that $\tilde{Y}_{x_n}^{n,\triangle}(t) = Y_{x_n}^{n,\triangle}(t) - Z^{n,\triangle}(t)$, (19) follows from (17).

By an argument similar to that used to prove Theorem 2, we can also derive the following result which states that solutions of (8) starting from different values will be sufficiently close as time tends to ∞.

Theorem 3. *Let the assumptions of Lemma 2 hold. If* $\triangle < \min\{1, (2\alpha+\gamma)^2/(4\rho_2^2)\}$ *then*

$$\lim_{t\to\infty}\sup_{x_n,y_n\in U}\mathbb{E}\|Y_{x_n}^{n,\triangle}(t) - Y_{y_n}^{n,\triangle}(t)\|_{\mathrm{H}}^2 = 0,$$

where $\rho_2 := 6(\lambda_n^2+\bar{L})(|2\alpha-\gamma|+1)+3+7\bar{L}+\lambda_n^2\bar{L}+6\lambda_n^2$ *and* U *is a bounded subset of* H_n*. Hence, property (P2) holds whenever the step size* $\triangle$ *is sufficiently small.*

4. Weak limit distribution

In Section 3 we gave some sufficient conditions for (7) to have a unique stationary distribution $\pi^{n,\triangle} \in \mathcal{P}(H_n)$ for a fixed n and a sufficiently small step size $\triangle \in (0,1)$. In this section we proceed to discuss the weak limit behavior of $\pi^{n,\triangle} \in \mathcal{P}(H_n)$ and answer the following questions.

- Will the stationary distribution $\pi^{n,\triangle}(\cdot)$ converge weakly to some probability measure in $\mathcal{P}(H)$ whenever $n\to\infty$ and $\triangle\to 0$?
- If yes, what is the weak limit probability measure?

Denote $\{X_x(t)\}_{t\geq 0,\,x\in H}$ by the mild solution of (1) starting from the point x at time $t=0$, which is a homogeneous Markov process. For any subset $\Gamma\subset\mathcal{B}(H)$ and arbitrary $t\geq 0$, let $\mathbb{P}_t(x,\Gamma) := \mathbb{P}(X_x(t)\in\Gamma)$.

Definition 3. We say that $\{X_x(t)\}_{t\geq 0,\,x\in H}$ has a stationary distribution $\pi(\cdot)\in\mathcal{P}(H)$ if $\lim_{t\to\infty}\mathrm{d}_{\mathbb{L}}(\mathbb{P}_t(x,\cdot),\pi(\cdot)) = 0$.

To study the limit behavior of $\pi^{n,\triangle}(\cdot)$, we first give several auxiliary lemmas.

Lemma 3. *Let (H1)–(H4) hold, and assume further that $2\alpha+\gamma>0$. Then the mild solution $\{X_x(t)\}_{t\geq 0,\, x\in H}$ of (1) has a unique stationary distribution $\pi(\cdot)\in\mathcal{P}(H)$.*

Proof. We note that Bao *et al.* [1, Theorem 3.1] investigated the stationary distribution of (1) with $\sigma^0=0$, i.e. the diffusion coefficient there is a Hilbert–Schmidt operator. For $\sigma^0\neq 0$, note that σ is not Hilbert–Schmidt. Therefore, [1, Theorem 3.1] does not apply to (1). Let

$$\bar{Z}(t):=\int_0^t \mathrm{e}^{(t-s)A}\sigma^0\,\mathrm{d}W(s) \quad\text{and}\quad \bar{X}_x(t):=X_x(t)-\bar{Z}(t).$$

Then (1) can be rewritten in the form

$$\mathrm{d}\bar{X}_x(t)=\{A\bar{X}_x(t)+b(X_x(t))\}\,\mathrm{d}t+\sigma^1(X_x(t))\,\mathrm{d}W(t). \tag{27}$$

To be precise, (27) is first meant in the mild sense. But, under (H1)–(H3), it also has a unique variation solution, and, therefore, the Itô formula applies to $\|\bar{X}_x(t)\|_{\mathrm{H}}^2$. By arguments similar to those used in the proofs of Theorem 2 and Theorem 3, respectively, for some bounded subset $U\subseteq H$, we deduce that

$$\sup_{t\geq 0}\sup_{x\in U}\mathbb{E}\|X_x(t)\|_{\mathrm{H}}^2<\infty \tag{28}$$

and

$$\lim_{t\to\infty}\sup_{x,y\in U}\mathbb{E}\|X_x(t)-X_y(t)\|_{\mathrm{H}}^2=0.$$

Then, following the argument used in [1, Proof of Theorem 3.1], we obtain the desired assertion.

Lemma 4. *Let (H1) and (H2) hold, and assume further that there exist $\delta_2>0$ and $\theta_2\in(0,1)$ such that*

$$\int_0^{\triangle}\|\mathrm{e}^{sA}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s\leq\delta_2\triangle^{\theta_2}. \tag{29}$$

Then

$$\sup_{t\geq 0}\mathbb{E}\|\bar{Z}(t)-\bar{Z}(t_\triangle)\|_{\mathrm{H}}^2\leq C\triangle^{\theta_1\wedge\theta_2}, \tag{30}$$

where $C>0$ is a constant independent of $\triangle$.

Proof. Recall from [21, Theorem 6.13] that there exists $C_1>0$ such that

$$\|(-A)^{\alpha_1}\mathrm{e}^{tA}\|\leq C_1t^{-\alpha_1},\qquad \|(-A)^{-\alpha_2}(1-\mathrm{e}^{tA})\|\leq C_1t^{\alpha_2}, \tag{31}$$

for arbitrary $\alpha_1\geq 0$ and $\alpha_2\in[0,1]$, and that

$$(-A)^{\alpha_3+\alpha_4}x=(-A)^{\alpha_3}(-A)^{\alpha_4}x,\qquad x\in\mathcal{D}((-A)^{\gamma}), \tag{32}$$

for any $\alpha_3,\alpha_4\in\mathbb{R}$, where $\gamma:=\max\{\alpha_3,\alpha_4,\alpha_3+\alpha_4\}$. In light of the independent increment of the Wiener process and Itô's isometry,

$$\mathbb{E}\|\bar{Z}(t)-\bar{Z}(t_\triangle)\|_{\mathrm{H}}^2=\int_0^{t_\triangle}\|(\mathrm{e}^{(t-t_\triangle)A}-\mathrm{id}_{\mathrm{H}})\mathrm{e}^{(t_\triangle-s)A}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s+\int_{t_\triangle}^t\|\mathrm{e}^{(t-s)A}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s.$$

This, combining (H2), (29), and (31) with (32), yields

$$\begin{aligned}\mathbb{E}\|\bar{Z}(t)-\bar{Z}(t_\triangle)\|_{\mathrm{H}}^2 &\le \int_0^{t_\triangle} \|(-A)^{-\theta_1}(\mathrm{e}^{(t-t_\triangle)A}-\mathrm{id}_{\mathrm{H}})\|^2 \cdot \|(-A)^{\theta_1}\mathrm{e}^{(t_\triangle-s)A}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \\ &\quad + \int_0^{\triangle} \|\mathrm{e}^{sA}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s \\ &\le (C_1^2\delta_1+\delta_2)\triangle^{\theta_1\wedge\theta_2},\end{aligned}$$

completing the proof.

Remark 5. Let $\sigma^0=\mathrm{id}_{\mathrm{H}}$, and let A be the Laplace operator defined in Remark 1. A straightforward computation shows that

$$\int_0^{\triangle} \|\mathrm{e}^{sA}\|_{\mathrm{HS}}^2\,\mathrm{d}s = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}(1-\mathrm{e}^{-2k^2\triangle}). \tag{33}$$

Recall that, for arbitrary $\delta\in(0,1)$ and $x,y\ge 0$,

$$|\mathrm{e}^{-x}-\mathrm{e}^{-y}| \le |x-y|^{\delta}. \tag{34}$$

It then follows from (33) and (34) that

$$\int_0^{\triangle} \|\mathrm{e}^{sA}\|_{\mathrm{HS}}^2\,\mathrm{d}s \le 2^{\delta-1}\triangle^{\delta}\sum_{k=1}^{\infty}\frac{1}{k^{2(1-\delta)}}.$$

Hence, (29) holds with $\delta_2=2^{\delta-1}\sum_{k=1}^{\infty}1/k^{2(1-\delta)}$ and $\theta_2=\delta\in(0,\frac{1}{2})$.

Lemma 5. *Let the assumptions of Lemma 3 hold, and let*

$$\tau := \alpha^{-1}L_1+(2\alpha)^{-1/2}L_2 \in (0,1). \tag{35}$$

Then

$$\sup_{t\ge 0}\mathbb{E}\|X_x(t)-Y_{x_n}^{n,\triangle}(t)\|_{\mathrm{H}}^2 \le C\{\lambda_n^{-(\theta_1\wedge 1/2)}+\triangle^{\theta_1\wedge\theta_2}\},$$

where $C>0$ is a constant dependent on $x\in H$ but independent of n and $\triangle$.

Proof. By (3) and (28), it follows that

$$\sup_{t\ge 0}\mathbb{E}\|b(X_x(t))\|_{\mathrm{H}}^2 + \sup_{t\ge 0}\mathbb{E}\|\sigma^1(X_x(t))\|_{\mathrm{HS}}^2 \le C. \tag{36}$$

Note that $(\mathbb{E}\|\cdot\|_{\mathrm{H}}^2)^{1/2}$ is a norm and recall from [11, Theorem 202] the Minkowski integral inequality,

$$\bigg(\mathbb{E}\bigg|\int_0^t F(s)\,\mathrm{d}s\bigg|^2\bigg)^{1/2} \le \int_0^t (\mathbb{E}|F(s)|^2)^{1/2}\,\mathrm{d}s, \qquad t\ge 0,$$

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where $F\colon [0,\infty)\times\Omega\to R$ is measurable and locally integrable. Then, applying the Itô isometry and using (H1), we obtain, from (2),

$$
\begin{aligned}
&(\mathbb{E}\|\bar{X}_x(t)-\bar{X}_x(t_\triangle)\|_{\mathrm{H}}^2)^{1/2}\\
&\quad\le \|\mathrm{e}^{t_\triangle A}\{\mathrm{e}^{(t-t_\triangle)A}-1\}x\|_{\mathrm{H}}\\
&\qquad+\int_0^{t_\triangle}(\mathbb{E}\|\mathrm{e}^{(t_\triangle-s)A}\{\mathrm{e}^{(t-t_\triangle)A}-\mathrm{id}_{\mathrm{H}}\}b(X_x(s))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s\\
&\qquad+\biggl(\int_0^{t_\triangle}\mathbb{E}\|\mathrm{e}^{(t_\triangle-s)A}\{\mathrm{e}^{(t-t_\triangle)A}-\mathrm{id}_{\mathrm{H}}\}\sigma^1(X_x(s))\|_{\mathrm{HS}}^2\,\mathrm{d}s\biggr)^{1/2}\\
&\qquad+\int_{t_\triangle}^{t}(\mathbb{E}\|b(X_x(s))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s+\biggl(\int_{t_\triangle}^{t}\mathbb{E}\|\sigma^1(X_x(s))\|_{\mathrm{HS}}^2\,\mathrm{d}s\biggr)^{1/2}\\
&\quad=: F_1(t)+F_2(t)+F_3(t)+F_4(t)+F_5(t).
\end{aligned}\tag{37}
$$

Let $\rho:=(\theta_1\wedge\theta_2)/2$. In view of (31), (32), (H1), and the boundedness of $(-A)^{-(1-\rho/2)}$, we have

$$
\begin{aligned}
F_1(t)&=\|(-A)^{-(1-\rho/2)}\mathrm{e}^{t_\triangle A}(-A)^{-\rho/2}\{\mathrm{e}^{(t-t_\triangle)A}-\mathrm{id}_{\mathrm{H}}\}(-A)x\|_{\mathrm{H}}^2\\
&\le C\|(-A)^{-(1-\rho/2)}\|^2\|Ax\|_{\mathrm{H}}^2\triangle^\rho.
\end{aligned}
$$

Also, by (31) and (32), from (36), we obtain, for $\tilde{\theta}\in(0,1)$,

$$
\sum_{k=2}^{5}F_k(t)\le C\triangle^{1/2}+C\triangle^\rho\int_0^{t_\triangle}(\tilde{\theta}s)^{-\rho}\mathrm{e}^{-\alpha(1-\tilde{\theta})s}\,\mathrm{d}s+C\triangle^\rho\biggl(\int_0^{t_\triangle}(\tilde{\theta}s)^{-2\rho}\mathrm{e}^{-2\alpha(1-\tilde{\theta})s}\,\mathrm{d}s\biggr)^{1/2}.\tag{38}
$$

Observe that

$$
\int_0^{t_\triangle}s^{-\rho}\mathrm{e}^{-\alpha(1-\tilde{\theta})s}\,\mathrm{d}s\le(\alpha(1-\tilde{\theta}))^{\rho-1}\int_0^{\infty}s^{-\rho}\mathrm{e}^{-s}\,\mathrm{d}s=(\alpha(1-\tilde{\theta}))^{\rho-1}\Gamma(1-\rho),
$$

and, similarly,

$$
\int_0^{t_\triangle}s^{-2\rho}\mathrm{e}^{-2\alpha(1-\tilde{\theta})s}\,\mathrm{d}s\le(2\alpha(1-\tilde{\theta}))^{2\rho-1}\Gamma(1-2\rho),
$$

where $\Gamma(\cdot)$ is the gamma function. Hence,

$$
\sum_{k=2}^{4}F_k(t)\le C\triangle^{(\theta_1\wedge\theta_2)/2}.
$$

This, together with the estimate of $F_1(t)$, gives

$$
\sup_{t\ge0}\mathbb{E}\|\bar{X}_x(t)-\bar{X}_x(t_\triangle)\|_{\mathrm{H}}^2\le C\triangle^{\theta_1\wedge\theta_2}.
$$

Noting that $\bar{X}_x(t)=X_x(t)-\bar{Z}(t)$ and utilizing (30), we have

$$
\sup_{t\ge0}\mathbb{E}\|X_x(t)-X_x(t_\triangle)\|_{\mathrm{H}}^2\le C\triangle^{\theta_1\wedge\theta_2}.\tag{39}
$$

Since

$$\|(\mathrm{id}_{\mathrm{H}}-\pi_n)(-A)^{-\theta_1}u\|_{\mathrm{H}}^2 \le \lambda_n^{-2\theta_1}\|u\|_{\mathrm{H}}^2, \qquad u\in H,$$

we arrive at

$$\|(\mathrm{id}_{\mathrm{H}}-\pi_n)(-A)^{-\theta_1}\|^2 \le \lambda_n^{-2\theta_1}. \tag{40}$$

By virtue of the Itô isometry, (H2), (40), (31), and (32), it follows that

$$\begin{aligned}\mathbb{E}\|\bar{Z}(t)-Z^{n,\triangle}(t)\|_{\mathrm{H}}^2 &\le 2\|(\mathrm{id}_{\mathrm{H}}-\pi_n)(-A)^{-\theta_1}\|^2\int_0^t \|(-A)^{\theta_1}\mathrm{e}^{sA}\sigma^0\|_{\mathrm{HS}}^2\,\mathrm{d}s\\ &\quad + C\triangle^{2\theta_1}\int_0^t \|(-A)^{\theta_1}\mathrm{e}^{sA}\sigma_n^0\|_{\mathrm{HS}}^2\,\mathrm{d}s\\ &\le C(\lambda_n^{-2\theta_1}+\triangle^{2\theta_1}).\end{aligned} \tag{41}$$

Following the argument used to obtain (37), we have

$$\begin{aligned}&(\mathbb{E}\|\bar{X}_x(t)-\tilde{Y}_{x_n}^{n,\triangle}(t)\|_{\mathrm{H}}^2)^{1/2}\\ &\quad\le \|\mathrm{e}^{tA}(\mathrm{id}_{\mathrm{H}}-\pi_n)x\|_{\mathrm{H}}\\ &\qquad + \int_0^t \|\mathrm{e}^{(t-s)A}(\mathrm{id}_{\mathrm{H}}-\pi_n)\|(\mathbb{E}\|b(X_x(s))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s\\ &\qquad + \bigg(\int_0^t \|\mathrm{e}^{(t-s)A}(\mathrm{id}_{\mathrm{H}}-\pi_n)\|^2\mathbb{E}\|\sigma^1(X_x(s))\|_{\mathrm{HS}}^2\,\mathrm{d}s\bigg)^{1/2}\\ &\qquad + \int_0^t \|\mathrm{e}^{(t-s)A}\|(\mathbb{E}\|b_n(X_x(s))-b_n(X_x(s_\triangle))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s\\ &\qquad + \bigg(\int_0^t \|\mathrm{e}^{(t-s)A}\|^2\mathbb{E}\|\sigma_n^1(X_x(s))-\sigma_n^1(X_x(s_\triangle))\|_{\mathrm{HS}}^2\,\mathrm{d}s\bigg)^{1/2}\\ &\qquad + \int_0^t \|\mathrm{e}^{(t-s)A}\|(\mathbb{E}\|b_n(X_x(s_\triangle))-b_n(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s\\ &\qquad + \bigg(\int_0^t \|\mathrm{e}^{(t-s)A}\|^2\mathbb{E}\|\sigma_n^1(X_x(s_\triangle))-\sigma_n^1(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{HS}}^2\,\mathrm{d}s\bigg)^{1/2}\\ &\qquad + \int_0^t \|\mathrm{e}^{(t-s)A}\{\mathrm{id}_{\mathrm{H}}-\mathrm{e}^{(s-s_\triangle)A}\}\|(\mathbb{E}\|b(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{H}}^2)^{1/2}\,\mathrm{d}s\\ &\qquad + \bigg(\int_0^t \|\mathrm{e}^{(t-s)A}\{\mathrm{id}_{\mathrm{H}}-\mathrm{e}^{(s-s_\triangle)A}\}\|^2\mathbb{E}\|\sigma^1(Y_{x_n}^{n,\triangle}(s_\triangle))\|_{\mathrm{HS}}^2\,\mathrm{d}s\bigg)^{1/2}\\ &\quad=: \sum_{i=1}^9 G_i(t).\end{aligned} \tag{42}$$

A straightforward computation shows that

$$\|\mathrm{e}^{tA}(\mathrm{id}_{\mathrm{H}}-\pi_n)u\|_{\mathrm{H}}^2 = \sum_{i=n+1}^{\infty}\mathrm{e}^{-2\lambda_i t}\langle u,e_i\rangle_{\mathrm{H}}^2, \qquad u\in H.$$

This further gives

$$\|\mathrm{e}^{tA}(\mathrm{id}_{\mathrm{H}}-\pi_n)\|^2 \le \mathrm{e}^{-2\lambda_n t} \tag{43}$$

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and

$$G_1(t) \le \left(\sum_{i=n+1}^{\infty} \frac{e^{-2\lambda_i t}}{\lambda_i^2} \lambda_i^2 \langle x, e_i \rangle_H^2 \right)^{1/2} \le \lambda_n^{-1} \|Ax\|_H \tag{44}$$

by recalling that $\{\lambda_i\}_{i\ge1}$ is a nondecreasing sequence. By (36) and (43), we have

$$\begin{aligned} \sum_{i=2}^{3} G_i(t) &\le C \int_0^t \|e^{(t-s)A}(\mathrm{id}_H - \pi_n)\| \, ds + C \left(\int_0^t \|e^{(t-s)A}(\mathrm{id}_H - \pi_n)\|^2 \, ds \right)^{1/2} \\ &\le C \int_0^t e^{-\lambda_n (t-s)} \, ds + C \left(\int_0^t e^{-2\lambda_n (t-s)} \, ds \right)^{1/2} \\ &\le C(\lambda_n^{-1} + \lambda_n^{-1/2}). \end{aligned} \tag{45}$$

Taking (H1), (H3), and (39) into account gives

$$G_4(t) + G_5(t) \le C \triangle^{(\theta_1 \wedge \theta_2)/2}. \tag{46}$$

Next, note from (H1) and (H3) that

$$\begin{aligned} &G_6(t) + G_7(t) \\ &\quad \le \alpha^{-1} \sup_{0\le s\le t} (\mathbb{E}\|b(X_x(s_\triangle)) - b(Y_{x_n}^{n,\triangle}(s_\triangle))\|_H^2)^{1/2} \\ &\qquad + (2\alpha)^{-1/2} \sup_{0\le s\le t} (\mathbb{E}\|\sigma^1(X_x(s_\triangle)) - \sigma^1(Y_{x_n}^{n,\triangle}(s_\triangle))\|_H^2)^{1/2} \\ &\quad \le \tau \sup_{0\le s\le t} (\mathbb{E}\|\bar{X}_x(s) - \tilde{Y}_{x_n}^{n,\triangle}(s)\|_H^2)^{1/2} + \tau \sup_{0\le s\le t} (\mathbb{E}\|\bar{Z}(s) - Z^{n,\triangle}(s)\|_H^2)^{1/2}, \end{aligned} \tag{47}$$

where $\tau \in (0, 1)$ is defined by (35). Following the argument used to obtain (38), we have

$$G_8(t) + G_9(t) \le C \triangle^{(\theta_1 \wedge \theta_2)/2}. \tag{48}$$

Substituting (44)–(48) into (42) yields

$$\sup_{t\ge0} (\mathbb{E}\|\bar{X}_x(t) - \tilde{Y}_{x_n}^{n,\triangle}(t)\|_H^2)^{1/2} \le C(\lambda_n^{-1/2} + \triangle^{(\theta_1 \wedge \theta_2)/2})$$

due to $\tau \in (0, 1)$. Consequently, the desired assertion follows from (41).

Theorem 4. *Assume that (H1)–(H4), (29), and (35) hold. Then there exists a* $\triangle_n$ *such that* $\lim_{n\to\infty} \triangle_n = 0$ *and*

$$\lim_{n\to\infty} d_{\mathbb{L}}(\pi^{n,\triangle_n}(\cdot), \pi(\cdot)) = 0.$$

Proof. Fix $x \in H$, and let $\varepsilon > 0$ be arbitrary. By Lemma 5, there exist a sufficiently large $n \in \mathbb{N}$ and a $\bar{\triangle}_n$ sufficiently small such that

$$d_{\mathbb{L}}(\mathbb{P}_{k\bar{\triangle}_n}(x, \cdot), \mathbb{P}_k^{n,\bar{\triangle}_n}(x_n, \cdot)) \le \tfrac{1}{3}\varepsilon.$$

For the previous $n \in \mathbb{N}$, by Theorem 1, there exist a sufficiently small $\tilde{\triangle}_n$ and $T_1 > 0$ such that

$$d_{\mathbb{L}}(\mathbb{P}_k^{n,\tilde{\triangle}_n}(x_n, \cdot), \pi^{n,\tilde{\triangle}_n}(\cdot)) \le \tfrac{1}{3}\varepsilon$$

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whenever $k\tilde{\triangle}_n \geq T_1$. Furthermore, owing to Lemma 3, there exists $T_2 > 0$ such that

$$\mathrm{d}_{\mathbb{L}}(\mathbb{P}_t(x,\cdot),\pi(\cdot)) \leq \tfrac{1}{3}\varepsilon, \qquad t \geq T_2.$$

Let $T := T_1 \vee T_2$, $\triangle_n = \bar{\triangle}_n \wedge \tilde{\triangle}_n$, and $k = [T/\triangle_n] + 1$. Then the desired assertion follows from the triangle inequality

$$\begin{aligned}\mathrm{d}_{\mathbb{L}}(\pi^{n,\triangle}(\cdot),\pi(\cdot)) &\leq \mathrm{d}_{\mathbb{L}}(\mathbb{P}_{k\triangle}(x,\cdot),\pi(\cdot)) + \mathrm{d}_{\mathbb{L}}(\mathbb{P}_{k\triangle}(x,\cdot),\mathbb{P}_k^{n,\triangle}(x_n,\cdot)) \\ &\quad + \mathrm{d}_{\mathbb{L}}(\mathbb{P}_k^{n,\triangle}(x_n,\cdot),\pi^{n,\triangle}(\cdot)).\end{aligned}$$

Remark 6. For the finite-dimensional case, finite-time convergence is enough to discuss the limit of the stationary distribution for the numerical scheme; see, e.g. [20, Theorem 6.23]. For the infinite-dimensional case, we need the *uniform convergence* of EI scheme (7) to get the limit behavior of $\pi^{n,\triangle}$, which is quite different from the finite-dimensional case. In fact, for finite-time convergence of EI scheme (8), condition (35) can be removed by checking the argument of Lemma 5 and taking the Gronwall inequality into consideration.

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