LIMIT THEOREMS FOR CONTINUOUS-TIME BRANCHING FLOWS

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Abstract

We construct a flow of continuous-time and discrete-state branching processes. Some scaling limit theorems for the flow are proved, which lead to the path-valued branching processes and nonlocal branching superprocesses, over the positive half line, studied in Li (2014).

Keywords: Stochastic flow; branching process; continuous time; superprocess; discrete state; nonlocal branching

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1. Introduction

A genealogical tree is naturally associated with a Galton–Watson branching process. A continuous-state branching process (CB process) can be obtained as the small particle limit of rescaled Galton–Watson processes; see, e.g. Lamperti (1967). The genealogical structures of binary branching CB processes were investigated by introducing *continuum random trees* in the pioneering work of Aldous (1991), (1993). Continuum random trees corresponding to general branching mechanisms were constructed in Le Gall and Le Jan (1998a), (1998b) and were studied further in Duquesne and Le Gall (2002). By pruning a Galton–Watson tree, Aldous and Pitman (1998) and Abraham *et al.* (2012) constructed a tree-valued Markov process. Tree-valued processes associated with general CB processes were studied in Abraham and Delmas (2012) by using pruning arguments.

Motivated by the study of genealogy trees for critical branching processes conditioned on nonextinction, Bakhtin (2011) studied a flow of binary branching continuous-state branching processes with immigration driven by a time-space Gaussian white noise. He also pointed out the connection of the model with a superprocess conditioned on nonextinction. In Li (2014), a class of path-valued branching processes were constructed and studied using the techniques of stochastic equations and superprocesses. The work is closely related to those of Bertoin and Le Gall (2006) and Dawson and Li (2014). In a special case, the path-valued branching processes in Li (2014) can be coded by the tree-valued processes of Abraham and Delmas (2012). In He and Ma (2014), two flows of discrete time and state Galton–Watson branching processes were introduced. There it was showed that suitable rescaled sequences of those flows converge to special forms of the flows of Dawson and Li (2012) and Li (2014). The limit theorems in He and Ma (2014) were given in the setting of the corresponding superprocesses. From those

limit theorems, the convergence of the finite-dimensional distributions of the corresponding path-valued processes was derived. The results give a better understanding of the connection between discrete and continuum tree-valued branching processes.

In this paper we introduce flows of continuous-time and discrete-state branching processes. We shall prove the scaling limit theorems, of the type given in He and Ma (2014), for those flows. In Section 2, a short review is given of the path-valued branching processes and nonlocal branching superprocesses studied in Li (2014). In Section 3 we construct a continuous-time and discrete-state branching process as the strong solution of a stochastic integral equation. In Section 4, the construction is extended to branching flows by considering systems of stochastic equations. In Section 5 we prove that suitable rescaled sequences of those flows converge to the nonlocal branching superprocess. From the limit theorem we also derive the convergence of the finite-dimensional distributions of the corresponding path-valued processes.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$. Let $M[0, 1]$ be the set of finite Borel measures on $[0, 1]$ endowed with the topology of weak convergence. We identify $M[0, 1]$ with the set $F[0, 1]$ of positive right continuous increasing functions on $[0, 1]$. Let $B[0, 1]$ be the Banach space of bounded Borel functions on $[0, 1]$ endowed with the supremum norm $\|\cdot\|$. Let $C[0, 1]$ denote its subspace of continuous functions. We use $B[0, 1]^+$ and $C[0, 1]^+$ to denote the subclasses of nonnegative elements and $C[0, 1]^{++}$ to denote the subset of $C[0, 1]^+$ of functions bounded away from 0. For $\mu \in M[0, 1]$ and $f \in B[0, 1]$, we write $\langle \mu, f \rangle = \int f \, \mathrm{d}\mu$ if the integral exists. Let $D([0, \infty), M[0, 1])$ denote the space of càdlàg paths from $[0, \infty)$ to $M[0, 1]$ endowed with the Skorokhod topology. Throughout this paper we only consider *continuous-time* processes, so we shall often omit this phrase in the sequel.

2. Preliminaries

In this section we recall some results established in Li (2014) on flows of CB processes and nonlocal branching superprocesses over the positive half line. By a *branching mechanism* ϕ we mean a function ϕ on $[0, \infty)$ with the representation

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)m(\mathrm{d}u), \tag{2.1}$$

where $\sigma \geq 0$ and b are constants and $(u \wedge u^2)m(\mathrm{d}u)$ is a finite measure on $(0, \infty)$. Consider a family of branching mechanisms $\{\phi_q : q \in [0, 1]\}$ that is *admissible* in the sense that each ϕ_q is given by (2.1) with parameters $(b, m) = (b_q, m_q)$ depending on $q \in [0, 1]$ and, for each $z \geq 0$, the function $q \mapsto \phi_q(z)$ is decreasing and continuously differentiable with the derivative $\psi_\theta(z) = -(\partial/\partial\theta)\phi_\theta(z)$ of the form

$$\psi_\theta(z) = h_\theta z + \int_0^\infty (1 - \mathrm{e}^{-zu})n_\theta(\mathrm{d}u),$$

where $h_\theta \geq 0$ and $n_\theta(\mathrm{d}u)$ is a σ-finite kernel from $[0, 1]$ to $(0, \infty)$ satisfying

$$\sup_{0 \leq \theta \leq 1} \left[h_\theta + \int_0^\infty un_\theta(\mathrm{d}u) \right] < \infty.$$

Let $m(\mathrm{d}z, \mathrm{d}\theta)$ be the measure on $(0, \infty) \times [0, 1]$ defined by

$$m([c, d] \times [0, q]) = m_q[c, d], \qquad q \in [0, 1], \ d > c > 0.$$

Let $W(\mathrm{d}s, \mathrm{d}u)$ be a white noise on $(0, \infty)^2$ based on the Lebesgue measure, and let

$$\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

be a compensated Poisson random measure on $(0, \infty)^2 \times [0, 1] \times (0, \infty)$ with intensity $\mathrm{d}s\, m(\mathrm{d}z, \mathrm{d}\theta)\, \mathrm{d}u$. By the results in Li (2014), the stochastic equation

$$Y_t(q) = Y_0(q) - b_q \int_0^t Y_{s-}(q)\, \mathrm{d}s + \sigma \int_0^t \int_0^{Y_{s-}(q)} W(\mathrm{d}s, \mathrm{d}u)$$
$$+ \int_0^t \int_0^\infty \int_{[0,q]} \int_0^{Y_{s-}(q)} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

has a unique solution flow $\{Y_t(q) \colon t \geq 0, q \in [0, 1]\}$. For each $q \in [0, 1]$, the one-dimensional process $\{Y_t(q) \colon t \geq 0\}$ is a CB process with branching mechanism ϕ_q. The flow is increasing in $q \in [0, 1]$. It was verified in Li (2014) that $\{(Y_t(q))_{t \geq 0} \colon q \in [0, 1]\}$ can be identified as a path-valued branching process. Moreover, the flow induces a càdlàg $M[0, 1]$-valued superprocess $\{Y_t \colon t \geq 0\}$ which is the unique solution of the following martingale problem: for every $G \in C^2(\mathbb{R})$ and $f \in C[0, 1]$,

$$G(\langle Y_t, f \rangle) = G(\langle Y_0, f \rangle) + \int_0^t G'(\langle Y_s, f \rangle)\, \mathrm{d}s \int_{[0,1]} Y_s(\mathrm{d}x) \int_{[0,1]} f(x \vee \theta) h_\theta\, \mathrm{d}\theta$$

$$- b_0 \int_0^t G'(\langle Y_s, f \rangle) \langle Y_s, f \rangle\, \mathrm{d}s + \frac{1}{2}\sigma^2 \int_0^t G''(\langle Y_s, f \rangle) \langle Y_s, f^2 \rangle\, \mathrm{d}s$$

$$+ \int_0^t \mathrm{d}s \int_{[0,1]} Y_s(\mathrm{d}x) \int_0^\infty [G(\langle Y_s, f \rangle + zf(x)) - G(\langle Y_s, f \rangle)$$
$$- zf(x)G'(\langle Y_s, f \rangle)] m_0(\mathrm{d}z)$$

$$+ \int_0^t \mathrm{d}s \int_{[0,1]} Y_s(\mathrm{d}x) \int_{[0,1]} \mathrm{d}\theta \int_0^\infty [G(\langle Y_s, f \rangle + zf(x \vee \theta)) - G(\langle Y_s, f \rangle)]$$
$$\times n_\theta(\mathrm{d}z) + \text{local martingale.} \quad (2.2)$$

Let $f \mapsto \Psi(\cdot, f)$ be the operator on $C^+[0, 1]$ defined by

$$\Psi(x, f) = \int_{[0,1]} f(x \vee \theta) h_\theta\, \mathrm{d}\theta + \int_{[0,1]} \mathrm{d}\theta \int_0^\infty (1 - \mathrm{e}^{-zf(x \vee \theta)}) n_\theta(\mathrm{d}z).$$

Then the superprocess $\{Y_t \colon t \geq 0\}$ has local branching mechanism ϕ_0 and nonlocal branching mechanism Ψ. The transition semigroup $(Q_t)_{t \geq 0}$ of $\{Y_t \colon t \geq 0\}$ is given by

$$\int_{M[0,1]} \mathrm{e}^{-\langle v, f \rangle} Q_t(\mu, \mathrm{d}v) = \exp\{-\langle \mu, V_t f \rangle\}, \qquad f \in C^+[0, 1], \quad (2.3)$$

where $t \mapsto V_t f$ is the unique locally bounded positive solution of

$$V_t f(x) = f(x) - \int_0^t [\phi_0(V_s f(x)) - \Psi(x, V_s f)]\, \mathrm{d}s, \qquad t \geq 0,\ x \in [0, 1]. \quad (2.4)$$

We refer the reader to Li (2014) for the derivations of the superprocess $\{Y_t \colon t \geq 0\}$.

Remark 2.1. Usually, we may only use the Lévy measure m in ϕ in (2.1) to integrate $1 \wedge u^2$, and an indicator function is added in the integral. In fact, the assumption that $(u \wedge u^2) m(\mathrm{d}u)$ is a finite measure on $(0, \infty)$ is equivalent to ϕ being locally Lipschitz; see Proposition 1.45

of Li (2011). This assumption is technically required to construct the scaling limits of Galton–Watson processes; see Proposition 3.40 of Li (2011) and Assumption 5.1 below.

3. Stochastic equations for discrete-state branching processes

In this section we give a construction of the continuous-time and discrete-state branching process as the solution of a stochastic integral equation driven by a Poisson random measure. Stochastic integral equations of this type were used in Li and Ma (2008) to construct catalytic branching processes. We here give all the details for completeness.

Let $g = g(z) = \sum_{i=0}^{\infty} p_i z^i$ be a probability generating function with $g'(1) < \infty$. Let $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $\sigma \, \mathrm{d}s \, \pi(\mathrm{d}z) \, \mathrm{d}u$, where $\sigma > 0$ is a constant and $\pi(\mathrm{d}z) := \sum_{i=0}^{\infty} p_i \delta_i(\mathrm{d}z)$. Suppose that X_0 is a nonnegative integer-valued random variable satisfying $\mathbb{E}[X_0] < \infty$. We assume that X_0 is independent of $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and consider the stochastic integral equation

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}} (z - 1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \tag{3.1}$$

By a *solution* of (3.1) we mean a nonnegative càdlàg progressive process $\{X_t : t \geq 0\}$ satisfying the equation almost surely (a.s.) for each $t \geq 0$. We say *pathwise uniqueness* of solution holds for (3.1) if any two solutions of the equation, with the same initial state, are indistinguishable.

Theorem 3.1. *Suppose that $\{X_t^1\}$ and $\{X_t^2\}$ are two solutions of (3.1) satisfying the inequality $\mathbb{E}[|X_0^1 + X_0^2|] < \infty$. Then we have*

$$\mathbb{E}[|X_t^2 - X_t^1|] \leq \mathbb{E}[|X_0^2 - X_0^1|] \exp\{\sigma t (g'(1) + 1)\}. \tag{3.2}$$

Consequently, the pathwise uniqueness of solution holds for (3.1).

Proof. The pathwise uniqueness for (3.1) follows from Theorem 2.1 of Dawson and Li (2012). We present a proof of the result here for completeness. Let $\xi_t = X_t^2 - X_t^1$ for $t \geq 0$. From (3.1) we have

$$\xi_t = X_0^2 - X_0^1 + \int_0^t \int_{\mathbb{N}} \int_{X_{s-}^1}^{X_{s-}^2} (z - 1) \mathbf{1}_{\{X_{s-}^1 \leq X_{s-}^2\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

$$- \int_0^t \int_{\mathbb{N}} \int_{X_{s-}^2}^{X_{s-}^1} (z - 1) \mathbf{1}_{\{X_{s-}^1 > X_{s-}^2\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Let $\tau_m = \inf\{t \geq 0 : X_t^1 \geq m \text{ or } X_t^2 \geq m\}$. Then we have

$$\mathbb{E}[|\xi_{t \wedge \tau_m}|] \leq \mathbb{E}[|\xi_0|] + \mathbb{E} \int_0^{t \wedge \tau_m} \int_{\mathbb{N}} \int_{X_{s-}^1}^{X_{s-}^2} (z + 1) \mathbf{1}_{\{X_{s-}^1 \leq X_{s-}^2\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

$$+ \mathbb{E} \int_0^{t \wedge \tau_m} \int_{\mathbb{N}} \int_{X_{s-}^2}^{X_{s-}^1} (z + 1) \mathbf{1}_{\{X_{s-}^1 > X_{s-}^2\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

$$= \mathbb{E}[|\xi_0|] + \mathbb{E} \int_0^{t \wedge \tau_m} \mathrm{d}s \int_{\mathbb{N}} \xi_{s-} \mathbf{1}_{\{\xi_{s-} \geq 0\}} (z + 1) \sigma \pi(\mathrm{d}z)$$

$$+ \mathbb{E} \int_0^{t \wedge \tau_m} \mathrm{d}s \int_{\mathbb{N}} (-\xi_{s-}) \mathbf{1}_{\{\xi_{s-} < 0\}} (z + 1) \sigma \pi(\mathrm{d}z)$$

$$\leq \mathbb{E}[|\xi_0|] + \int_0^t \mathbb{E}[|\xi_{s \wedge \tau_m}|] \sigma (g'(1) + 1) \, \mathrm{d}s.$$

By Gronwall's inequality we obtain

$$\mathbb{E}[|\xi_{t \wedge \tau_m}|] \le \mathbb{E}[|\xi_0|] \exp\{\sigma t (g'(1) + 1)\}.$$

Then (3.2) follows by Fatou's lemma, completing the proof.

By Theorem 2.5 of Dawson and Li (2012), there is a unique strong solution to (3.1). Here we give a simple direct proof of the existence of the solution. We first take an $n \in \mathbb{N}_+$ and consider the following stochastic equation

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-} \wedge n} (z - 1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \tag{3.3}$$

Proposition 3.1. *For each $n \ge 1$, there is a solution $\{X_t^n : t \ge 0\}$ of (3.3).*

Proof. Let $\{S_k : k = 1, 2, \ldots\}$ be the set of jump times of the Poisson process

$$t \mapsto \int_0^t \int_{\mathbb{N}} \int_0^n N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

We clearly have $S_k \to \infty$ as $k \to \infty$. For $0 \le t < S_1$, set $X_t^n = X_0$. Suppose that X_t^n has been defined for $0 \le t < S_k$ and let

$$X_t^n = X_{S_k-}^n + \int_{\{S_k\}} \int_{\mathbb{N}} \int_0^{X_{S_k-}^n \wedge n} (z - 1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \qquad S_k \le t < S_{k+1}.$$

From the construction of $X_{S_k}^n$ we see that $X_{S_k}^n - X_{S_{k-1}}^n \ge -1$. Furthermore, since $X_{S_{k-1}}^n = 0$ implies that $X_{S_k}^n = 0$, we have $X_{S_k}^n \in \mathbb{N}$. By induction we have defined a nonnegative process $\{X_t^n : t \ge 0\}$ which is clearly a solution to (3.3).

Proposition 3.2. *Let $\{X_t^n\}$ be a solution of (3.3). Then we have*

$$\mathbb{E}\left[\sup_{0 \le s \le t} X_s^n\right] \le \mathbb{E}[X_0] \exp\{\sigma g'(1) t\}, \qquad t \ge 0.$$

Proof. From (3.3) we have

$$\mathbb{E}\left[\sup_{0 \le s \le t} X_s^n\right] \le \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}^n \wedge n} z N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\right]$$
$$= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{\mathbb{N}} (X_{s-}^n \wedge n) z \sigma \pi(\mathrm{d}z)\right].$$

Thus $t \mapsto \mathbb{E}[\sup_{0 \le s \le t} X_s^n]$ is a locally bounded function. Moreover,

$$\mathbb{E}\left[\sup_{0 \le s \le t} X_s^n\right] \le \mathbb{E}[X_0] + \int_0^t \mathrm{d}s \int_{\mathbb{N}} \mathbb{E}\left[\sup_{0 \le r \le s} X_r^n\right] z \sigma \pi(\mathrm{d}z)$$
$$= \mathbb{E}[X_0] + \sigma g'(1) \int_0^t \mathbb{E}\left[\sup_{0 \le r \le s} X_r^n\right] \mathrm{d}s.$$

By Gronwall's lemma we obtain the result, completing the proof.

By a modification of the proof of Theorem 3.1 we obtain the following Proposition.

Proposition 3.3. *Suppose that $\{X_t^{n,1}\}$ and $\{X_t^{n,2}\}$ are two solutions of (3.3). Then we have*

$$\mathbb{E}[|X_t^{n,2} - X_t^{n,1}|] \leq \mathbb{E}[|X_0^{n,2} - X_0^{n,1}|]\exp\{\sigma t(g'(1) + 1)\}.$$

Consequently, the pathwise uniqueness of solution holds for (3.3).

Proposition 3.4. *Let $\{X_t^n : t \geq 0\}$ be the solution of (3.3) with $n = 1, 2, \ldots$. Then the sequence $\{X_t^n : t \geq 0\}$ is tight in $D([0, \infty), \mathbb{N})$.*

Proof. By Proposition 3.2, it is easy to see that

$$t \mapsto C_t := \sup_{n \geq 1} \mathbb{E}\left[\sup_{0 \leq s \leq t} X_s^n\right]$$

is locally bounded. Then, for every fixed $t \geq 0$, the sequence of random variables X_t^n is tight. Moreover, in view of (3.3), if $\{\tau_n\}$ is a sequence of stopping times bounded above by $T \geq 0$, we have

$$\mathbb{E}[|X_{t+\tau_n}^n - X_{\tau_n}^n|] = \mathbb{E}\left[\left|\int_{\tau_n}^{t+\tau_n}\int_{\mathbb{N}}\int_0^{X_{s-}^n \wedge n}(z - 1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\right|\right]$$

$$\leq \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{\mathbb{N}}(X_{(s+\tau_n)-}^n \wedge n)(z + 1)\sigma\pi(\mathrm{d}z)\right]$$

$$\leq \sigma(g'(1) + 1)\int_0^t \mathbb{E}[X_{(s+\tau_n)-}^n]\,\mathrm{d}s$$

$$\leq \mathbb{E}[X_0]\exp\{\sigma g'(1)(t + T)\}\sigma(g'(1) + 1)t,$$

where the last inequality follows by Proposition 3.2. Consequently, as $t \to 0$,

$$\sup_{n \geq 1}\mathbb{E}[|X_{t+\tau_n}^n - X_{\tau_n}^n|] \to 0.$$

Then $\{X_t^n : t \geq 0\}$ is tight in $D([0, \infty), \mathbb{N})$ by the criterion of Aldous (1978); see also Ethier and Kurtz (1986, pp. 137–138).

Theorem 3.2. *There is a solution $\{X_t : t \geq 0\}$ of (3.1).*

Proof. For each $n \geq 1$, let $\{X_t^n : t \geq 0\}$ be the solution of (3.3). Furthermore, define $\tau_n = \inf\{t \geq 0 : X_t^n \geq n\}$. From Proposition 3.2, it follows that

$$\mathbb{E}[X_{t \wedge \tau_n}^n] \leq \mathbb{E}\left[\sup_{0 \leq s \leq t} X_s^n\right] \leq \mathbb{E}[X_0]\exp\{\sigma g'(1)t\}, \qquad t \geq 0.$$

Then we have

$$\mathbb{E}[X_{t \wedge \tau_n}^n \mathbf{1}_{\{\tau_n \leq t\}}] \leq \mathbb{E}[X_0]\exp\{\sigma g'(1)t\}.$$

By the right continuity of $\{X_t^n\}$ we have $X_{\tau_n}^n \geq n$, so

$$n\mathbb{P}[\{\tau_n \leq t\}] \leq \mathbb{E}[X_0]\exp\{\sigma g'(1)t\}, \qquad t \geq 0.$$

This implies that $\tau_n \to \infty$ almost surely as $n \to \infty$. On the other hand, $\{X_t^n\}$ satisfies (3.1) for $0 \leq t < \tau_n$. By the pathwise uniqueness of the solution of (3.1) we obtain, for any $i, j \in \mathbb{N}$,

$$X_t^i = X_t^j, \qquad t < \tau_i \wedge \tau_j.$$

Let $\{X_t\}$ be the process such that $X_t = X_t^n$ for all $0 \leq t < \tau_n$ and $n \geq 1$. It is easily seen that $\{X_t\}$ is a solution of (3.1). This completes the proof.

Theorem 3.1 and Theorem 3.2 imply that (3.1) has a unique *strong solution* and the solution $\{X_t : t \geq 0\}$ is a strong Markov process; see, e.g. Ikeda and Watanabe (1989, pp. 163–166, 215). Let $B(\mathbb{N})$ denote the set of bounded functions on $\mathbb{N}$. By Itô's formula, it is easy to see that $\{X_t : t \geq 0\}$ has a generator A defined by

$$Af(x) = \sigma x \sum_{i=0}^{\infty} [f(x + i - 1) - f(x)] p_i, \qquad x \in \mathbb{N}, \ f \in B(\mathbb{N}).$$

Hence, $\{X_t : t \geq 0\}$ is a Galton–Watson branching process with σ-exponentially distributed life time and offspring distribution $\{p_i : i \geq 0\}$.

In fact, let $N^{(1)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N^{(2)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be two mutually independent Poisson random measures on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with the same intensity $\sigma \, \mathrm{d}s \pi(\mathrm{d}z) \, \mathrm{d}u$. Consider the two stochastic equations

$$X_t^{(1)} = X_0^{(1)} + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}^{(1)}} (z - 1) N^{(1)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and

$$X_t^{(2)} = X_0^{(2)} + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}^{(2)}} (z - 1) N^{(2)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Clearly, $X_t^{(1)}$ and $X_t^{(2)}$ are mutually independent. Set $X_t = X_t^{(1)} + X_t^{(2)}$. Since the random measure

$$N'(\mathrm{d}s, \mathrm{d}z) := \int_{\{0 < u \leq X_{s-}^{(1)}\}} N^{(1)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_{\{0 < u \leq X_{s-}^{(2)}\}} N^{(2)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

has the predictable compensator $\sigma X_{s-} \, \mathrm{d}s \pi(\mathrm{d}z)$, by representation theorems for semimartingales, on an extension of the original probability space, there is a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $\sigma \, \mathrm{d}s \pi(\mathrm{d}z) \, \mathrm{d}u$ such that

$$X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}} (z - 1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u);$$

see, e.g. Ikeda and Watanabe (1989, p. 93). Hence, the solution of (3.1) is a branching process (continuous time and discrete state). This gives another derivation of the branching property of $\{X_t : t \geq 0\}$.

4. The flow of discrete-state branching processes

In this section we give a formulation of the discrete-state branching flow as the solution flow of a set of stochastic integral equations. Let $\{g_\theta : \theta \geq 0\}$ be a family of probability generating functions, that is, for each $\theta \geq 0$,

$$g_\theta(z) = \sum_{i=0}^{\infty} p_i(\theta) z^i, \qquad |z| \leq 1,$$

where $p_i(\theta) \geq 0$ and $\sum_{k=0}^{\infty} p_i(\theta) = 1$. Moreover, we assume that $\theta \mapsto g'_\theta(1)$ is continuous and $p_i(\theta_2) \geq p_i(\theta_1)$ holds for all $\theta_2 \geq \theta_1 \geq 0$ and $i \in \mathbb{N}_+$. Define a family of probability measures $\{\pi_\theta : \theta \geq 0\}$ on $\mathbb{N}$ by

$$\pi_\theta(\mathrm{d}z) = \sum_{i=0}^{\infty} p_i(\theta) \delta_i(\mathrm{d}z).$$

Then we have $\pi_{\theta_2}|_{\mathbb{N}_+} \geq \pi_{\theta_1}|_{\mathbb{N}_+}$ for all $\theta_2 \geq \theta_1 \geq 0$. Let $\bar{\pi}(\mathrm{d}z, \mathrm{d}\theta)$ be the measure on $\mathbb{N}_+ \times [0, \infty)$ defined by

$$\bar{\pi}(A \times [0, \theta]) = \pi_\theta(A), \qquad A \subset \mathbb{N}_+, \ \theta \geq 0.$$

Note that the positive function $\theta \mapsto b(\theta) := \pi_\theta(\{0\})$ is decreasing.

Let $q \mapsto X_0(q)$ be a deterministic, nonnegative, right-continuous, nondecreasing function on $[0, \infty)$ that takes values in $\mathbb{N}$. Let $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$ be a Poisson random measure on $(0, \infty) \times \mathbb{N} \times [0, \infty) \times (0, \infty)$ with intensity $\sigma \, \mathrm{d}s \bar{\pi}(\mathrm{d}z, \mathrm{d}\theta) \, \mathrm{d}u$ and $N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$ a Poisson random measure on $(0, \infty)^3$ with intensity $\sigma \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}u$. Suppose that $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$ and $N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$ are independent of each other. Consider the stochastic integral equation

$$X_t(q) = X_0(q) + \int_0^t \int_{\mathbb{N}_+} \int_{[0,q]} \int_0^{X_{s-}(q)} (z-1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$
$$- \int_0^t \int_0^{b(q)} \int_0^{X_{s-}(q)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u). \tag{4.1}$$

Note that, for each $q \geq 0$,

$$\int_{\{0 < \theta \leq b(q)\}} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$$

is a Poisson random measure with intensity $\sigma b(q) \, \mathrm{d}s \, \mathrm{d}u = \sigma \bar{\pi}_0(\mathbb{N} \times [0, q]) \, \mathrm{d}s \, \mathrm{d}u$, where $\bar{\pi}_0(\mathrm{d}z, \mathrm{d}\theta)$ is a measure on $\mathbb{N} \times [0, \infty)$ defined by

$$\bar{\pi}_0(A \times [0, q]) = \pi_q(\{0\})\delta_0(A), \qquad A \subset \mathbb{N}, \ \theta \geq 0.$$

By representation theorems for semimartingales, there exists a Poisson random measure $N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$ on $(0, \infty) \times \mathbb{N} \times [0, \infty) \times (0, \infty)$ with intensity $\sigma \, \mathrm{d}s \bar{\pi}_0(\mathrm{d}z, \mathrm{d}\theta) \, \mathrm{d}u$ such that, for every $E \in \mathcal{B}(0, \infty)$,

$$\int_0^t \int_0^{b(q)} \int_E N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u) = \int_0^t \int_{\mathbb{N}} \int_{[0,q]} \int_E N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u);$$

see, e.g. Ikeda and Watanabe (1989, p. 93). Define $N_2(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ by

$$N_2(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \int_{\{0 \leq \theta \leq q\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u) + \int_{\{0 \leq \theta \leq q\}} N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u).$$

Then N_2 is a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $\sigma \, \mathrm{d}s \pi_q(\mathrm{d}z) \, \mathrm{d}u$, and (4.1) can be rewritten as

$$X_t(q) = X_0(q) + \int_0^t \int_{\mathbb{N}} \int_0^{X_{s-}(q)} (z-1) N_2(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

By Theorem 3.2 we see that, for each $q \geq 0$, (4.1) has a unique strong solution $\{X_t(q) : t \geq 0\}$.

Theorem 4.1. *Suppose that $q \geq p \geq 0$. Let $\{X_t(q)\}$ be the solution of (4.1) and $\{X_t(p)\}$ be the solution of the equation with q replaced by p. Then we have*

$$\mathbb{P}\{X_t(q) \geq X_t(p) \text{ for all } t \geq 0\} = 1.$$

Proof. Let $\zeta_t = X_t(p) - X_t(q)$ for $t \geq 0$. From (4.1) we have

$$\zeta_t = \zeta_0 + \int_0^t \int_{\mathbb{N}_+} \int_{[0,p]} \int_{X_{s-}(q)}^{X_{s-}(p)} (z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \int_0^t \int_{\mathbb{N}_+} \int_{(p,q]} \int_0^{X_{s-}(q)} (z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \int_0^t \int_0^{b(q)} \int_{X_{s-}(q)}^{X_{s-}(p)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u) - \int_0^t \int_{b(q)}^{b(p)} \int_0^{X_{s-}(p)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u). \quad (4.2)$$

Let $\tau_m = \inf\{t \geq 0 : X_t(q) \geq m \text{ or } X_t(p) \geq m\}$. It is easy to construct a sequence of functions $\{f_n\}$ on $\mathbb{R}$ such that $0 \leq f_n'(z) \leq 1$ for $z \geq 0$, and $f_n(z) = f_n'(z) = 0$ for $z \leq 0$. Moreover, $f_n(z) \to z^+ := 0 \vee z$ increasingly as $n \to \infty$. By (4.2) and Itô's formula,

$$f_n(\zeta_{t \wedge \tau_m}) = \int_0^{t \wedge \tau_m} \int_{\mathbb{N}_+} \int_{[0,p]} \int_{X_{s-}(q)}^{X_{s-}(p)} [f_n(\zeta_{s-} + z - 1) - f_n(\zeta_{s-})]$$

$$\times \mathbf{1}_{\{\zeta_{s-} > 0\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$+ \int_0^{t \wedge \tau_m} \int_{\mathbb{N}_+} \int_{(p,q]} \int_0^{X_{s-}(q)} [f_n(\zeta_{s-} - z + 1) - f_n(\zeta_{s-})]N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$+ \int_0^{t \wedge \tau_m} \int_0^{b(q)} \int_{X_{s-}(p)}^{X_{s-}(q)} [f_n(\zeta_{s-} - 1) - f_n(\zeta_{s-})] \mathbf{1}_{\{\zeta_{s-} > 0\}} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$$

$$+ \int_0^{t \wedge \tau_m} \int_{b(q)}^{b(p)} \int_0^{X_{s-}(p)} [f_n(\zeta_{s-} - 1) - f_n(\zeta_{s-})]N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$$

$$\leq \sigma \int_0^{t \wedge \tau_m} \zeta_{s-} \mathbf{1}_{\{\zeta_{s-} > 0\}} \, \mathrm{d}s \int_{\mathbb{N}_+} (z-1)\pi_p(\mathrm{d}z) + \text{martingale}.$$

Taking the expectation in both sides and letting $n \to \infty$ gives

$$\mathbb{E}[\zeta_{t \wedge \tau_m}^+] \leq \sigma(g_p'(1) - 1 + b(p)) \int_0^t \mathbb{E}[\zeta_{s \wedge \tau_m}^+] \, \mathrm{d}s.$$

Hence, $\mathbb{E}[\zeta_{t \wedge \tau_m}^+] = 0$ for all $t \geq 0$. Since $\tau_m \to \infty$ as $m \to \infty$, this proves the desired comparison result.

Proposition 4.1. *There is a locally bounded positive function* $(t, u) \mapsto C(t, u)$ *on* $[0, \infty)^2$ *satisfying, for any* $t \geq 0$ *and* $p \leq q \leq u < \infty$,

$$\mathbb{E}\left\{\sup_{0 \leq s \leq t} [X_s(q) - X_s(p)]\right\} \leq C(t, u)\{X_0(q) - X_0(p) + g_q'(1) - g_p'(1)\}.$$

Proof. Let $\xi_t = X_t(q) - X_t(p)$. From (4.1) we obtain

$$\sup_{0 \leq s \leq t} \xi_s \leq \xi_0 + \int_0^t \int_{\mathbb{N}_+} \int_{[0,q]} \int_{X_{s-}(p)}^{X_{s-}(q)} (z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$+ \int_0^t \int_{\mathbb{N}_+} \int_{(p,q]} \int_0^{X_{s-}(p)} (z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$+ \int_0^t \int_{b(q)}^{b(p)} \int_0^{X_{s-}(p)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u).$$

Hence,

$$\mathbb{E}\Big[\sup_{0\le s\le t}\xi_s\Big] \le \xi_0 + \sigma[g'_p(1) - 1 + b(q)]\int_0^t \mathbb{E}[\xi_s]\,\mathrm{d}s + \sigma[g'_q(1) - g'_p(1)]\int_0^t \mathbb{E}[X_s(p)]\,\mathrm{d}s.$$

Since $t \mapsto \mathbb{E}[X_t(p)]$ is locally bounded, by Gronwall's inequality, we obtain the desired estimate, completing the proof.

From the discussion above, given a constant $\sigma > 0$ and a family of probability generating functions $\{g_\theta \colon \theta \ge 0\}$, we obtain a continuous-time and discrete-state branching process flow $\{X_t(q) \colon t \ge 0, q \ge 0\}$ as the solution of equation (4.1). For every $t \ge 0$, define the random function $\tilde{X}_t \in F[0,1]$ by $\tilde{X}_t(1) = X_t(1)$ and

$$\tilde{X}_t(q) = \inf\{X_t(u)\colon \text{rational } u \in (q, 1]\}, \qquad 0 \le q < 1. \tag{4.3}$$

By Proposition 4.1, for each $q \in [0, 1]$, we have

$$\mathbb{P}\{\tilde{X}_t(q) = X_t(q) \text{ for all } t \ge 0\} = 1.$$

Therefore, $\{\tilde{X}_t(q)\colon t \ge 0\}$ is also càdlàg and solves (4.1) for every $q \in [0, 1]$.

5. Scaling limits of the discrete branching flows

In this section we prove some limit theorems for the discrete-state branching flows, which will lead to the continuous-state branching flows of Li (2014). We shall present the limit theorems in the settings of measure-valued processes and path-valued processes.

Suppose that, for each $k \ge 1$, there is a positive constant σ_k and a family of generating functions $\{g_\theta^{(k)}\colon \theta \ge 0\}$ satisfying the assumptions specified at the beginning of Section 4. Then we can define $\pi_\theta^{(k)}(\mathrm{d}z)$ and $\bar{\pi}^{(k)}(\mathrm{d}z, \mathrm{d}\theta)$ in the same way as there. Let $\{X_t^{(k)}(q)\colon t \ge 0\}$ be the corresponding solution of (4.1) and $\{\tilde{X}_t^{(k)}(q)\colon t \ge 0, q \in [0, k]\}$ be defined in the same way as in (4.3). Define

$$Y_t^{(k)}(q) = \frac{1}{k}\tilde{X}_t^{(k)}(kq), \qquad q \in [0, 1].$$

From (4.1) we have

$$Y_t^{(k)}(q) = Y_0^{(k)}(q) + \frac{1}{k}\int_0^t \int_{\mathbb{N}_+} \int_{[0,kq]} \int_0^{kY_{s-}^{(k)}(q)} (z - 1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \frac{1}{k}\int_0^t \int_0^{b_k(kq)} \int_0^{kY_{s-}^{(k)}(q)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u). \tag{5.1}$$

We can use a standard stopping time argument to show that, for any $q \in [0, 1]$, the function $t \mapsto \mathbb{E}[Y_t^{(k)}(q)]$ is locally bounded. Then, by an argument similar to the proof of Proposition 3.2, we have the following proposition.

Proposition 5.1. *For any $t \ge 0$ and $q \in [0, 1]$, we have*

$$\mathbb{E}\Big[\sup_{0\le s\le t} Y_s^{(k)}(q)\Big] \le Y_0^{(k)}(q)\exp\{t\sigma_k((g_{kq}^{(k)})'(1) - 1 + b_k(kq))\}.$$

The random function $Y_t^{(k)} \in F[0, 1]$ induces a random measure $Y_t^{(k)} \in M[0, 1]$ so that $Y_t^{(k)}([0, q]) = Y_t^{(k)}(q)$ for $q \in [0, 1]$. We are interested in the asymptotic behavior of the

process $\{Y_t^{(k)} : t \geq 0\}$ as $k \to \infty$. For any $f \in C^1[0, 1]$, we can use Fubini's theorem to see that

$$\langle Y_t^{(k)}, f \rangle = f(1)Y_t^{(k)}(1) - \int_0^1 f'(q)Y_t^{(k)}(q)\,\mathrm{d}q. \tag{5.2}$$

Fix an integer $n \geq 1$ and let $q_i = i/2^n$ for $i = 0, 1, \ldots, 2^n$. By (5.1), we have

$$\sum_{i=1}^{2^n} f'(q_i)Y_t^{(k)}(q_i) = \sum_{i=1}^{2^n} f'(q_i)Y_0^{(k)}(q_i)$$

$$+ \frac{1}{k}\sum_{i=1}^{2^n} f(q_i) \int_0^t \int_{\mathbb{N}_+} \int_{[0, kq_i]} \int_0^{kY_{s-}^{(k)}(q_i)} (z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \frac{1}{k}\sum_{i=1}^{2^n} f'(q_i) \int_0^t \int_0^{b_k(kq_i)} \int_0^{kY_{s-}^{(k)}(q_i)} N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u)$$

$$= \sum_{i=1}^{2^n} f'(q_i)Y_0^{(k)}(q_i)$$

$$+ \frac{1}{k}\int_0^t \int_{\mathbb{N}_+} \int_{[0, k]} \int_0^{kY_{s-}^{(k)}(1)} F_n^{(k)}(s, \theta, u)(z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \frac{1}{k}\int_0^t \int_0^{b_k(0)} \int_0^{kY_{s-}^{(k)}(1)} \tilde{F}_n^{(k)}(s, \theta, u)N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u), \tag{5.3}$$

where

$$F_n^{(k)}(s, \theta, u) = \sum_{i=1}^{2^n} f'(q_i)\,\mathbf{1}_{\{\theta \leq kq_i\}}\,\mathbf{1}_{\{u \leq kY_{s-}^{(k)}(q_i)\}}$$

$$\text{and} \quad \tilde{F}_n^{(k)}(s, \theta, u) = \sum_{i=1}^{2^n} f'(q_i)\,\mathbf{1}_{\{\theta \leq b_k(kq_i)\}}\,\mathbf{1}_{\{u \leq kY_{s-}^{(k)}(q_i)\}}.$$

By the right continuity of $q \mapsto Y_t^{(k)}(q)$, it is easy to see that, as $n \to \infty$,

$$2^{-n}F_n^{(k)}(s, \theta, u) \to F^{(k)}(s, \theta, u) := \int_0^1 f'(q)\,\mathbf{1}_{\{\theta \leq kq\}}\,\mathbf{1}_{\{u \leq kY_{s-}^{(k)}(q)\}}\,\mathrm{d}q$$

and

$$2^{-n}\tilde{F}_n^{(k)}(s, \theta, u) \to \tilde{F}^{(k)}(s, \theta, u) := \int_0^1 f'(q)\,\mathbf{1}_{\{\theta \leq b_k(kq)\}}\,\mathbf{1}_{\{u \leq kY_{s-}^{(k)}(q)\}}\,\mathrm{d}q.$$

Then, by (5.3), we have, almost surely,

$$\int_0^1 f'(q)Y_t^{(k)}(q)\,\mathrm{d}q = \int_0^1 f'(q)Y_0^{(k)}(q)\,\mathrm{d}q$$

$$+ \frac{1}{k}\int_0^t \int_{\mathbb{N}_+} \int_{[0, k]} \int_0^{kY_{s-}^{(k)}(1)} F^{(k)}(s, \theta, u)(z-1)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$$

$$- \frac{1}{k}\int_0^t \int_0^{b_k(0)} \int_0^{kY_{s-}^{(k)}(1)} \tilde{F}^{(k)}(s, \theta, u)N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u). \tag{5.4}$$

From (5.1), (5.2), and (5.4), it follows that, almost surely,

$$
\begin{aligned}
\langle Y_t^{(k)}, f \rangle &= \langle Y_0^{(k)}, f \rangle \\
&\quad + \frac{1}{k} \int_0^t \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s-}^{(k)}(1)} [f(1) - F^{(k)}(s, \theta, u)](z-1) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u) \\
&\quad - \frac{1}{k} \int_0^t \int_0^{b_k(k)} \int_0^{kY_{s-}^{(k)}(1)} [f(1) - \tilde{F}^{(k)}(s, \theta, u)] N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u) \\
&\quad + \frac{1}{k} \int_0^t \int_{b_k(k)}^{b_k(0)} \int_0^{kY_{s-}^{(k)}(1)} \tilde{F}^{(k)}(s, \theta, u) N_0(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}u).
\end{aligned}
\tag{5.5}
$$

Proposition 5.2. *Suppose that* $Y_0^{(k)}(1)$ *converges to some* $Y_0(1)$ *as* $k \to \infty$ *and*

$$
\sup_{k \geq 1} \sigma_k [(g_k^{(k)})'(1) - 1 + b_k(0)] < \infty.
$$

Then $\{Y_t^{(k)} : t \geq 0\}$, $k = 1, 2, \ldots$ *is a tight sequence in* $D([0, \infty), M[0,1])$.

Proof. For any $t \geq 0$ and $f \in C[0,1]$, by Proposition 5.1, it is easy to see that

$$
t \mapsto C_t := \sup_{k \geq 1} \mathbb{E}\left[\sup_{0 \leq s \leq t} \langle Y_s^{(k)}, f \rangle \right]
$$

is locally bounded. Hence, for every fixed $t \geq 0$, the sequence $\langle Y_t^{(k)}, f \rangle$ is tight. Let τ_k be a bounded stopping time for $\{Y_t^{(k)} : t \geq 0\}$ and assume that the sequence $\{\tau_k : k = 1, 2, \ldots\}$ is bounded above by $T \geq 0$. Let $f \in C^1[0,1]$. By (5.5), we see that

$$
\begin{aligned}
\mathbb{E}&[|\langle Y_{\tau_k+t}^{(k)}, f \rangle - \langle Y_{\tau_k}^{(k)}, f \rangle|] \\
&\leq \frac{\sigma_k}{k} \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s+\tau_k}^{(k)}(1)} (z-1)|f(1) - F^{(k)}(s+\tau_k, \theta, u)| \bar{\pi}^{(k)}(\mathrm{d}z, \mathrm{d}\theta)\, \mathrm{d}u \right] \\
&\quad + \frac{\sigma_k}{k} \mathbb{E}\left[\int_0^t \mathrm{d}s \int_0^{b_k(k)} \mathrm{d}\theta \int_0^{kY_{s+\tau_k}^{(k)}(1)} |f(1) - \tilde{F}^{(k)}(s+\tau_k, \theta, u)|\, \mathrm{d}u \right] \\
&\quad + \frac{\sigma_k}{k} \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{b_k(k)}^{b_k(0)} \mathrm{d}\theta \int_0^{kY_{s+\tau_k}^{(k)}(1)} |\tilde{F}^{(k)}(s+\tau_k, \theta, u)|\, \mathrm{d}u \right].
\end{aligned}
\tag{5.6}
$$

For $s, \theta, u > 0$, let $Y_{s,k}^{-1}(u) = \inf\{q \geq 0 : Y_s^{(k)}(q) > u\}$ and $b_k^{-1}(u) = \inf\{q \geq 0 : b_k(q) > u\}$. It is easy to see that

$$
\{q \geq 0 : u \leq kY_s^{(k)}(q)\} = [Y_{s,k}^{-1}(u/k), \infty), \qquad \{q \geq 0 : \theta \leq b_k(kq)\} = [0, b_k^{-1}(\theta)/k]
$$

except for at most countably many $u > 0$ and $\theta > 0$, respectively. Hence, in (5.6) we can replace $f(1) - F^{(k)}(s, \theta, u)$ by

$$
f(1) - \int_{\theta/k}^1 f'(q) \mathbf{1}_{\{Y_{s,k}^{-1}(u/k) \leq q\}}\, \mathrm{d}q = f\left(Y_{s,k}^{-1}\left(\frac{u}{k}\right) \vee \frac{\theta}{k} \right)
$$

and $\tilde{F}^{(k)}(s, \theta, u)$ can be replaced by

$$
\begin{aligned}
\int_0^1 f'(q) &\mathbf{1}_{\{q \leq b_k^{-1}(\theta)/k\}} \mathbf{1}_{\{Y_{s,k}^{-1}(u/k) \leq q\}}\, \mathrm{d}q \\
&= [f(1 \wedge (b_k^{-1}(\theta)/k)) - f(Y_{s,k}^{-1}(u/k))] \mathbf{1}_{\{Y_{s,k}^{-1}(u/k) \leq b_k^{-1}(\theta)/k\}}.
\end{aligned}
$$

Then, from (5.6), we have

$$
\begin{aligned}
\mathbb{E}[|\langle Y_{\tau_k+t}^{(k)}, f\rangle &- \langle Y_{\tau_k}^{(k)}, f\rangle|] \\
&\leq \sigma_k \mathbb{E}\left[\int_0^t \mathrm{d}s \int_0^1 Y_{s+\tau_k}^{(k)}(\mathrm{d}x) \int_{\mathbb{N}_+} \int_{[0,k]} (z-1)|f(x\vee\theta)|\bar{\pi}^{(k)}(\mathrm{d}z,\mathrm{d}\theta)\right] \\
&\quad + \sigma_k \mathbb{E}\left[\int_0^t \mathrm{d}s \int_0^{b_k(k)} \mathrm{d}\theta \int_0^1 |f(x)|Y_{s+\tau_k}^{(k)}(\mathrm{d}x)\right] \\
&\quad + \sigma_k \mathbb{E}\left[\int_0^t \mathrm{d}s \int_{b_k(k)}^{b_k(0)} \mathrm{d}\theta \int_0^1 \left|f\left(\frac{b_k^{-1}(\theta)}{k}\right) - f(x)\right|Y_{s+\tau_k}^{(k)}(\mathrm{d}x)\right] \\
&\leq \|f\|\sigma_k \int_0^t \mathbb{E}[Y_{s+\tau_k}^{(k)}(1)]\,\mathrm{d}s \int_{\mathbb{N}_+} (z-1)\pi_k^{(k)}(\mathrm{d}z) \\
&\quad + \|f\|\sigma_k b_k(k)\mathbb{E}\int_0^t [Y_{s+\tau_k}^{(k)}(1)]\,\mathrm{d}s \\
&\quad + 2\|f\|\sigma_k[b_k(0)-b_k(k)]\int_0^t \mathbb{E}[Y_{s+\tau_k}^{(k)}(1)]\,\mathrm{d}s \\
&\leq \|f\|\sigma_k((g_k^{(k)})'(1) - 1 + 2b_k(0))\int_0^t \mathbb{E}[Y_{s+\tau_k}^{(k)}(1)]\,\mathrm{d}s \\
&\leq 2\|f\|Y_0^{(k)}(1)t\sigma_k A_k \exp\{\sigma_k A_k(t+T)\},
\end{aligned}
\tag{5.7}
$$

where $A_k = (g_k^{(k)})'(1) - 1 + b_k(0)$ and the last inequality follows by Proposition 5.1. For $f \in C[0,1]$, the above inequality follows by an approximation argument. From (5.7) we have

$$
\lim_{t\to 0}\sup_{k\geq 1}\mathbb{E}[|\langle Y_{\tau_k+t}^{(k)}, f\rangle - \langle Y_{\tau_k}^{(k)}, f\rangle|] = 0.
$$

By a criterion of Aldous (1978), the sequence $\{\langle Y_t^{(k)}, f\rangle\colon t\geq 0\}$ is tight in $D([0,\infty),\mathbb{R})$; see also Ethier and Kurtz (1986, pp. 137–138). Hence, the tightness criterion of Roelly (1986) implies that $\{Y_t^{(k)}\colon t\geq 0\}$ is tight in $D([0,\infty), M[0,1])$.

For $z\geq 0$, define

$$
\phi_\theta^{(k)}(z) = k\sigma_k[g_{k\theta}^{(k)}(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}].
$$

Let us consider the following assumption.

Assumption 5.1. *For each $l\geq 0$, the sequence $\{\phi_\theta^{(k)}(z)\}$ is Lipschitz with respect to z uniformly on $[0,1]\times[0,l]$ and there is an admissible family of branching mechanisms $\{\phi_\theta(z)\colon \theta\geq 0\}$ with $(\partial/\partial\theta)\phi_\theta(z) = -\psi_\theta(z)$ such that $\phi_\theta^{(k)}(z) \to \phi_\theta(z)$ uniformly on $[0,1]\times[0,l]$ as $k\to\infty$.*

Let $\{Y_t\colon t\geq 0\}$ be the càdlàg superprocess with the transition semigroup defined by (2.3) and (2.4).

Theorem 5.1. *Suppose that Assumption 5.1 holds and $\sup_{k\geq 1}\sigma_k b_k(0) < \infty$. If $Y_0^{(k)}$ converges weakly to $Y_0 \in M[0,1]$ then $\{Y_t^{(k)}\colon t\geq 0\}$ converges in distribution to the superprocess $\{Y_t\colon t\geq 0\}$ on $D([0,\infty), M[0,1])$.*

Proof. Under the assumption, we have

$$
\sup_{k\geq 1}\sigma_k[(g_k^{(k)})'(1) - 1 + b_k(0)] < \infty.
$$

By Proposition 5.2 and Skorokhod's representation theorem, to simplify the notation we pass to a subsequence and simply assume $\{Y_t^{(k)} : t \geq 0\}$ converges a.s. to a process $\{Z_t : t \geq 0\}$ in the topology of $D([0, \infty), M[0, 1])$. Since the solution of the martingale problem (2.2) is unique, it suffices to prove that the weak limit point $\{Z_t : t \geq 0\}$ of the sequence $\{Y_t^{(k)} : t \geq 0\}$ is the solution of the martingale problem. Let $Y_{s,k}^{-1}(u)$ and $b_k^{-1}(u)$ be defined as in Proposition 5.2. For every $G \in C^2(\mathbb{R})$ and $f \in C^1[0, 1]$, we use (5.5) and Itô's formula to obtain

$$
\begin{aligned}
&G(\langle Y_t^{(k)}, f \rangle) \\
&\quad = G(\langle Y_0^{(k)}, f \rangle) \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}(z-1)[f(1) - F^{(k)}(s, \theta, u)]) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\} \bar{\pi}^{(k)}(\mathrm{d}z, \mathrm{d}\theta)\,\mathrm{d}u \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_0^{b_k(k)} \mathrm{d}\theta \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle - k^{-1}[f(1) - \tilde{F}^{(k)}(s, \theta, u)]) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\}\,\mathrm{d}u \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_{b_k(k)}^{b_k(0)} \mathrm{d}\theta \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}\tilde{F}^{(k)}(s, \theta, u)) - G(\langle Y_{s-}^{(k)}, f \rangle)\}\,\mathrm{d}u \\
&\qquad + \text{local martingale} \\
&\quad = G(\langle Y_0^{(k)}, f \rangle) \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_{\mathbb{N}_+} \int_{[0,k]} \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}(z-1)f(Y_{s,k}^{-1}(u/k) \vee (\theta/k))) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\} \bar{\pi}^{(k)}(\mathrm{d}z, \mathrm{d}\theta)\,\mathrm{d}u \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_0^{b_k(k)} \mathrm{d}\theta \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle - k^{-1}f(Y_{s,k}^{-1}(u/k))) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\}\,\mathrm{d}u \\
&\qquad + \sigma_k \int_0^t \mathrm{d}s \int_{b_k(k)}^{b_k(0)} \mathrm{d}\theta \int_0^{kY_{s-}^{(k)}(1)} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}[f(b_k^{-1}(\theta)/k) - f(Y_{s,k}^{-1}(u/k))] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathbf{1}_{\{Y_{s,k}^{-1}(u/k) \leq b_k^{-1}(\theta)/k\}}) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\}\,\mathrm{d}u + \text{local martingale} \\
&\quad = G(\langle Y_0^{(k)}, f \rangle) \\
&\qquad + k\sigma_k \int_0^t \mathrm{d}s \int_{[0,1]} Y_{s-}^{(k)}(\mathrm{d}x) \int_{\mathbb{N}_+} \int_{[0,1]} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}(z-1)f(x \vee \theta)) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\}\bar{\pi}^{(k)}(\mathrm{d}z, k\,\mathrm{d}\theta) \\
&\qquad + k\sigma_k b_k(k) \int_0^t \mathrm{d}s \int_{[0,1]} \{G(\langle Y_{s-}^{(k)}, f \rangle - k^{-1}f(x)) - G(\langle Y_{s-}^{(k)}, f \rangle)\}Y_{s-}^{(k)}(\mathrm{d}x) \\
&\qquad + k\sigma_k \int_0^t \mathrm{d}s \int_{[0,1]} Y_{s-}^{(k)}(\mathrm{d}x) \int_1^x \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}[f(\theta) - f(x)]) \\
&\qquad\qquad - G(\langle Y_{s-}^{(k)}, f \rangle)\}b_k(k\,\mathrm{d}\theta) + \text{local martingale}
\end{aligned}
$$

$$= G(\langle Y_0^{(k)}, f \rangle)$$

$$+ k\sigma_k \int_0^t \mathrm{d}s \int_{[0,1]} Y_{s-}^{(k)}(\mathrm{d}x) \int_{\mathbb{N}} \int_{[0,1]} \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}(z-1)f(x \vee \theta))$$

$$- G(\langle Y_{s-}^{(k)}, f \rangle)\} \bar{\pi}^{(k)}(\mathrm{d}z, k\,\mathrm{d}\theta)$$

$$+ k\sigma_k \int_0^t \mathrm{d}s \int_{[0,1]} Y_{s-}^{(k)}(\mathrm{d}x) \int_{\{0\}} \int_x^1 \varepsilon_k(s, x, \theta) \bar{\pi}^{(k)}(\mathrm{d}z, k\,\mathrm{d}\theta)$$

$$+ \text{local martingale}, \tag{5.8}$$

where

$$\varepsilon_k(s, \theta, x) = \{G(\langle Y_{s-}^{(k)}, f \rangle - k^{-1}f(x)) - G(\langle Y_{s-}^{(k)}, f \rangle - k^{-1}f(\theta))\}$$

$$- \{G(\langle Y_{s-}^{(k)}, f \rangle + k^{-1}[f(\theta) - f(x)]) - G(\langle Y_{s-}^{(k)}, f \rangle)\}.$$

It is easy to see that

$$k\sigma_k \int_{\{0\}} \int_x^1 \varepsilon_k(s, x, \theta) \bar{\pi}^{(k)}(\mathrm{d}z, k\,\mathrm{d}\theta)$$

tends to 0 uniformly as $k \to \infty$. Let $G(x) = \mathrm{e}^{-x}$. By letting $k \to \infty$ in (5.8) we obtain (2.2) for $f \in C^1[0, 1]$. A simple approximation shows that the martingale problem (2.2) actually holds for all $f \in C[0, 1]$. By the proof of Theorem 7.13 in Li (2011), we obtain the result completing the proof.

Let $\{0 \le a_1 < a_2 < \cdots < a_n = 1\}$ be an ordered set of constants. Denote by $\{Y_{t,a_i} : t \ge 0\}$ and $\{Y_{t,a_i}^{(k)} : t \ge 0\}$ the restriction of $\{Y_t : t \ge 0\}$ and $\{Y_t^{(k)} : t \ge 0\}$ to $[0, a_i]$, respectively. Let $Y_t(a_i) := Y_t[0, a_i]$ and $Y_t^{(k)}(a_i) := Y_t^{(k)}[0, a_i]$ for every $t \ge 0$, $i = 1, 2, \ldots, n$. By arguments similar to those in He and Ma (2014), we have the following results.

Theorem 5.2. *Suppose that Assumption 5.1 is satisfied and* $\sup_{k \ge 1} \sigma_k b_k(0) < \infty$. *If* $Y_0^{(k)}$ *converges weakly to* $Y_0 \in M[0, 1]$ *then* $\{(Y_{t,a_1}^{(k)}, \ldots, Y_{t,a_n}^{(k)}) : t \ge 0\}$ *converges in distribution to* $\{(Y_{t,a_1}, \ldots, Y_{t,a_n}) : t \ge 0\}$ *on* $D([0, \infty), M[0, a_1] \times \cdots \times M[0, a_n])$.

Corollary 5.1. *Suppose that Assumption 5.1 is satisfied and* $\sup_{k \ge 1} \sigma_k b_k(0) < \infty$. *If*

$$(Y_0^{(k)}(a_1), \ldots, Y_0^{(k)}(a_n))$$

converges to $(Y_0(a_1), \ldots, Y_0(a_n))$ *then it follows that* $\{(Y_t^{(k)}(a_1), \ldots, Y_t^{(k)}(a_n)) : t \ge 0\}$ *converges in distribution to* $\{(Y_t(a_1), \ldots, Y_t(a_n)) : t \ge 0\}$ *on* $D([0, \infty), \mathbb{R}_+^n)$.

Example 5.1. Suppose that ϕ is defined in (2.1). Let Θ_ϕ be the set of $\theta \ge 0$ such that

$$\int_1^\infty u\mathrm{e}^{\theta u} m(\mathrm{d}u) < \infty.$$

Then a particular choice of ϕ_θ is

$$\phi_\theta(\cdot) = \phi(\cdot - \theta) - \phi(-\theta), \qquad \theta \in \Theta_\phi.$$

Suppose that $[0, 1] \subset \Theta_\phi$. Since $\phi_1(\cdot)$ is a branching mechanism, Li (2011, p.93) implies that there is a sequence of generating functions $\{g_k^{(k)} : k \ge 1\}$ and some σ_k such that

$$k\sigma_k[g_k^{(k)}(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}] \to \phi(z-1) - \phi(-1) \quad \text{as } k \to \infty.$$

Hence, for $\theta \in [0, 1]$, define

$$g_{k\theta}^{(k)}(z) = 1 - e^{(1-\theta)/k} g_k^{(k)}(e^{-(1-\theta)/k}) + e^{(1-\theta)/k} g_k^{(k)}(e^{-(1-\theta)/k} z).$$

The reader can check that Assumption 5.1 holds with $\phi_\theta(\cdot) = \phi(\cdot - \theta) - \phi(-\theta)$. In fact, if $g_k^{(k)}$ corresponds to a probability measure $\{p_i^{(k)} : i \geq 0\}$ then, for each $\theta \in [0, 1]$, $g_{k\theta}^{(k)}$ is the generating function of the probability measure

$$p_i^{(k)}(\theta) = p_i^{(k)} e^{-(1-\theta)(i-1)/k}, \qquad i \geq 1,$$

and $p_0^{(k)}(\theta) = 1 - \sum_{i \geq 1} p_i^{(k)}(\theta)$.

Acknowledgements

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