Compositio Math. 143 (2007) 1127–1163
doi:10.1112/S0010437X07002710

The first Pontryagin class

Paul Bressler

Abstract

We give a natural obstruction theoretic interpretation to the first Pontryagin class in terms of Courant algebroids. As an application we calculate the class of the stack of algebras of chiral differential operators. In particular, we establish the existence and uniqueness of the chiral de Rham complex.

1. Introduction

1.1 The first Pontryagin class

The first Pontryagin class, for the purposes of the present paper, is a characteristic class associated to a pair consisting of a principal G-bundle, G a Lie group, over a manifold X and an invariant symmetric bilinear form $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ of G. For a G-bundle P on X the Pontryagin class, denoted by $\pi(P, \langle\ ,\ \rangle)$ takes values in $H^2(X; \Omega^2 \to \Omega^{3,cl})$.

Incarnations of the first Pontryagin class corresponding to particular choices of $(G, \langle\ ,\ \rangle)$ are quite familiar. For example, the class $2\,\mathrm{ch}_2$ is the Pontryagin class corresponding to $GL_n(\mathbb{C})$ and the canonical pairing on $\mathfrak{gl}_n$ given by the trace of the product of matrices.

More generally, the first Pontryagin class with values as above may be associated to a transitive Lie algebroid (see § A.1), say, $\mathcal{A}$, together with an invariant symmetric pairing $\langle\ ,\ \rangle$ on the kernel of the anchor map and will be denoted $\pi(\mathcal{A}, \langle\ ,\ \rangle)$. (The construction of characteristic classes in this general context and the calculation of the Čech–de Rham representative of the first Pontryagin class are recalled in Appendix A.) The Pontryagin class of a principal bundle is defined as the Pontryagin class of the Atiyah algebra of the bundle.

1.2 The first Pontryagin class as an obstruction

It turns out that the Pontryagin class $\pi(\mathcal{A}, \langle\ ,\ \rangle)$ appears naturally in the context of a certain classification problem which is canonically associated with the pair $(\mathcal{A}, \langle\ ,\ \rangle)$.

Just as the degree one cohomology classifies torsors (principal bundles), degree two cohomology classifies certain stacks [Bre94]. To each transitive Lie algebroid $\mathcal{A}$ on X and an invariant symmetric bilinear form $\langle\ ,\ \rangle$ on the kernel of the anchor map $\pi : \mathcal{A} \to \mathcal{T}_X$ we will associate the stack $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ with the corresponding class in $H^2(X; \Omega^2 \to \Omega^{3,cl})$ equal to $-\frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$. The latter equality is the content of Theorem 4.1.

The stack $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ associates to an open subset U of X the category (groupoid) $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}(U)$ of certain *Courant extensions* of $\mathcal{A}$. Courant extensions of Lie algebroids are, in particular, *Courant algebroids*. Consequently, a significant part of the paper is devoted to the definition, basic properties and classification of Courant algebroids, the main result (Theorem 4.1) being the identification of the class of $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ with $-\frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$.

Received 10 January 2006, accepted in final form 24 October 2006.
2000 Mathematics Subject Classification 14F10 (primary), 57N65 (secondary).
Keywords: Courant algebroid, vertex algebroid, gerbe, chiral differential operators.

Thus, $(\mathcal{A}, \langle\ ,\ \rangle)$ admits a (globally defined) Courant extension if and only if the Pontryagin class of $(\mathcal{A}, \langle\ ,\ \rangle)$ vanishes. This fact (without the terminology of the present paper) was originally discovered by Pavol Ševera.

1.3 Courant extensions of Atiyah algebras

The following is intended to convey in an informal fashion the differential geometric meaning of a Courant extension of the Atiyah algebra of a principal bundle.

Suppose that G is a reductive group with Lie algebra $\mathfrak{g}$. Let $\underline{G}$ denote the sheaf of groups represented by G. An invariant symmetric pairing $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which satisfies certain integrality conditions gives rise to a central extension, say $\widehat{\underline{G}}_{\langle\ ,\ \rangle}$, of $\underline{G}$ by (the sheaf) $\underline{K}_2$ so that there is a short exact sequence of sheaves of groups

$$1 \longrightarrow \underline{K}_2 \longrightarrow \widehat{\underline{G}}_{\langle\ ,\ \rangle} \longrightarrow \underline{G} \longrightarrow 1$$

which leads to the short exact sequence of tangent spaces at the identity

$$0 \longrightarrow \Omega^1 \longrightarrow \widehat{\underline{\mathfrak{g}}}_{\langle\ ,\ \rangle} \longrightarrow \underline{\mathfrak{g}} \longrightarrow 0$$

where $T_e\underline{K}_2 = \Omega^1$, $\widehat{\underline{\mathfrak{g}}}_{\langle\ ,\ \rangle}$ denotes $T_e\widehat{\underline{G}}_{\langle\ ,\ \rangle}$ and $\underline{\mathfrak{g}} = T_e\underline{G}$ is the sheaf of Lie algebras represented by $\mathfrak{g}$. As was pointed out by Bloch [Blo81] (at least in the case of the Steinberg group), the usual construction does not yield a Lie bracket on $\widehat{\underline{\mathfrak{g}}}_{\langle\ ,\ \rangle}$. Thus, $\widehat{\underline{\mathfrak{g}}}_{\langle\ ,\ \rangle}$ is not a sheaf of Lie algebras in a way compatible with the projection to $\underline{\mathfrak{g}}$. It *is* however a Courant algebroid (with the trivial anchor map) and, in particular, a sheaf of Leibniz algebras (the Leibniz bracket, however, is *not* $\mathcal{O}$-linear). Examples of this kind are studied in § 3.2.

Now suppose that P is a $\underline{G}$-torsor (i.e. a principal G-bundle) on X. Lifts of P to $\widehat{\underline{G}}_{\langle\ ,\ \rangle}$ (i.e. pairs $(\widehat{P}, \phi)$ comprising a $\widehat{\underline{G}}_{\langle\ ,\ \rangle}$-torsor $\widehat{P}$ and a $\widehat{\underline{G}}_{\langle\ ,\ \rangle}$-equivariant map $\phi : \widehat{P} \to P$) exist locally on X and form a $\underline{K}_2$-gerbe whose class in $H^2(X; \underline{K}_2)$ is the class of the central extension.

Given a lift $\widehat{P}$ as above one might ask what a connection on $\widehat{P}$ might be, or, better, what sort of structure would serve as the 'Atiyah algebra' in this context. Denoting this hypothetical, for the moment, object by $\widehat{\mathcal{A}}_P$ we note that it may be expected, at the very least, to fit into the commutative diagram

$$\begin{array}{ccccc}
\Omega^1 & \longrightarrow & \widehat{\underline{\mathfrak{g}}}_{\langle\ ,\ \rangle} & \longrightarrow & \underline{\mathfrak{g}} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\
\Omega^1 & \longrightarrow & \widehat{\mathcal{A}}_P & \longrightarrow & \mathcal{A}_P \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{T}_X & \xrightarrow{\mathrm{id}} & \mathcal{T}_X
\end{array}$$

(where $\mathcal{A}_P$ denotes the Atiyah algebra of P) and carry a (Leibniz) bracket compatible with all of the maps. The above picture is encapsulated in the notion of a *Courant extension* of the Lie algebroid $\mathcal{A}_P$.

Courant extensions of $\mathcal{A}_P$ which fit into the above diagram exist locally on X (due to local triviality of P) and form the gerbe which we denoted by $\mathcal{CE}xt_{\mathcal{O}_X}(\mathcal{A}_P)_{\langle\ ,\ \rangle}$ above.

1.4 Vertex algebroids

The rest of the paper is devoted to an application of the thus far developed theory of Courant algebroids and classification thereof to the question of classification of *exact vertex algebroids* or,

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

equivalently, sheaves of chiral differential operators. Vertex algebroids were defined and classified in terms of local data in [GMS04] (and other papers by the same authors) and, later, in the language of chiral algebras, in [BD04].

The approach to the classification of vertex algebroids carried out in the present paper is suggested by some results contained in [BD04].[1] It turns out that the latter problem reduces (in the sense of equivalence of stacks, Proposition 5.2) to the classification problem for Courant extensions of $\mathcal{A}_{\Omega^1_X})_{\langle\ ,\ \rangle}$ (the Atiyah algebra of the cotangent bundle with the symmetric pairing $\langle\ ,\ \rangle$ on the Lie algebra $\underline{\mathrm{End}}_{\mathcal{O}_X}(\Omega^1_X)$ given by the trace of the product of endomorphisms). The result (Theorem 5.1) is that the stack $\mathcal{EVA}_{\mathcal{O}_X}$ of vertex algebroids on X gives rise to a class (obstruction to existence of a globally defined vertex algebroid) in $H^2(X;\Omega^2 \to \Omega^{3,cl})$, and that class is equal to $\mathrm{ch}_2(\Omega^1_X)$.

This reduction is achieved with the aid of (the degree zero component of) the differential graded vertex algebroid over the de Rham complex of X (which gives rise to the chiral de Rham complex of [MSV99]). The existence and uniqueness of this object was demonstrated in [MSV99] and [GMS03] (and [BD04] in the language of chiral algebras). We give a 'coordinate-free' proof of this result. To this end we apply the results of the preceding sections to the differential graded manifold $X^\sharp$ whose underlying space is X and the structure sheaf is the de Rham complex of X. We show (Proposition 6.1) that every (differential graded) exact Courant algebroid on $X^\sharp$ is canonically trivialized. This implies (Corollary 6.2) that an exact vertex algebroid on $X^\sharp$ exists and is unique up to a unique isomorphism. This is a differential graded object whose degree zero constituent is a vertex extension of the Atiyah algebra of the cotangent bundle.

2. Courant algebroids

2.1 Leibniz algebras

Suppose that k is a commutative ring.

Definition 2.1. A *Leibniz k-algebra* is a k-module $\mathfrak{g}$ equipped with a bilinear operation

$$[\ ,\ ] : \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$$

(the Leibniz bracket) which satisfies the Jacobi type identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

A morphism of Leibniz k-algebras is a k-linear map which commutes with the respective Leibniz brackets.

Example 2.1. Suppose that $\mathfrak{g}$ is a Lie algebra, $\widehat{\mathfrak{g}}$ is a $\mathfrak{g}$-module, and $\pi : \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a morphism of $\mathfrak{g}$-modules. The bilinear operation on $\widehat{\mathfrak{g}}$ defined by

$$[a,b] = \pi(a)(b)$$

for $a, b \in \widehat{\mathfrak{g}}$ satisfies the Jacobi identity and thus defines a structure of a Leibniz algebra on $\widehat{\mathfrak{g}}$.

2.2 Courant algebroids

Courant algebroids, as defined below, appear as quasi-classical limits of the vertex algebroids (see Definitions 5.3 and 5.4 for discussion of quantization). The format of the definition given below follows that of the corresponding non-commutative notion ('vertex algebroid') which, in turn, is distilled from the structure of a vertex operator algebra in § 5.3.

[1]The author is grateful to Alexander Beilinson for sending him an early preprint of *Chiral algebras*.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Definition 2.2. A *Courant $\mathcal{O}_X$-algebroid* is an $\mathcal{O}_X$-module $\mathcal{Q}$ equipped with the following:

(i) a structure of a Leibniz $\mathbb{C}$-algebra

$$[\ ,\ ] : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \to \mathcal{Q};$$

(ii) an $\mathcal{O}_X$-linear map of Leibniz algebras (the anchor map)

$$\pi : \mathcal{Q} \to \mathcal{T}_X;$$

(iii) a symmetric $\mathcal{O}_X$-bilinear pairing

$$\langle\ ,\ \rangle : \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{Q} \to \mathcal{O}_X;$$

(iv) a derivation

$$\partial : \mathcal{O}_X \to \mathcal{Q}.$$

These satisfy

$$\pi \circ \partial = 0, \tag{2.2.1}$$

$$[q_1, fq_2] = f[q_1, q_2] + \pi(q_1)(f)q_2, \tag{2.2.2}$$

$$\langle [q, q_1], q_2\rangle + \langle q_1, [q, q_2]\rangle = \pi(q)(\langle q_1, q_2\rangle), \tag{2.2.3}$$

$$[q, \partial(f)] = \partial(\pi(q)(f)), \tag{2.2.4}$$

$$\langle q, \partial(f)\rangle = \pi(q)(f), \tag{2.2.5}$$

$$[q_1, q_2] + [q_2, q_1] = \partial(\langle q_1, q_2\rangle), \tag{2.2.6}$$

for $f \in \mathcal{O}_X$ and $q, q_1, q_2 \in \mathcal{Q}$.

A morphism of Courant $\mathcal{O}_X$-algebroids is an $\mathcal{O}_X$-linear map of Leibnitz algebras which commutes with the respective anchor maps and derivations and preserves the respective pairings.

Remark 2.1. The definition of *Courant algebroid* given below reduces to Definition 2.1 of [LWX97] under the additional hypotheses of that work (that $\mathcal{Q}$ is locally free of finite rank and the symmetric pairing is non-degenerate).

Courant algebroids are to *vertex Poisson algebras* (coisson algebras in the terminology of [BD04]) what vertex algebroids are to vertex algebras in the sense of the analysis carried out at the outset of § 5.

Remark 2.2. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ let $\mathcal{F}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

The derivation $\partial : \mathcal{O}_X \to \mathcal{Q}$ factors canonically as $\mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \mathcal{Q}$, the second map given by $fdg \mapsto f\partial g$ for $f, g \in \mathcal{O}_X$. The symmetric pairing $\langle\ ,\ \rangle$ on $\mathcal{Q}$ induces the map $\widetilde{\langle\ ,\ \rangle} : \mathcal{Q} \to \mathcal{Q}^\vee$. Formula (2.2.5) says that the transpose of the anchor map, $\pi^\vee : \Omega^1_X \cong \mathcal{T}_X{}^\vee \to Q^\vee$ factors as

$$\Omega^1_X \xrightarrow{fdg \mapsto f\partial g} \mathcal{Q} \xrightarrow{\widetilde{\langle\ ,\ \rangle}} Q^\vee,$$

i.e. the map

$$\Omega^1_X \xrightarrow{fdg \mapsto f\partial g} \mathcal{Q}$$

is *a transpose* of the anchor map π with respect to the symmetric pairing $\langle\ ,\ \rangle$ on $\mathcal{Q}$.

Notation 2.1. We will denote the map $\Omega^1_X \xrightarrow{fdg \mapsto f\partial g} \mathcal{Q}$ by $\pi^\dagger$, where π is the anchor map; the justification of the notation (transpose) is provided in Remark 2.2.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

2.3 Twisting by 3-forms

Suppose that $\mathcal{Q}$ is a Courant algebroid with Leibniz bracket denoted $[\ ,\ ]$, and H is a 3-form on X. Let $[\ ,\ ]_H$ denote the bilinear operation on $\mathcal{Q}$ defined by the formula

$$[q_1, q_2]_H = [q_1, q_2] + \pi^\dagger(\iota_{\pi(q_2)}\iota_{\pi(q_1)}H). \tag{2.3.1}$$

Recall that the Jacobiator $J(\{\ ,\ \})$ of a binary operation $\{\ ,\ \}$ is defined by the formula

$$J(\{\ ,\ \})(a, b, c) = \{a, \{b, c\}\} - \{\{a, b\}, c\} - \{b, \{a, c\}\}. \tag{2.3.2}$$

LEMMA 2.1. *The map $J([\ ,\ ]_H) : \mathcal{Q}^{\otimes 3} \to \mathcal{Q}$ factors into the composition*

$$\mathcal{Q}^{\otimes 3} \xrightarrow{\pi^{\otimes 3}} \mathcal{T}_X^{\otimes 3} \xrightarrow{\widetilde{dH}} \Omega_X^1 \xrightarrow{\pi^\dagger} \mathcal{Q},$$

where $\widetilde{dH}$ is the image of dH under the canonical map $\widetilde{(\bullet)} : \Omega_X^4 \to \underline{\mathrm{Hom}}(\mathcal{T}_X^{\otimes 3}, \Omega_X^1)$.

Proof. This is shown by a direct calculation. □

Notation 2.2. For a Courant algebroid $\mathcal{Q}$ and a closed 3-form H on X we denote by $\mathcal{Q}_H$ the Courant algebroid with the underlying $\mathcal{O}_X$-module $\mathcal{Q}$ equipped with the same symmetric pairing and derivation, and with the Leibniz bracket $[\ ,\ ]_H$ given by (2.3.1).

We refer to $\mathcal{Q}_H$ as *the H-twist of $\mathcal{Q}$*.

It is clear that twisting by a 3-form is a functorial operation: a morphism of Courant algebroids is also a morphism of their respective twists (by the same form).

2.4 The associated Lie algebroid

Suppose that $\mathcal{Q}$ is a Courant $\mathcal{O}_X$-algebroid.

Notation 2.3. Let $\Omega_{\mathcal{Q}}$ denote the $\mathcal{O}_X$-submodule of $\mathcal{Q}$ generated by the image of the derivation ∂. Let $\overline{\mathcal{Q}} = \mathcal{Q}/\Omega_{\mathcal{Q}}$.

For $q \in \mathcal{Q}$, $f, g \in \mathcal{O}_X$, we have

$$\begin{aligned}[q, f\partial(g)] &= f[q, \partial(g)] + \pi(q)(f)\partial(g) \\ &= f\partial(\pi(q)(g)) + \pi(q)(f)\partial(g),\end{aligned}$$

which shows that $[\mathcal{Q}, \Omega_{\mathcal{Q}}] \subseteq \Omega_{\mathcal{Q}}$. Therefore, the Leibniz bracket on $\mathcal{Q}$ descends to the bilinear operation

$$[\ ,\ ] : \overline{\mathcal{Q}} \otimes_{\mathbb{C}} \overline{\mathcal{Q}} \to \overline{\mathcal{Q}}. \tag{2.4.1}$$

Since π is $\mathcal{O}_X$-linear and $\pi \circ \partial = 0$, π vanishes on $\Omega_{\mathcal{Q}}$, hence factors through the map

$$\pi : \overline{\mathcal{Q}} \to \mathcal{T}_X. \tag{2.4.2}$$

LEMMA 2.2. *The bracket (2.4.1) and the anchor (2.4.2) determine a structure of a Lie $\mathcal{O}_X$-algebroid on $\overline{\mathcal{Q}}$.*

Proof. According to (2.2.6) the symmetrization of the Leibniz bracket on $\mathcal{Q}$ takes values in $\Omega_{\mathcal{Q}}$. Therefore, the induced bracket is skew-symmetric. The Leibniz rule and the Jacobi identity for $\overline{\mathcal{Q}}$ follow from those for $\mathcal{Q}$. □

In what follows we refer to the Lie algebroid $\overline{\mathcal{Q}}$ as *the Lie algebroid associated to the Courant algebroid $\mathcal{Q}$*.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Definition 2.3. A *Courant extension* of a Lie algebroid $\mathcal{A}$ is a Courant algebroid $\mathcal{Q}$ together with an isomorphism $\overline{\mathcal{Q}} = \mathcal{A}$ of Lie algebroids.

A morphism $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ of Courant extensions of $\mathcal{A}$ is a morphism of Courant algebroids which is compatible with the identifications $\overline{\mathcal{Q}_i} = \mathcal{A}$.

Suppose that $\mathcal{A}$ is a Lie algebroid. For each open subset U of X there is category of Courant extensions of $\mathcal{A}|_U$. Together with the obvious restriction functors these form a stack.

Notation 2.4. We denote the stack of Courant extensions of $\mathcal{A}$ by $\mathcal{CEXT}_{\mathcal{O}_X}(\mathcal{A})$.

2.5 From Leibniz to Lie

For a Lie algebroid $\mathcal{A}$ (respectively, Courant algebroid $\mathcal{Q}$) we will denote by $\mathfrak{g}(\mathcal{A})$ (respectively, $\mathfrak{g}(\mathcal{Q})$) the kernel of the anchor map of $\mathcal{A}$ (respectively, $\mathcal{Q}$). Note that $\mathfrak{g}(\mathcal{Q})$ is, naturally, a Courant algebroid with the trivial anchor map and $\overline{\mathfrak{g}(\mathcal{Q})} = \mathfrak{g}(\overline{\mathcal{Q}})$. Since $\langle \mathfrak{g}(\mathcal{Q}), \Omega_{\mathcal{Q}} \rangle = 0$, the pairing $\langle\ ,\ \rangle$ on $\mathcal{Q}$ induces the pairings

$$\langle\ ,\ \rangle : \mathfrak{g}(\mathcal{Q}) \otimes_{\mathcal{O}_X} \overline{\mathcal{Q}} \to \mathcal{O}_X, \tag{2.5.1}$$

$$\langle\ ,\ \rangle : \mathfrak{g}(\overline{\mathcal{Q}}) \otimes_{\mathcal{O}_X} \mathfrak{g}(\overline{\mathcal{Q}}) \to \mathcal{O}_X. \tag{2.5.2}$$

The (restriction of the left) adjoint action $\mathcal{Q} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathfrak{g}(\mathcal{Q}))$ is a morphism of Leibniz algebras (this is equivalent to the Jacobi identity) which annihilates $\Omega_{\mathcal{Q}}$, hence factors through the morphism of Lie algebras

$$\overline{\mathcal{Q}} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathfrak{g}(\mathcal{Q})) \tag{2.5.3}$$

or, equivalently, induces a canonical structure of a Lie $\overline{\mathcal{Q}}$-module on $\mathfrak{g}(\mathcal{Q})$.

2.6 Transitive Courant algebroids

Definition 2.4. A Courant $\mathcal{O}_X$-algebroid is called *transitive* if the anchor is surjective.

Remark 2.3. A Courant $\mathcal{O}_X$-algebroid is transitive if and only if the associated Lie algebroid is.

Lemma 2.3. *Suppose that $\mathcal{Q}$ is a transitive Courant algebroid. Then, the sequence*

$$0 \to \Omega^1_X \to \mathcal{Q} \to \overline{\mathcal{Q}} \to 0, \tag{2.6.1}$$

where the first map is induced by the derivation ∂ and the second map is the canonical projection, is exact. Moreover, the embedding $\Omega^1_X \to \mathcal{Q}$ is isotropic with respect to the symmetric pairing.

Proof. It suffices to note that the composition $\Omega^1_X \to \mathcal{Q} \to \overline{\mathcal{Q}}$ is equal to zero by (2.2.1) and check that the map $\Omega^1_X \to \mathcal{Q}$ induced by ∂ is a monomorphism. Since, for $q \in \mathcal{Q}$, $f, g \in \mathcal{O}_X$, we have $\langle q, f\partial g\rangle = \iota_{\pi(q)} f dg$, it follows that the map $\Omega^1_X \to \mathcal{Q}$ is adjoint to the anchor map π. The surjectivity of the latter implies that $\Omega^1_X \to \mathcal{Q}$ is injective. □

In what follows, when dealing with a transitive Courant algebroid $\mathcal{Q}$ we will, in view of Lemma 2.3, regard Ω^1_X as a subsheaf of $\mathcal{Q}$, i.e. identify the former with its image under the embedding into the latter. This should not lead to confusion since morphisms of transitive Courant algebroids will, under the above identification, induce the identity map on Ω^1_X.

Remark 2.4. The exact sequence (2.6.1) is functorial. Thus, a morphism of Courant extensions of a transitive Lie algebroid $\mathcal{A}$ induces a morphism of associated extensions of $\mathcal{A}$ by Ω^1_X. A morphism of extensions is necessarily an isomorphism on the respective middle terms, and it is clear that the inverse isomorphism is a morphism of Courant extensions of $\mathcal{A}$. Hence, the category of Courant extensions of a transitive Lie algebroid is, in fact, a groupoid.

Definition 2.5. A *connection* on a transitive Courant $\mathcal{O}_X$-algebroid $\mathcal{Q}$ is an $\mathcal{O}_X$-linear isotropic (with respect to the symmetric pairing on $\mathcal{Q}$) section of the anchor map $\mathcal{Q} \to \mathcal{T}_X$.

A *flat connection* on a transitive Courant $\mathcal{O}_X$-algebroid $\mathcal{Q}$ is an $\mathcal{O}_X$-linear section of the anchor map which is a morphism of Leibniz algebras.

Lemma 2.4. *A flat connection is a connection.*

Proof. Suppose that ∇ is a flat connection on $\mathcal{Q}$. Then, $\mathrm{Im}(\nabla)$ is a *Lie* subalgebra in $\mathcal{Q}$, i.e. the restriction of the bracket to $\mathrm{Im}(\nabla)$ is skew-symmetric. It follows from (2.2.6) that the ($\mathbb{C}$-linear) composition $\partial \circ \langle\ ,\ \rangle \circ (\nabla \otimes \nabla) : \mathcal{T}_X \otimes_{\mathbb{C}} \mathcal{T}_X \to \mathcal{Q}$ is equal to zero. Therefore, the ($\mathcal{O}_X$-linear) composition $\langle\ ,\ \rangle \circ (\nabla \otimes \nabla) : \mathcal{T}_X \otimes \mathcal{T}_X \to \mathcal{O}_X$ factors through the inclusion $\mathbb{C} \to \mathcal{O}_X$, hence is trivial.

Thus, $\langle\ ,\ \rangle$ vanishes on $\mathrm{Im}(\nabla)$, i.e. the latter is isotropic, which means that ∇ is a connection. □

Lemma 2.5. *A transitive Courant algebroid which is a locally free $\mathcal{O}_X$-module admits a connection locally on X.*

Proof. Suppose that $\mathcal{Q}$ is a Courant algebroid as above. Let $s : \mathcal{T}_X \to \mathcal{Q}$ denote a locally defined section of the anchor map (such exist in a neighborhood of every point of X). Let $\phi : \mathcal{T}_X \to \Omega^1_X$ be defined by

$$\iota_\eta \phi(\xi) = -\tfrac{1}{2}\langle s(\xi), s(\eta)\rangle.$$

Then, as is easy to check, $s + \phi$ is a connection. □

Notation 2.5. We denote by $\mathcal{C}(\mathcal{Q})$ (respectively, $\mathcal{C}^\flat(\mathcal{Q})$) the sheaf of (locally defined) connections (respectively, flat connections) on $\mathcal{Q}$.

Definition 2.6. For a connection ∇ on $\mathcal{Q}$ the *curvature of* ∇, denoted $c(\nabla)$, is defined by the formula

$$c(\nabla)(\xi, \eta) = [\nabla(\xi), \nabla(\eta)] - \nabla([\xi, \eta]),$$

for $\xi, \eta \in \mathcal{T}_X$.

Remark 2.5. The curvature of connections on Courant algebroids shares basic properties with the corresponding notion for Lie algebroids (§ A.1.2).

(i) It is clear that $c(\nabla)$ takes values in $\mathfrak{g}(\mathcal{Q})$. The usual calculations show that $c(\nabla)$ is $\mathcal{O}_X$-bilinear. Since it is clearly alternating, it determines a map $c(\nabla) : \wedge^2 \mathcal{T}_X \to \mathfrak{g}(\mathcal{Q})$ or, equivalently, a section $c(\nabla) \in \Gamma(X; \Omega^2_X \otimes \mathfrak{g}(\mathcal{Q}))$.

(ii) A connection is flat if and only if its curvature is equal to zero.

(iii) Suppose that ∇ is a connection on $\mathcal{Q}$. Let $\overline{\nabla}$ denote the composition of ∇ with the projection $\overline{(\bullet)} : \mathcal{Q} \to \overline{\mathcal{Q}}$. Then, $\overline{\nabla}$ is a connection on the Lie algebroid $\overline{\mathcal{Q}}$ and $c(\overline{\nabla}) = \overline{c(\nabla)}$.

3. Courant extensions

3.1 Courant extensions of transitive Lie algebroids

From now on we assume that $\mathcal{A}$ is a *transitive* Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$. By Remark 2.4, $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})$ is a stack in groupoids.

Suppose that $\widehat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. Let

$$\exp : \Omega^2_X \to \underline{\mathrm{Aut}}_{\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\mathcal{A}, \Omega^1_X)}(\widehat{\mathcal{A}}) \tag{3.1.1}$$

denote the map defined by $\exp(B)(a) = a + \iota_{\pi(a)} B$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Lemma 3.1. *The map (3.1.1) establishes an isomorphism*

$$\exp : \Omega_X^{2,cl} \to \underline{\mathrm{Aut}}_{\mathcal{CEXT}_{\mathcal{O}_X}(\mathcal{A})}(\widehat{\mathcal{A}}).$$

Proof. Suppose that $\phi \in \underline{\mathrm{Aut}}_{\mathcal{CEXT}_{\mathcal{O}_X}(\mathcal{A})}(\widehat{\mathcal{A}})$. Then, ϕ restricts to the identity on Ω^1 and induces the identity on $\mathcal{A}$. Therefore, $\phi(a) = a + \phi'(\pi(a))$, where $\phi' : \mathcal{T}_X \to \Omega_X^1$. Since, for $a_1, a_2 \in \widehat{\mathcal{A}}$, we have

$$\begin{aligned}\langle \phi(a_1), \phi(a_2)\rangle &= \langle a_1, a_2\rangle + \langle \phi'(\pi(a_1)), a_2\rangle + \langle a_1, \phi'(\pi(a_2))\rangle \\ &= \langle a_1, a_2\rangle + \iota_{\pi(a_2)}\phi'(\pi(a_1)) + \iota_{\pi(a_1)}\phi'(\pi(a_2)),\end{aligned}$$

it follows that ϕ preserves the symmetric pairing if and only if ϕ', viewed as a section of $\Omega_X^1 \otimes_{\mathcal{O}_X} \Omega_X^1$, is alternating, i.e. $\phi' = B \in \Omega_X^2$ and $\phi = \exp(B)$.

The formula[2]

$$\begin{aligned}[\phi(a_1), \phi(a_2)] &= [a_1 + \iota_{\pi(a_1)}B, a_2 + \iota_{\pi(a_2)}B] \\ &= [a_1, a_2] + L_{\pi(a_1)}\iota_{\pi(a_2)}B - L_{\pi(a_2)}\iota_{\pi(a_1)}B + d\iota_{\pi(a_2)}\iota_{\pi(a_1)}B \\ &= [a_1, a_2] + \iota_{[\pi(a_1),\pi(a_2)]}B + \iota_{\pi(a_2)}\iota_{\pi(a_1)}dB \\ &= \phi([a_1, a_2]) + \iota_{\pi(a_2)}\iota_{\pi(a_1)}dB \end{aligned} \tag{3.1.2}$$

shows that $\phi = \exp(B)$ is a morphism of Leibniz algebras if and only if B is closed. □

Remark 3.1. The calculation (3.1.2) (combined with the equality $\pi(a_i) = \pi(\phi(a_i))$) says that

$$\phi([a_1, a_2]) = [\phi(a_1), \phi(a_2)] + \iota_{\pi(\phi(a_2))}\iota_{\pi(\phi(a_1))}(-dB) = [\phi(a_1), \phi(a_2)]_{-dB}$$

(the latter operation being the $-dB$-twisted bracket on $\widehat{\mathcal{A}}$ as defined in § 2.3, formula (2.3.1)). In other words, for $B \in \Omega_X^2$ (not necessarily closed), the map $\exp(B)$ is a morphism of Courant extensions $\widehat{\mathcal{A}}_{dB} \to \widehat{\mathcal{A}}$ (the former being the dB-twist of $\widehat{\mathcal{A}}$).

3.2 From Lie to Leibniz

For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules let $\mathcal{F}^\vee$ denote $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Suppose that $\widehat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. The pairings (2.5.1) and (2.5.2) yield, respectively, the maps $\mathfrak{g}(\widehat{\mathcal{A}}) \to \mathcal{A}^\vee$ and $\mathfrak{g} \to \mathfrak{g}^\vee$. Together with the projection $\mathfrak{g}(\widehat{\mathcal{A}}) \to \mathfrak{g}$ and the map $\mathcal{A}^\vee \to \mathfrak{g}^\vee$ adjoint to the inclusion $\mathfrak{g} \to \mathcal{A}$ they fit into the following diagram.

$$\begin{array}{ccc} \mathfrak{g}(\widehat{\mathcal{A}}) & \longrightarrow & \mathcal{A}^\vee \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}^\vee \end{array} \tag{3.2.1}$$

Lemma 3.2. *The diagram (3.2.1) is Cartesian.*

Proof. The diagram (3.2.1) is commutative since (2.5.2) is induced from the restriction of (2.5.1) to $\mathfrak{g}(\widehat{\mathcal{A}}) \otimes_{\mathcal{O}_X} \mathfrak{g}$. In fact, it extends to the morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \mathfrak{g}(\widehat{\mathcal{A}}) & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_X^\vee & \longrightarrow & \mathcal{A}^\vee & \longrightarrow & \mathfrak{g}^\vee & \longrightarrow & 0 \end{array}$$

[2]Computed using the identity $d\iota_\eta\iota_\xi = (L_\eta - \iota_\eta d)\iota_\xi = L_\eta\iota_\xi - \iota_\eta(L_\xi - \iota_\xi d) = L_\eta\iota_\xi - \iota_\eta L_\xi + \iota_\eta\iota_\xi d = L_\eta\iota_\xi - L_\xi\iota_\eta + \iota_{[\xi,\eta]} + \iota_\eta\iota_\xi d$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

induced by the pairing. In particular, the map $\Omega^1_X \to \mathcal{T}_X^\vee$ is the canonical isomorphism (by (2.2.5)) and the claim follows. □

Corollary 3.1. *The map* $\mathfrak{g}(\widehat{\mathcal{A}})$ *is canonically isomorphic to* $\mathcal{A}^\vee \times_{\mathfrak{g}(\mathcal{A})^\vee} \mathfrak{g}(\mathcal{A})$.

Since $\mathcal{A}$ is transitive there is an exact sequence

$$0 \to \mathfrak{g} \xrightarrow{i} \mathcal{A} \to \mathcal{T}_X \to 0.$$

Suppose in addition that $\mathfrak{g}$ is equipped with a symmetric $\mathcal{O}_X$-bilinear pairing

$$\langle\ ,\ \rangle : \mathfrak{g} \otimes_{\mathcal{O}_X} \mathfrak{g} \to \mathcal{O}_X$$

which is invariant under the adjoint action of $\mathcal{A}$, i.e. for $a \in \mathcal{A}$ and $b, c \in \mathfrak{g}$

$$\pi(a)(\langle b, c\rangle) = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$$

holds.

The map $i : \mathfrak{g} \to \mathcal{A}$ and the pairing on $\mathfrak{g}$ give rise to the maps

$$\mathcal{A}^\vee \xrightarrow{i^\vee} \mathfrak{g}^\vee \xleftarrow{\langle\ ,\ \rangle} \mathfrak{g}.$$

Let $\widehat{\mathfrak{g}} = \mathcal{A}^\vee \times_{\mathfrak{g}^\vee} \mathfrak{g}$ and let $\mathrm{pr} : \widehat{\mathfrak{g}} \to \mathfrak{g}$ denote the canonical projection. A section of $\widehat{\mathfrak{g}}$ is a pair $(a^\vee, b)$, where $a^\vee \in \mathcal{A}^\vee$ and $b \in \mathfrak{g}$, which satisfies $i^\vee(a^\vee)(c) = \langle b, c\rangle$ for $c \in \mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ acts on $\mathcal{A}$ (by the restriction of the adjoint action) by $\mathcal{O}_X$-linear endomorphisms and the map $i : \mathfrak{g} \to \mathcal{A}$ is a map of $\mathfrak{g}$-modules. Therefore, $\mathcal{A}^\vee$ and $\mathfrak{g}^\vee$ are $\mathfrak{g}$-modules[3] in a natural way and the map $i^\vee$ is a morphism of such. Hence, $\widehat{\mathfrak{g}}$ is a $\mathfrak{g}$-module in a natural way and the map pr is a morphism of $\mathfrak{g}$-modules.

As a consequence, $\widehat{\mathfrak{g}}$ acquires the canonical structure of a Leibniz algebra with the Leibniz bracket $[\widehat{a}, \widehat{b}]$ of two sections $\widehat{a}, \widehat{b} \in \widehat{\mathfrak{g}}$ given by the formula $[\widehat{a}, \widehat{b}] = \mathrm{pr}(\widehat{a})(\widehat{b})$. Explicitly, for $a_1^\vee, a_2^\vee \in \mathcal{A}^\vee$, $b_1, b_2 \in \mathfrak{g}$, we have

$$[(a_1^\vee, b_1), (a_2^\vee, b_2)] = (b_1(a_2^\vee), [b_1, b_2]) = (\langle [\bullet, b_1], b_2\rangle, [b_1, b_2]).$$

We define a symmetric $\mathcal{O}_X$-bilinear pairing

$$\langle\ ,\ \rangle : \widehat{\mathfrak{g}} \otimes_{\mathcal{O}_X} \widehat{\mathfrak{g}} \to \mathcal{O}_X$$

as the composition of $\mathrm{pr} \otimes \mathrm{pr}$ with the pairing on $\mathfrak{g}$.

The inclusion $\pi^\vee : \Omega^1_X \to \mathcal{A}^\vee$ gives rise to the derivation $\partial : \mathcal{O}_X \to \widehat{\mathfrak{g}}$.

Lemma 3.3. *The Leibniz bracket, the symmetric pairing and the derivation defined above endow* $\widehat{\mathfrak{g}}$ *with the structure of a Courant extension of* $\mathfrak{g}$ *(in particular, a Courant* $\mathcal{O}_X$*-algebroid with the trivial anchor map).*

The isomorphism of Corollary 3.1 is an isomorphism of Courant extensions of $\mathfrak{g}(\mathcal{A})$.

Proof. This is left to the reader. □

Starting with a transitive Lie algebroid $\mathcal{A}$ and an $\mathcal{A}$-invariant symmetric pairing $\langle\ ,\ \rangle$ on $\mathfrak{g} = \mathfrak{g}(\mathcal{A})$ we have constructed a Courant extension $\widehat{\mathfrak{g}}$ with the property that for any Courant extension $\widehat{\mathcal{A}}$ which induces the pairing $\langle\ ,\ \rangle$ on $\mathfrak{g}$ there is a canonical isomorphism (Corollary 3.1) $\widehat{\mathfrak{g}} \cong \mathfrak{g}(\widehat{\mathcal{A}})$.

In what follows we will suppress the dependence of the extension $\widehat{\mathfrak{g}}$ on the pairing $\langle\ ,\ \rangle$ whenever the pairing is fixed by the context. In this case we will use the canonical isomorphism of Corollary 3.1 to identify $\widehat{\mathfrak{g}}$ and $\mathfrak{g}(\widehat{\mathcal{A}})$.

[3] The action of $b \in \mathfrak{g}$ on $a^\vee \in \mathcal{A}^\vee$ is determined by $b(a^\vee)(c) = -a^\vee([b, c]) = a^\vee([c, b])$ for $c \in \mathcal{A}$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

3.3 Leibniz extensions from connections

Suppose that ∇ is a connection on $\mathcal{A}$. Then ∇ determines:

(i) the isomorphism $\mathfrak{g} \oplus \mathcal{T}_X \cong \mathcal{A}$ by $(a, \xi) \mapsto i(a) + \nabla(\xi)$, where $a \in \mathfrak{g}$ and $\xi \in \mathcal{T}_X$;

(ii) the isomorphism $\phi_\nabla : \widehat{\mathfrak{g}} \to \Omega^1_X \oplus \mathfrak{g}$ by $(a^\vee, b) \mapsto (\nabla^\vee(a^\vee), b)$, where $a^\vee \in \mathcal{A}^\vee$, $b \in \mathfrak{g}$, $i^\vee(a^\vee) = \langle b, \bullet \rangle$, $\nabla^\vee : \mathcal{A}^\vee \to \Omega^1_X$ is the transpose of ∇ and $i^\vee : \mathcal{A}^\vee \to \mathfrak{g}^\vee$ is the transpose of i.

Let $[\ ,\ ]_\nabla$ denote the Leibniz bracket on $\Omega^1_X \oplus \mathfrak{g}$ induced by ϕ_∇. Let $c_\nabla(\ ,\ ) : \mathfrak{g} \otimes \mathfrak{g} \to \Omega^1_X$ be (the Ω^1_X-valued Leibniz cocycle) determined by $\iota_\xi c_\nabla(a, b) = \langle [\nabla(\xi), a], b \rangle$, where $\xi \in \mathcal{T}_X$, $a, b \in \mathfrak{g}$, and the bracket is computed in $\mathcal{A}$.

Lemma 3.4.

(i) *The Leibniz bracket* $[\ ,\ ]_\nabla$ *is the extension of the Lie bracket on* $\mathfrak{g}$ *by the* Ω^1_X*-valued (Leibniz) cocycle* $c_\nabla(\ ,\ )$.

(ii) *Suppose that* $A \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$. *The automorphism of* $\Omega^1_X \oplus \mathfrak{g}$ *defined by*

$$(\alpha, a) \mapsto (\alpha + \langle A, a \rangle, a)$$

is the isomorphism of Courant algebroids $(\Omega^1_X \oplus \mathfrak{g}, [\ ,\ ]_\nabla) \to (\Omega^1_X \oplus \mathfrak{g}, [\ ,\ ]_{\nabla + A})$ *which corresponds to the identity map on* $\widehat{\mathfrak{g}}$ *under the identifications* ϕ_∇ *and* $\phi_{\nabla + A}$.

3.4 Courant extensions with connection

Suppose that $\mathcal{A}$ is a transitive Lie algebroid locally free of finite rank over $\mathcal{O}_X$ and $\widehat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$.

For a connection ∇ on $\mathcal{A}$ let $\mathcal{E}_\nabla \subset \widehat{\mathcal{A}}$ denote the 'inverse image' of $\nabla(\mathcal{T}_X)$ under the projection. Thus, $\mathcal{E}_\nabla$ contains (the image of) Ω^1_X and the anchor map induces the isomorphism $\mathcal{E}_\nabla / \Omega^1_X \to \mathcal{T}_X$ so that there is a short exact sequence

$$0 \to \Omega^1_X \to \mathcal{E}_\nabla \to \mathcal{T}_X \to 0. \tag{3.4.1}$$

It is clear that the restriction of the symmetric pairing to $\mathcal{E}_\nabla$ is non-degenerate. A connection $\widehat{\nabla} : \mathcal{T}_X \to \widehat{\mathcal{A}}$ on $\widehat{\mathcal{A}}$ lifting ∇ (i.e. ∇ is the composition of $\widehat{\nabla}$ and the projection $\widehat{\mathcal{A}} \to \mathcal{A}$) determines a Lagrangian splitting of (3.4.1).

Let $\mathcal{E}^\perp_\nabla \subset \widehat{\mathcal{A}}$ denote the annihilator of $\mathcal{E}_\nabla$ with respect to the symmetric pairing.

Lemma 3.5. *We have*

(i) $\mathcal{E}_\nabla \cap \mathcal{E}^\perp_\nabla = 0$.

(ii) *The projection* $\widehat{\mathcal{A}} \to \mathcal{A}$ *restricts to an isomorphism* $\mathcal{E}^\perp_\nabla \to \mathfrak{g}$.

(iii) *The sheaf* $\mathfrak{g}(\widehat{\mathcal{A}})$ *decomposes into the orthogonal direct sum* $\Omega^1_X + \mathcal{E}^\perp_\nabla$.

(iv) *The induced isomorphism* $\Omega^1_X \oplus \mathfrak{g} \overset{\sim}{=} \widehat{\mathfrak{g}}$ *coincides with the one in Lemma 3.4.*

(v) *Suppose in addition that* ∇ *is flat. Then, the Leibniz bracket and the symmetric pairing restrict to a structure of a Courant algebroid on* $\mathcal{E}_\nabla$.

Proof. Since the symmetric pairing on $\mathcal{E}_\nabla$ is non-degenerate, $\mathcal{E}_\nabla \cap \mathcal{E}^\perp_\nabla = 0$. This means that the natural map $\mathcal{E}_\nabla \oplus \mathcal{E}^\perp_\nabla \to \widehat{\mathcal{A}}$ is an isomorphism.

Since $\mathcal{E}^\perp_\nabla \perp \Omega^1_X (\subset \mathcal{E}_\nabla)$ it follows (from (2.2.5)) that the composition $\mathcal{E}^\perp_\nabla \to \mathcal{A} \to \mathcal{T}_X$ is trivial. Hence, the map $\mathcal{E}^\perp_\nabla \to \mathcal{A}$ factors through $\mathcal{E}^\perp_\nabla \to \mathfrak{g}$ which, clearly, is an isomorphism.

It follows that the composition

$$\Omega^1_X \oplus \mathcal{E}^\perp_\nabla \to \mathcal{E}_\nabla \oplus \mathcal{E}^\perp_\nabla \to \widehat{\mathcal{A}}$$

is an isomorphism onto $\mathfrak{g}(\widehat{\mathcal{A}})$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

For $a, b \in \mathfrak{g}$ let $\widetilde{a}, \widetilde{b} \in \mathcal{E}_{\widehat{\nabla}}^{\perp}$ denote their respective lifts. For $q \in \mathcal{E}_{\nabla}$, viewed as a section of $\widehat{\mathcal{A}}$,

$$\langle q, [\widetilde{a}, \widetilde{b}]\rangle_{\widehat{\mathcal{A}}} = -\langle [\widetilde{a}, q], \widetilde{b}\rangle_{\widehat{\mathcal{A}}} = -\langle \overline{[\widetilde{a}, q]}, b\rangle_{\mathfrak{g}} = -\langle [a, \nabla(\pi(q))], b\rangle_{\mathfrak{g}} = \langle [\nabla(\pi(q)), a], b\rangle_{\mathfrak{g}},$$

i.e. the bracket on $\Omega^1_X \oplus \mathfrak{g}$ is given precisely by the cocycle $c_{\nabla}(\ ,\ )$ of Lemma 3.4.

If ∇ is flat, then $\nabla(\mathcal{T})$ is closed under the Lie bracket in $\mathcal{A}$, hence $\mathcal{E}_{\nabla}$ is closed under the Leibniz bracket in $\widehat{\nabla}$. □

Suppose that $\widehat{\nabla}$ is a connection on $\widehat{\mathcal{A}}$ which lifts the connection ∇ on $\mathcal{A}$. According to Lemma 3.5, the curvature $c(\widehat{\nabla}) \in \Omega^2_X \otimes_{\mathcal{O}_X} \widehat{\mathfrak{g}}$ of the connection $\widehat{\nabla}$ (see Definition 2.6 and Remark 2.5) decomposes uniquely as $c(\widehat{\nabla}) = \widetilde{c(\nabla)} + c_{\mathrm{rel}}(\widehat{\nabla})$, where $\widetilde{c(\nabla)} \in \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{E}_{\widehat{\nabla}}^{\perp}$ is the lift of $c(\nabla)$ (the curvature of the connection ∇) and $c_{\mathrm{rel}}(\widehat{\nabla}) \in \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\wedge^2 \mathcal{T}_X, \Omega^1_X) = \Omega^2_X \otimes_{\mathcal{O}_X} \Omega^1_X$.

Lemma 3.6. *In the notation introduced above, we have:*

(i) $\iota_\xi(c_{\mathrm{rel}}(\widehat{\nabla})(\xi_1, \xi_2)) = \langle [\widehat{\nabla}(\xi_1), \widehat{\nabla}(\xi_2)], \widehat{\nabla}(\xi)\rangle$;

(ii) c_{rel} *is totally skew-symmetric, i.e.* $c_{\mathrm{rel}}(\widehat{\nabla}) \in \Omega^3_X$.

Proof. Both claims follow from the calculation

$$\begin{aligned}
\iota_\xi(c_{\mathrm{rel}}(\widehat{\nabla})(\xi_1, \xi_2)) &= \langle c_{\mathrm{rel}}(\widehat{\nabla})(\xi_1, \xi_2), \widehat{\nabla}(\xi)\rangle \\
&= \langle c(\widehat{\nabla}), \widehat{\nabla}(\xi)\rangle \\
&= \langle [\widehat{\nabla}(\xi_1), \widehat{\nabla}(\xi_2)], \widehat{\nabla}(\xi)\rangle \\
&= \xi_1(\langle \widehat{\nabla}(\xi_2), \widehat{\nabla}(\xi)\rangle) - \langle [\widehat{\nabla}(\xi_1), \widehat{\nabla}(\xi)], \widehat{\nabla}(\xi_2)\rangle \\
&= -\langle [\widehat{\nabla}(\xi_1), \widehat{\nabla}(\xi)], \widehat{\nabla}(\xi_2)\rangle \\
&= -\iota_{\xi_2}(c_{\mathrm{rel}}(\widehat{\nabla})(\xi_1, \xi))
\end{aligned}$$

(using $\langle \widetilde{c(\nabla)}, \widehat{\nabla}(\xi)\rangle = \langle \widehat{\nabla}([\xi_1, \xi_2]), \widehat{\nabla}(\xi)\rangle = \langle \widehat{\nabla}(\xi_2), \widehat{\nabla}(\xi)\rangle = 0$). □

The choice of the lifting $\widehat{\nabla}$ of ∇ gives rise to the identification

$$\Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X \to \widehat{\mathcal{A}},$$

with $(\alpha, a, \xi) \mapsto \alpha + \widetilde{a} + \widehat{\nabla}(\xi)$, which:

(i) maps the flag $\Omega^1_X \subset \Omega^1_X \oplus \mathfrak{g} \subset \Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X$ isomorphically onto the flag $\Omega^1_X \subset \widehat{\mathfrak{g}} \subset \widehat{\mathcal{A}}$;

(ii) projects (modulo Ω^1_X) to the identification $\mathfrak{g} \oplus \mathcal{T}_X \to \mathcal{A}$ induced by the connection ∇ on $\mathcal{A}$.

Let $\langle\ ,\ \rangle_{\widehat{\nabla}}$ and $[\ ,\ ]_{\widehat{\nabla}}$ denote, respectively, the induced symmetric pairing and Leibniz bracket on $\Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X$.

Lemma 3.7. *The induced symmetric pairing* $\langle\ ,\ \rangle_{\widehat{\nabla}}$ *and the Leibniz bracket* $[\ ,\ ]_{\widehat{\nabla}}$ *satisfy:*

(i) $\langle \Omega^1_X \oplus \mathcal{T}_X, \mathfrak{g}\rangle_{\widehat{\nabla}} = 0$;

(ii) *the restriction of* $\langle\ ,\ \rangle_{\widehat{\nabla}}$ *to* $\Omega^1_X \oplus \mathcal{T}_X$ *(respectively,* $\mathfrak{g}$*) is induced by the duality pairing (respectively, is the pairing induced by the one on* $\widehat{\mathcal{A}}$*);*

(iii) *the restriction of the Leibniz bracket to* $\Omega^1_X \oplus \mathfrak{g}$ *is given by Lemma 3.5 (and Lemma 3.4);*

(iv) $[\xi_1, \xi_2]_{\widehat{\nabla}} = [\nabla(\xi_1), \nabla(\xi_2)]_{\mathcal{A}} + c_{\mathrm{rel}}(\xi_1, \xi_2)$;

(v) $[\xi, a]_{\widehat{\nabla}} = [\nabla(\xi), a]_{\mathcal{A}} - \langle c(\nabla)(\xi, \bullet), a\rangle_{\mathfrak{g}}$;

where $a \in \mathfrak{g}$, *and* $\xi, \xi_i \in \mathcal{T}_X$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Proof. Only the last formula has not been proven earlier. In terms of the orthogonal direct sum decomposition (Lemma 3.5) $\widehat{\mathcal{A}} = \mathcal{E}_\nabla^\perp + \mathcal{E}_\nabla$, we have

$$[\widehat{\nabla}(\xi), \widetilde{a}] = \widetilde{[\nabla(\xi), a]} + \Phi(\xi, a),$$

where $\Phi(\xi, a) \in \Omega^1_X$ satisfies

$$\begin{aligned} \iota_\eta \Phi(\xi, a) &= \langle \widehat{\nabla}(\eta), \Phi(\xi, a)\rangle \\ &= \langle \widehat{\nabla}(\eta), [\widehat{\nabla}(\xi), \widetilde{a}]\rangle \\ &= \xi(\langle \widehat{\nabla}(\eta), \widetilde{a}\rangle) - \langle [\widehat{\nabla}(\xi), \widehat{\nabla}(\eta)], \widetilde{a}\rangle \\ &= -\langle \widehat{\nabla}([\xi,\eta]) + \widetilde{c(\nabla)(\xi,\eta)} + c_{\mathrm{rel}}(\widehat{\nabla})(\xi,\eta), \widetilde{a}\rangle \\ &= -\langle \widetilde{c(\nabla)(\xi,\eta)}, \widetilde{a}\rangle. \end{aligned}$$ □

3.5 Construction of Courant extensions

Suppose that $\mathcal{A}$ is a transitive Lie algebroid locally free of finite rank over $\mathcal{O}_X$, $\mathfrak{g} = \mathfrak{g}(\mathcal{A})$, and $\langle\ ,\ \rangle$ is a symmetric $\mathcal{A}$-invariant pairing on $\mathfrak{g}$.

We extend the pairing $\langle\ ,\ \rangle$ to a symmetric pairing on $\Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X$ by the rules:

(i) $\langle \mathfrak{g}, \Omega^1_X \oplus \mathcal{T}_X\rangle = 0$;

(ii) the restriction of $\langle\ ,\ \rangle$ to $\Omega^1_X \oplus \mathcal{T}_X$ coincides with the canonical symmetric pairing.

For a connection ∇ on $\mathcal{A}$ and $H \in \Omega^3_X$ let $\widehat{\mathcal{A}}_{\nabla,H}$ denote $\Omega^1_X \oplus \mathcal{A}$ identified with $\Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X$ via ∇ with the induced symmetric pairing, denoted $\langle\ ,\ \rangle_{\nabla,H}$, the map (derivation) $\partial : \mathcal{O}_X \to \widehat{\mathcal{A}}$, defined as the composition of the exterior differentiation with the inclusion of Ω^1_X, the (anchor) map $\pi : \widehat{\mathcal{A}} \to \mathcal{T}_X$, denoting the projection onto $\mathcal{T}_X$ and the operation

$$[\ ,\ ]_{\nabla,H} : \widehat{\mathcal{A}}_{\nabla,H} \times \widehat{\mathcal{A}}_{\nabla,H} \to \widehat{\mathcal{A}}_{\nabla,H}$$

defined by:

(i) the formulas

$$\begin{aligned} [\xi, df]_{\nabla,H} &= \partial(\xi(f)), \\ [\xi, a]_{\nabla,H} &= [\nabla(\xi), a]_{\mathcal{A}} - \langle c(\nabla)(\xi, \bullet), a\rangle_{\mathfrak{g}}, \\ [\xi_1, \xi_2]_{\nabla,H} &= [\xi_1, \xi_2] + c(\nabla)(\xi_1, \xi_2) + H(\xi_1, \xi_2, \bullet), \\ [a_1, a_2]_{\nabla,H} &= [a_1, a_2]_{\mathfrak{g}} + \langle [\nabla(\bullet), a_1]_{\mathcal{A}}, a_2\rangle_{\mathfrak{g}}, \\ [a, \partial(f)]_{\nabla,H} &= [\partial(f), \partial(g)]_{\nabla,H} = 0, \end{aligned}$$

where $f, g \in \mathcal{O}_X$, $a, a_i \in \mathfrak{g}$, $\xi, \xi_i \in \mathcal{T}_X$;

(ii) the Leibniz rule

$$[\widehat{a}, f\widehat{b}]_{\nabla,H} = [\widehat{a}, \widehat{b}]_{\nabla,H} + \pi(\widehat{a})(f)\widehat{b};$$

(iii) the relation

$$[\widehat{a}, \widehat{b}]_{\nabla,H} = -[\widehat{b}, \widehat{a}]_{\nabla,H} + \partial\langle \widehat{a}, \widehat{b}\rangle_{\widehat{\mathcal{A}}_{\nabla,H}};$$

where $f \in \mathcal{O}_X$, and $\widehat{a}, \widehat{b} \in \widehat{\mathcal{A}}_{\nabla,H}$.

Remark 3.2. It is clear from the above formulas that:

(i) $[\widehat{\mathcal{A}}_{\nabla,H}, \Omega^1_X \oplus \mathfrak{g}]_{\nabla,H} \subseteq \Omega^1_X \oplus \mathfrak{g}$;

(ii) the (restriction of) $[\ ,\ ]_{\nabla,H}$ to $(\Omega^1_X \oplus \mathfrak{g})^{\otimes 2}$ coincides with the bracket $[\ ,\ ]_\nabla$ of § 3.3;

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

(iii) the induced map $\widehat{\mathcal{A}}_{\nabla,H} \to \underline{\mathrm{End}}_{\mathbb{C}}(\Omega^1_X \oplus \mathfrak{g})$ is trivial on Ω^1_X, hence induces the map $\mathcal{A}_{\nabla,H} \to \underline{\mathrm{End}}_{\mathbb{C}}(\Omega^1_X \oplus \mathfrak{g})$; the latter map is the direct sum of the Lie derivative action on Ω^1_X (through the quotient $\mathcal{T}_X$) and the adjoint action of $\mathcal{A}$ on $\mathfrak{g}$.

LEMMA 3.8. *The Jacobiator of the operation* $[\ ,\ ]_{\nabla,H}$ *(see (2.3.2)),* $J([\ ,\ ]_{\nabla,H}) : \widehat{\mathcal{A}}^{\otimes 3}_{\nabla,H} \to \widehat{\mathcal{A}}_{\nabla,H}$, *factors into the composition*

$$\widehat{\mathcal{A}}^{\otimes 3}_{\nabla,H} \xrightarrow{\pi^{\otimes 3}} \mathcal{T}_X^{\otimes 3} \xrightarrow{J_{\nabla,H}} \Omega^1_X \xrightarrow{i} \widehat{\mathcal{A}}_{\nabla,H},$$

where

$$J_{\nabla,H}(\xi_0,\xi_1,\xi_2) = \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0}(-\tfrac{1}{2}\langle c(\nabla)\wedge c(\nabla)\rangle + dH).$$

Proof. It follows from

(i) $(\widehat{\mathcal{A}}_{\nabla,H} \to \mathcal{A}) \circ J([\ ,\ ]_{\nabla,H}) = J([\ ,\ ]_{\mathcal{A}}) \circ (\widehat{\mathcal{A}}_{\nabla,H} \to \mathcal{A})^{\otimes 3}$ and

(ii) $J([\ ,\ ]_{\mathcal{A}}) = 0$ (i.e. the Lie bracket on $\mathcal{A}$ satisfies the Jacobi identity)

that $J([\ ,\ ]_{\nabla,H})$ takes values in Ω^1_X. By Remark 3.2, $J([\ ,\ ]_{\nabla,H})$ vanishes on $\ker(\pi^{\otimes 3})$. This proves the first claim and, hence,

$$\begin{aligned} J_{\nabla,H}(\xi_0,\xi_1,\xi_2) &= J([\ ,\ ]_{\nabla,H})(\xi_0,\xi_1,\xi_2) \\ &= [\xi_0, [\xi_1,\xi_2] + c(\nabla)(\xi_1,\xi_2) + \iota_{\xi_2}\iota_{\xi_1}H]_{\nabla,H} \\ &\quad - [[\xi_0,\xi_1] + c(\nabla)(\xi_0,\xi_1) + \iota_{\xi_1}\iota_{\xi_0}H, \xi_2]_{\nabla,H} \\ &\quad - [\xi_1, [\xi_0,\xi_2] + c(\nabla)(\xi_0,\xi_2) + \iota_{\xi_2}\iota_{\xi_0}H]_{\nabla,H}. \end{aligned}$$

The three summands expand, respectively, to

$$\begin{aligned} &[\xi_0, [\xi_1,\xi_2] + c(\nabla)(\xi_1,\xi_2) + \iota_{\xi_2}\iota_{\xi_1}H]_{\nabla,H} \\ &\quad = [\xi_0,[\xi_1,\xi_2]]_{\nabla,H} + [\xi_0, c(\nabla)(\xi_1,\xi_2)]_{\nabla,H} + [\xi_0, \iota_{\xi_2}\iota_{\xi_1}H]_{\nabla,H} \\ &\quad = [\xi_0,[\xi_1,\xi_2]] + c(\nabla)(\xi_0,[\xi_1,\xi_2]) + \iota_{[\xi_1,\xi_2]}\iota_{\xi_0}H \\ &\qquad + [\nabla(\xi_0), c(\nabla)(\xi_1,\xi_2)]_{\mathcal{A}} - \langle c(\nabla)(\xi_0,\bullet), c(\nabla)(\xi_1,\xi_2)\rangle + L_{\xi_0}\iota_{\xi_2}\iota_{\xi_1}H, \\ &[[\xi_0,\xi_1] + c(\nabla)(\xi_0,\xi_1) + \iota_{\xi_1}\iota_{\xi_0}H, \xi_2]_{\nabla,H} \\ &\quad = [[\xi_0,\xi_1],\xi_2]_{\nabla,H} + [c(\nabla)(\xi_0,\xi_1),\xi_2]_{\nabla,H} + [\iota_{\xi_1}\iota_{\xi_0}H,\xi_2]_{\nabla,H} \\ &\quad = [[\xi_0,\xi_1],\xi_2] + c(\nabla)([\xi_0,\xi_1],\xi_2) + \iota_{\xi_2}\iota_{[\xi_0,\xi_1]}H \\ &\qquad - [\nabla(\xi_2), c(\nabla)(\xi_0,\xi_1)]_{\mathcal{A}} + \langle c(\nabla)(\xi_2,\bullet), c(\nabla)(\xi_0,\xi_1)\rangle - \iota_{\xi_2}d\iota_{\xi_1}\iota_{\xi_0}H \end{aligned}$$

and

$$\begin{aligned} &[\xi_1, [\xi_0,\xi_2] + c(\nabla)(\xi_0,\xi_2) + \iota_{\xi_2}\iota_{\xi_0}H]_{\nabla,H} \\ &\quad = [\xi_1,[\xi_0,\xi_2]]_{\nabla,H} + [\xi_1, c(\nabla)(\xi_0,\xi_2)]_{\nabla,H} + [\xi_1,\iota_{\xi_2}\iota_{\xi_0}H]_{\nabla,H} \\ &\quad = [\xi_1,[\xi_0,\xi_2]] + c(\nabla)(\xi_1,[\xi_0,\xi_2]) + \iota_{[\xi_0,\xi_2]}\iota_{\xi_1}H \\ &\qquad + [\nabla(\xi_1), c(\nabla)(\xi_0,\xi_2)]_{\mathcal{A}} - \langle c(\xi_1,\bullet), c(\nabla)(\xi_0,\xi_2)\rangle + L_{\xi_1}\iota_{\xi_2}\iota_{\xi_0}H. \end{aligned}$$

Summing these up one obtains

$$\begin{aligned} J_{\nabla,H}(\xi_0,\xi_1,\xi_2) &= -\langle c(\nabla)(\xi_0,\bullet), c(\nabla)(\xi_1,\xi_2)\rangle - \langle c(\nabla)(\xi_0,\xi_1), c(\nabla)(\xi_2,\bullet)\rangle \\ &\quad + \langle c(\nabla)(\xi_1,\bullet), c(\nabla)(\xi_0,\xi_2)\rangle + \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0}dH \\ &= \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0}(-\tfrac{1}{2}\langle c(\nabla)\wedge c(\nabla)\rangle + dH). \end{aligned}$$

□

COROLLARY 3.2. *In the notation introduced above,* $\widehat{\mathcal{A}}_{\nabla,H}$ *is a Courant extension of* $\mathcal{A}$ *if and only if*

$$dH = \tfrac{1}{2}\langle c(\nabla)\wedge c(\nabla)\rangle.$$

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

If the latter condition is fulfilled the canonical connection $\widehat{\nabla}$ on $\widehat{\mathcal{A}}_{\nabla,H}$ (given the inclusion of the direct summand) satisfies $c_{\mathrm{rel}}(\widehat{\nabla}) = H$.

Corollary 3.3. *In the notation of § 3.4, $dc_{\mathrm{rel}}(\widehat{\nabla}) = \frac{1}{2}\langle c(\nabla) \wedge c(\nabla)\rangle$ and the map*

$$\phi_{\widehat{\nabla}} : \widehat{\mathcal{A}}_{\nabla, c_{\mathrm{rel}}(\widehat{\nabla})} \to \widehat{\mathcal{A}} \tag{3.5.1}$$

induced by $\widehat{\nabla}$ is an isomorphism of Courant extensions of $\mathcal{A}$.

3.6 Change of connection

Suppose that $\mathcal{A}$ is a Lie algebroid, $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant pairing on $\mathfrak{g} := \mathfrak{g}(\mathcal{A})$, ∇ is a connection on $\mathcal{A}$, and $H \in \Omega^3_X$ satisfies $dH = \frac{1}{2}\langle c(\nabla) \wedge c(\nabla)\rangle$.

Suppose that ∇' is another connection on $\mathcal{A}$. Then, the formula

$$\widehat{\nabla}'(\xi) = -\tfrac{1}{2}\langle A(\xi), A(\bullet)\rangle + A(\xi) + \xi,$$

where $A = \nabla' - \nabla \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$, determines a connection on $\widehat{\mathcal{A}}_{\nabla,H}$ which induced the connection ∇' on $\mathcal{A}$.

Lemma 3.9. *In the notation as above,*

$$c_{\mathrm{rel}}(\widehat{\nabla}') = H + \langle c(\nabla) \wedge A\rangle + \tfrac{1}{2}\langle [\nabla, A] \wedge A\rangle + \tfrac{1}{6}\langle [A, A] \wedge A\rangle.$$

Proof. We have

$$\begin{aligned}
&[\widehat{\nabla}'(\xi_0), \widehat{\nabla}'(\xi_1)]_{\nabla,H} \\
&\quad = [-\tfrac{1}{2}\langle A(\xi_0), A(\bullet)\rangle + A(\xi_0) + \xi_0, -\tfrac{1}{2}\langle A(\xi_1), A(\bullet)\rangle + A(\xi_1) + \xi_1]_{\nabla,H} \\
&\quad = \tfrac{1}{2}\iota_{\xi_1} d\langle A(\xi_0), A(\bullet)\rangle + [A(\xi_0), A(\xi_1)] + \langle [\nabla(\bullet), A(\xi_0)], A(\xi_1)\rangle \\
&\qquad + [A(\xi_0), \nabla(\xi_1)] + \langle c(\nabla)(\xi_1, \bullet), A(\xi_0)\rangle - \tfrac{1}{2} L_{\xi_0}\langle A(\xi_1), A(\bullet)\rangle \\
&\qquad + [\nabla(\xi_0), A(\xi_1)] - \langle c(\nabla)(\xi_0, \bullet), A(\xi_1)\rangle + [\xi_0, \xi_1] + c(\nabla)(\xi_0, \xi_1) + \iota_{\xi_1}\iota_{\xi_0} H.
\end{aligned}$$

Pairing the result with $\widehat{\nabla}'(\xi_2) = -\frac{1}{2}\langle A(\xi_2), A(\bullet)\rangle + A(\xi_2) + \xi_2$ gives

$$\begin{aligned}
\iota_{\xi_2} c_{\mathrm{rel}}(\widehat{\nabla}')(\xi_0, \xi_1) &= \langle [\widehat{\nabla}'(\xi_0), \widehat{\nabla}'(\xi_1)]_{\nabla,H}, \widehat{\nabla}'(\xi_2)\rangle \\
&= +\tfrac{1}{2}\iota_{\xi_2}\iota_{\xi_1} d\langle A(\xi_0), A(\bullet)\rangle + \langle [A(\xi_0), A(\xi_1)], A(\xi_2)\rangle \\
&\quad + \langle [\nabla(\xi_2), A(\xi_0)], A(\xi_1)\rangle + \langle [A(\xi_0), \nabla(\xi_1)], A(\xi_2)\rangle + \langle c(\nabla)(\xi_1, \xi_2), A(\xi_0)\rangle \\
&\quad - \tfrac{1}{2}\iota_{\xi_2} L_{\xi_0}\langle A(\xi_1), A(\bullet)\rangle + \langle [\nabla(\xi_0), A(\xi_1)], A(\xi_2)\rangle \\
&\quad - \langle c(\nabla)(\xi_0, \xi_2), A(\xi_1)\rangle - \tfrac{1}{2}\langle A(\xi_2), A([\xi_0, \xi_1])\rangle \\
&\quad + \langle c(\nabla)(\xi_0, \xi_1), A(\xi_2)\rangle + \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0} H.
\end{aligned}$$

The identities

$$\begin{aligned}
&\tfrac{1}{2}\iota_{\xi_2}\iota_{\xi_1} d\langle A(\xi_0), A(\bullet)\rangle + \langle [\nabla(\xi_2), A(\xi_0)], A(\xi_1)\rangle + \langle [A(\xi_0), \nabla(\xi_1)], A(\xi_2)\rangle \\
&\qquad - \tfrac{1}{2}\iota_{\xi_2} L_{\xi_0}\langle A(\xi_1), A(\bullet)\rangle + \langle [\nabla(\xi_0), A(\xi_1)], A(\xi_2)\rangle - \tfrac{1}{2}\langle A(\xi_2), A([\xi_0, \xi_1])\rangle \\
&\quad = \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0} \tfrac{1}{2}\langle [\nabla, A] \wedge A\rangle, \\
&\qquad - \langle c(\nabla)(\xi_2, \xi_1), A(\xi_0)\rangle - \langle c(\nabla)(\xi_0, \xi_2), A(\xi_1)\rangle + \langle c(\nabla)(\xi_0, \xi_1), A(\xi_2)\rangle \\
&\quad = \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0}\langle c(\nabla) \wedge A\rangle
\end{aligned}$$

and

$$\langle [A(\xi_0), A(\xi_1)], A(\xi_2)\rangle = \tfrac{1}{2}\langle [A, A](\xi_0, \xi_1), A(\xi_2)\rangle = \iota_{\xi_2}\iota_{\xi_1}\iota_{\xi_0}\tfrac{1}{6}\langle [A, A] \wedge A\rangle$$

give

$$c_{\mathrm{rel}}(\widehat{\nabla}') = H + (\langle c(\nabla) \wedge A\rangle + \tfrac{1}{2}\langle [\nabla, A] \wedge A\rangle + \tfrac{1}{6}\langle [A, A] \wedge A\rangle). \qquad \square$$

Notation 3.1. Suppose that $\mathcal{A}$ is a transitive Lie algebroid and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. For connections ∇, ∇' on $\mathcal{A}$ let

$$\mathcal{P}(\nabla, \nabla') \overset{\text{def}}{=} \langle c(\nabla) \wedge A\rangle + \tfrac{1}{2}\langle[\nabla, A] \wedge A\rangle + \tfrac{1}{6}\langle[A, A], A\rangle, \tag{3.6.1}$$

where $A = \nabla' - \nabla \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$.

LEMMA 3.10. *The isomorphism*

$$\widehat{\mathcal{A}}_{\nabla', H+\mathcal{P}(\nabla,\nabla')} \to \widehat{\mathcal{A}}_{\nabla, H}$$

of Corollary 3.3 is given by

$$\begin{aligned}\alpha + a + \xi &\mapsto \alpha - \langle a, A(\bullet)\rangle + a + \widehat{\nabla}'(\xi)\\ &= (\alpha - \langle a, A(\bullet)\rangle - \tfrac{1}{2}\langle A(\xi), A(\bullet)\rangle) + (a + A(\xi)) + \xi.\end{aligned} \tag{3.6.2}$$

Proof. In the orthogonal decomposition $\widehat{\mathcal{A}}_{\nabla,H} = \mathcal{E}_{\nabla'} \oplus \mathcal{E}^{\perp}_{\nabla'}$, the summands are given by

$$\mathcal{E}_{\nabla'} = \{\alpha + A(\xi) + \xi \mid \alpha \in \Omega^1_X,\ \xi \in \mathcal{T}_X\}$$

and

$$\mathcal{E}^{\perp}_{\nabla'} = \{-\langle a, A(\bullet)\rangle + a \mid a \in \mathfrak{g}\}.$$ □

Notation 3.2. Suppose that $\mathcal{A}$ is a transitive Lie algebroid and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. For connections ∇, ∇' on $\mathcal{A}$ we denote by $\phi(\nabla, \nabla')$ the isomorphism of Lemma 3.10, where $A = \nabla' - \nabla$.

LEMMA 3.11. *Suppose that $\nabla, \nabla', \nabla''$ are connections on $\mathcal{A}$. Then,*

$$\phi(\nabla, \nabla') \circ \phi(\nabla', \nabla'') \circ \phi(\nabla'', \nabla) = \exp(-\tfrac{1}{2}\langle A \wedge A'\rangle), \tag{3.6.3}$$

where $A = \nabla' - \nabla$, $A' = \nabla'' - \nabla'$ and $\exp(\bullet)$ is as in (3.1.1).

Proof. It is clear that the left-hand side of (3.6.3) is of the form $\exp(B)$ for suitable $B \in \Omega^2_X$, i.e. its value on an element of $\widehat{\mathcal{A}}_{\nabla,H}$ depends only on the projection of that element to $\mathcal{T}_X$. Hence, it suffices to calculate the left-hand side of (3.6.3) for the case $\alpha = 0$, $a = 0$ in the notation introduced above.

Since $\nabla - \nabla'' = -(A + A')$, the formula (3.6.2) gives

$$\phi(\nabla'', \nabla)(\xi) = -\tfrac{1}{2}\langle A(\xi) + A'(\xi), A(\bullet) + A'(\bullet)\rangle - (A(\xi) + A'(\xi)) + \xi,$$

$$\begin{aligned}&\phi(\nabla', \nabla'') \circ \phi(\nabla'', \nabla)\\ &\quad= -\tfrac{1}{2}\langle A(\xi) + A'(\xi), A(\bullet) + A'(\bullet)\rangle + \langle A(\xi) + A'(\xi), A'(\bullet)\rangle\\ &\qquad - \tfrac{1}{2}\langle A'(\xi), A'(\bullet)\rangle - (A(\xi) + A'(\xi)) + A'(\xi) + \xi\\ &\quad= -\tfrac{1}{2}\langle A(\xi), A(\bullet)\rangle - \tfrac{1}{2}\langle A'(\xi), A(\bullet)\rangle + \tfrac{1}{2}\langle A(\xi), A'(\bullet)\rangle - A(\xi) + \xi\end{aligned}$$

and, finally,

$$\begin{aligned}&\phi(\nabla, \nabla') \circ \phi(\nabla', \nabla'') \circ \phi(\nabla'', \nabla)\\ &\quad= -\tfrac{1}{2}\langle A(\xi), A(\bullet)\rangle - \tfrac{1}{2}\langle A'(\xi), A(\bullet)\rangle + \tfrac{1}{2}\langle A(\xi), A'(\bullet)\rangle\\ &\qquad + \langle A(\xi), A(\bullet)\rangle - \tfrac{1}{2}\langle A(\xi), A(\bullet)\rangle - A(\xi) + A(\xi) + \xi\\ &\quad= \iota_\xi(-\tfrac{1}{2}\langle A \wedge A'\rangle) + \xi,\end{aligned}$$

as desired. □

3.7 Exact Courant algebroids

Definition 3.1. The Courant algebroid $\mathcal{Q}$ is called *exact* if the anchor map $\pi : \overline{\mathcal{Q}} \to \mathcal{T}_X$ is an isomorphism. Equivalently, an exact Courant algebroid is a Courant extension of the Lie algebroid $\mathcal{T}_X$.

We denote the stack of exact Courant $\mathcal{O}_X$-algebroids by $\mathcal{ECA}_{\mathcal{O}_X}$. As was pointed out in Remark 2.4, $\mathcal{ECA}_{\mathcal{O}_X}$ is a stack in groupoids.

For an exact Courant algebroid $\mathcal{Q}$ the exact sequence (2.6.1) takes the shape

$$0 \to \Omega^1_X \to \mathcal{Q} \to \mathcal{T}_X \to 0. \tag{3.7.1}$$

An isotropic splitting of (3.7.1) (i.e. a connection on $\mathcal{Q}$) is necessarily Lagrangian.

Lemma 3.12. *Suppose that $\mathcal{Q}$ is an exact Courant algebroid.*

(i) *For $\nabla \in \mathcal{C}(\mathcal{Q})$ and $\omega \in \Omega^2_X$ the map $\mathcal{T}_X \to \mathcal{Q}$ defined by $\xi \mapsto \nabla(\xi) + \iota_\xi\omega$ is a connection.*

(ii) *The assignment $\nabla \mapsto \nabla + \omega$ (where $\nabla + \omega$ is the connection defined by the formula above) is an action of (the sheaf of groups) Ω^2_X on $\mathcal{C}(\mathcal{Q})$ which endows the latter with the structure of an Ω^2_X-torsor.*

Proof. The difference of two sections of the anchor map $\mathcal{Q} \to \mathcal{T}_X$ is a map $\mathcal{T}_X \to \Omega^1_X$ or, equivalently, a section of $\Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X$. The difference of two isotropic sections gives rise to a skew-symmetric tensor, i.e. a section of Ω^2_X. Indeed, suppose that ∇ is a connection and $\phi : \mathcal{T}_X \to \Omega^1_X$, for example, $\phi(\xi) = \iota_\xi\omega$, where $\omega \in \Omega^2_X$. Then, for $\xi, \eta \in \mathcal{T}_X$, we have

$$\langle(\nabla + \phi)(\xi), (\nabla + \phi)(\eta)\rangle = \langle\nabla(\xi), \nabla(\eta)\rangle + \langle\nabla(\xi), \phi(\eta)\rangle + \langle\phi(\xi), \nabla(\eta)\rangle + \langle\phi(\xi), \phi(\eta)\rangle.$$

Since ∇ and the inclusion $\Omega^1_X \to \mathcal{Q}$ are isotropic, $\nabla + \phi$ is isotropic if and only if

$$\langle\nabla(\xi), \nabla(\eta)\rangle + \langle\nabla(\xi), \phi(\eta)\rangle = 0$$

which is equivalent to $\iota_\xi\phi(\eta) = -\iota_\eta\phi(\xi)$ by (2.2.5).

By Lemma 2.5, $\mathcal{C}(\mathcal{Q})$ is locally non-empty, hence a torsor. □

For a connection ∇ on an exact Courant algebroid, curvature coincides with relative curvature, is a differential 3-form by Lemma 3.6, and will be denoted $c(\nabla)$.

Lemma 3.13.

(i) *The curvature form $c(\nabla)$ is closed.*

(ii) *For $\alpha \in \Omega^2_X$, $c(\nabla + \alpha) = c(\nabla) + d\alpha$.*

Proof. The curvature form is closed since its derivative is the form $\amalg_{\mathcal{T}_X}$ and the latter vanishes. The second claim follows from Lemma 3.9 with $A = 0$, $H = c(\nabla)$ and $B = \alpha$. □

Corollary 3.4.

(i) *The action of Ω^2_X on $\mathcal{C}(\mathcal{Q})$ restricts to an action of $\Omega^{2,cl}_X$ on $\mathcal{C}^\flat(\mathcal{Q})$.*

(ii) *If (locally) non-empty, $\mathcal{C}^\flat(\mathcal{Q})$ is an $\Omega^{2,cl}_X$-torsor.*

Proof. Suppose that ∇ is a flat connection and $\alpha \in \Omega^2_X$. It follows from Lemma 3.13 that $\nabla + \alpha$ is flat if and only if α is closed. □

Remark 3.3. The sheaf of flat connections $\mathcal{C}^\flat(\mathcal{Q})$ is locally non-empty if the Poincaré lemma is satisfied. This the case in the C^∞ and the analytic setting.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Example 3.1. The sheaf $\Omega^1_X \oplus \mathcal{T}_X$ endowed with the canonical symmetric bilinear form deduced from the duality pairing carries the canonical structure of an exact Courant algebroid with the obvious anchor map and the derivation, and the unique Leibniz bracket, such that the inclusion of $\mathcal{T}_X$ is a flat connection. We denote this exact Courant algebroid by $\mathcal{Q}_0$.

We leave it as an exercise for the reader to write down the explicit formula for the Leibniz bracket. The skew-symmetrization of this bracket was discovered by Courant [Cou90] and is usually referred to as 'the Courant bracket'.

3.8 Classification of exact Courant algebroids

Definition 3.2. A $\Omega^2_X \to \Omega^{3,cl}_X$-torsor is a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is a Ω^2_X-torsor and c is a map $\mathcal{C} \to \Omega^{3,cl}_X$ which satisfies $c(s+\alpha) = c(s) + d\alpha$. A morphism of $\Omega^2_X \to \Omega^{3,cl}_X$-torsors is a morphism of Ω^2_X-torsors which commutes with the respective maps to $\Omega^{3,cl}_X$.

Suppose that $\mathcal{Q}$ is an exact Courant $\mathcal{O}_X$-algebroid. The assignment $\nabla \mapsto c(\nabla)$ gives rise to the morphism

$$c : \mathcal{C}(\mathcal{Q}) \to \Omega^{3,cl}_X,$$

which satisfies $c(\nabla + \alpha) = c(\nabla) + d\alpha$ by Lemma 3.13. Thus, the pair $(\mathcal{C}(\mathcal{Q}), c)$ is a $(\Omega^2_X \to \Omega^{3,cl}_X)$-torsor.

Lemma 3.14. *The correspondence $\mathcal{Q} \mapsto (\mathcal{C}(\mathcal{Q}), c)$ establishes an equivalence*

$$\mathcal{ECA}_{\mathcal{O}_X} \longrightarrow (\Omega^2_X \to \Omega^{3,cl}_X)\text{-torsors}. \tag{3.8.1}$$

Proof. It is clear that the association $\mathcal{Q} \mapsto (\mathcal{C}(\mathcal{Q}), c)$ determines a functor. We construct a quasi-inverse to the latter.

Suppose that $(\mathcal{C}, c)$ is a $\Omega^2_X \to \Omega^{3,cl}_X$-torsor. We associate to it the exact Courant algebroid which is the $(\mathcal{C}, c)$-twist of the Courant algebroid $\mathcal{Q}_0$ of Example 3.1 constructed as follows.

The underlying extension of $\mathcal{T}_X$ by Ω^1_X is the $\mathcal{C}$-twist $\mathcal{Q}_0^{\mathcal{C}}$ of the trivial extension $\mathcal{Q}_0 = \Omega^1_X \oplus \mathcal{T}_X$, i.e. $\mathcal{Q}_0^{\mathcal{C}} = \mathcal{C} \times_{\Omega^2_X} \mathcal{Q}_0$. Since the action of Ω^2_X on $\mathcal{Q}_0$ preserves the symmetric pairing, it follows that $\mathcal{Q}_0^{\mathcal{C}}$ has the induced symmetric pairing.

The Leibniz bracket on $\mathcal{Q}_0^{\mathcal{C}}$ is defined by the formula

$$[(s_1, q_1), (s_2, q_2)] = (s_1, [q_1, q_2 + \iota_{\pi(q_2)}(s_1 - s_2)]_0 + \iota_{\pi(q_1)\wedge\pi(q_2)} c(s_1)),$$

where $s_i \in \mathcal{C}$, $q_i \in \mathcal{Q}_0$, $s_1 - s_2 \in \Omega^2_X$ is the unique form such that $s_1 = s_2 + (s_1 - s_2)$ and $[\ ,\ ]_0$ denotes the (Courant) bracket on $\mathcal{Q}_0$.

Next, we verify that the bracket is, indeed, well defined on $\mathcal{Q}_0^{\mathcal{C}}$, i.e. is independent of the choice of particular representatives. For $\omega \in \Omega^2_X$, we have

$$\begin{aligned}
&[(s_1, q_1), (s_2, q_2 + \iota_{\pi(q_2)}\omega)] \\
&\quad = (s_1, [q_1, q_2 + \iota_{\pi(q_2)}\omega + \iota_{\pi(q_2)}(s_1 - s_2)]_0 + \iota_{\pi(q_1)\wedge\pi(q_2)} c(s_1)) \\
&\quad = (s_1, [q_1, q_2 + \iota_{\pi(q_2)}(s_1 - (s_2 - \omega))]_0 + \iota_{\pi(q_1)\wedge\pi(q_2)} c(s_1)) \\
&\quad = [(s_1, q_1), (s_2 - \omega, q_2)]
\end{aligned}$$

(using $\pi(q_2) = \pi(q_2 + \iota_{\pi(q_2)}\omega)$). This shows that the bracket is well defined in the second variable and we can assume that $s_2 = s_1$ after modifying q_2, in which case the bracket is given by the simplified formula

$$[(s, q_1), (s, q_2)] = (s, [q_1, q_2]_0 + \iota_{\pi(q_1)\wedge\pi(q_2)} c(s)).$$

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Since $\iota_{\pi(q_1)\wedge\pi(q_2)}c(s)$ is skew-symmetric in q_1 and q_2 it follows that

$$\begin{aligned}[(s,q_1),(s,q_2)]+[(s,q_2),(s,q_1)] &= (s,[q_1,q_2]_0+[q_2,q_1]_0)\\ &= (s,d\langle q_1,q_2\rangle).\end{aligned}$$

This shows that the symmetrized bracket is well defined (in both variables). Since, as we established earlier, the bracket is well defined in the second variable, it follows that it is well defined (in both variables) and, moreover, satisfies (2.2.6). Since $c(s)$ is a closed form, the bracket satisfies the Jacobi identity. We leave the remaining verifications to the reader. □

Pairs $(\mathcal{Q},\nabla)$, where $\mathcal{Q}\in\mathcal{ECA}_{\mathcal{O}_X}$ and ∇ is a connection on $\mathcal{Q}$, with morphisms of pairs defined as morphisms of algebroids which commute with respective connections give rise to a stack which we denote $\mathcal{ECA}\nabla_{\mathcal{O}_X}$. It is clear that $\mathcal{ECA}\nabla_{\mathcal{O}_X}$ is a stack in groupoids. Note that the pair $(\mathcal{Q},\nabla)$ has no non-trivial automorphisms.

The assignment $(\mathcal{Q},\nabla)\mapsto c(\nabla)$ gives rise to the morphism of stacks

$$c:\mathcal{ECA}\nabla_{\mathcal{O}_X}\to\Omega_X^{3,cl}, \tag{3.8.2}$$

where $\Omega_X^{3,cl}$ is viewed as discrete, i.e. the only morphisms are the identity maps.

Lemma 3.15. *The morphism (3.8.2) is an equivalence.*

Proof. The quasi-inverse associates to $H\in\Omega_X^{3,cl}$ the H-twist (see § 2.3) $\mathcal{Q}_H$ of the algebroid $\mathcal{Q}_0$ of Example 3.1. The obvious connection on $\mathcal{Q}_H$ has curvature H. □

4. Linear algebra

4.1 $\mathcal{ECA}$ as a vector space

Let $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)$ denote the stack of extensions of $\mathcal{T}_X$ by Ω^1_X (in the category of $\mathcal{O}_X$-modules). The passage from an exact Courant algebroid to the associated extension as above gives rise to the faithful functor

$$\mathcal{ECA}_{\mathcal{O}_X}\to\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X).$$

The morphism (of complexes) $(\Omega^2_X\to\Omega^{3,cl}_X)\to\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)$ defined by $\Omega^2_X\ni\omega\mapsto(\xi\mapsto\iota_\xi\omega)$ induces the 'change of the structure group' functor

$$(\Omega^2_X\to\Omega^{3,cl}_X)\text{-torsors}\to\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)\text{-torsors}$$

and the diagram

$$\begin{array}{ccc}\mathcal{ECA}_{\mathcal{O}_X} & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)\\ {\scriptstyle(3.8.1)}\downarrow & & \downarrow\\ (\Omega^2_X\to\Omega^{3,cl}_X)\text{-torsors} & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)\text{-torsors}\end{array}$$

is, clearly, commutative: both compositions (from the upper left corner to the lower right corner) consist of taking the torsor of (locally defined) splittings of the underlying extension of $\mathcal{T}_X$ by Ω^2_X. Note that, with the exception of the upper left corner, all stacks in the above diagram have canonical structures of stacks in '$\mathbb{C}$-vector spaces in categories', and that the morphisms between them respect these structures. Below we will explicate the structure of a stack in '$\mathbb{C}$-vector spaces in categories' on $\mathcal{ECA}_{\mathcal{O}_X}$ such that the equivalence of Lemma 3.14 as well as the forgetful functor to $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{T}_X,\Omega^1_X)$ are morphisms of such.

Namely, given exact Courant algebroids $\mathcal{Q}_1,\ldots,\mathcal{Q}_n$ and complex numbers $\lambda_1,\ldots,\lambda_n$, the 'linear combination' $\lambda_1\mathcal{Q}_1+\cdots+\lambda_n\mathcal{Q}_n$ is as an exact Courant algebroid $\mathcal{Q}$ together with an

$\mathcal{O}_X$-linear map of Leibniz algebras

$$\mathcal{Q}_1 \times_{\mathcal{T}_X} \cdots \times_{\mathcal{T}_X} \mathcal{Q}_n \to \mathcal{Q}$$

(with respect to the componentwise bracket on $\mathcal{Q}_1 \times_{\mathcal{T}_X} \cdots \times_{\mathcal{T}_X} \mathcal{Q}_n$) which commutes with the respective projections to $\mathcal{T}_X$ and satisfies

$$(\partial_1(f_1), \dots, \partial_n(f_n)) \mapsto \lambda_1\partial(f_1) + \cdots + \lambda_n\partial(f_n),$$

where $f_i \in \mathcal{O}_X$ and ∂_i (respectively, ∂) is the derivation $\mathcal{O}_X \to \mathcal{Q}_i$ (respectively, $\mathcal{O}_X \to \mathcal{Q}$).

4.2 The action of $\mathcal{ECA}_{\mathcal{O}_X}$

As before, $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$, $\mathfrak{g}$ denotes $\mathfrak{g}(\mathcal{A})$, $\langle\ ,\ \rangle$ is an $\mathcal{O}_X$-bilinear symmetric $\mathcal{A}$-invariant pairing on $\mathfrak{g}$, and $\widehat{\mathfrak{g}}$ is the Courant extension of $\mathfrak{g}$ constructed in § 3.2.

Let $\mathcal{CExt}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ denote the substack of Courant extensions of $\mathcal{A}$ which induce the given pairing $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Note that, if $\widehat{\mathcal{A}}$ is in $\mathcal{CExt}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$, then $\mathfrak{g}(\widehat{\mathcal{A}})$ is canonically isomorphic to $\widehat{\mathfrak{g}}$.

Suppose that $\mathcal{Q}$ is an exact Courant $\mathcal{O}_X$-algebroid and $\widehat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. The 'translate by $\mathcal{Q}$ of $\widehat{\mathcal{A}}$' is a Courant extension $\mathcal{Q}+\widehat{\mathcal{A}}$ of $\mathcal{A}$ together with an $\mathcal{O}_X$-linear map of Leibniz algebras

$$\mathcal{Q} \times_{\mathcal{T}_X} \widehat{\mathcal{A}} \to \mathcal{Q} + \widehat{\mathcal{A}}$$

which commutes with respective projections to $\mathcal{T}_X$ and satisfies

$$(\partial_{\mathcal{Q}}(f), \partial_{\widehat{\mathcal{A}}}(g)) \mapsto \partial(f) + \partial(g),$$

where $f, g \in \mathcal{O}_X$, and $\partial_{\widehat{\mathcal{A}}}$, $\partial_{\mathcal{Q}}$, ∂ are the derivations of $\widehat{\mathcal{A}}$, $\mathcal{Q}$, and $\mathcal{Q}+\widehat{\mathcal{A}}$ respectively. In other words, $\mathcal{Q}+\widehat{\mathcal{A}}$ is the push-out of $\mathcal{Q}\times_{\mathcal{T}_X}\widehat{\mathcal{A}}$ by the addition map $\Omega^1_X \times \Omega^1_X \xrightarrow{+} \Omega^1_X$. Thus, a section of $\mathcal{Q}+\widehat{\mathcal{A}}$ is represented by a pair (q, a) with $a \in \widehat{\mathcal{A}}$ and $q \in \mathcal{Q}$ satisfying $\pi(a) = \pi(q) \in \mathcal{T}_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form $(\alpha, -\alpha)$ for some $\alpha \in \Omega^1_X$.

For $a_i \in \widehat{\mathcal{A}}$, $q_i \in \mathcal{Q}$ with $\pi(a_i) = \pi(q_i)$ let

$$\begin{aligned} [(q_1, a_1), (q_2, a_2)] &= ([q_1, q_2], [a_1, a_2]), \\ \langle (q_1, a_1), (q_2, a_2)\rangle &= \langle q_1, q_2\rangle + \langle a_1, a_2\rangle. \end{aligned} \tag{4.2.1}$$

These operations are easily seen to descend to $\mathcal{Q}+\widehat{\mathcal{A}}$. The derivation $\partial : \mathcal{O}_X \to \mathcal{Q}+\widehat{\mathcal{A}}$ is defined as the composition

$$\mathcal{O}_X \xrightarrow{\Delta} \mathcal{O}_X \times \mathcal{O}_X \xrightarrow{\partial\times\partial} \mathcal{Q}\times_{\mathcal{T}_X}\widehat{\mathcal{A}} \to \mathcal{Q}+\widehat{\mathcal{A}}. \tag{4.2.2}$$

LEMMA 4.1.

(i) *The formulas (4.2.1) and the map (4.2.2) determine a structure of Courant extension of $\mathcal{A}$ on $\mathcal{Q}+\widehat{\mathcal{A}}$.*

(ii) *The map $\mathfrak{g}(\widehat{\mathcal{A}}) \to \mathcal{Q}+\widehat{\mathcal{A}}$ defined by $a \mapsto (0, a)$ induces an isomorphism $\mathfrak{g}(\mathcal{Q}+\widehat{\mathcal{A}}) \overset{\sim}{=} \mathfrak{g}(\widehat{\mathcal{A}})$ of Courant extensions of $\mathfrak{g}(\mathcal{A})$ (by Ω^1_X).*

(iii) *Suppose that $\widehat{\mathcal{A}}^{(1)}$, $\widehat{\mathcal{A}}^{(2)}$ are in $\mathcal{CExt}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$. Then, there exists a unique $\mathcal{Q}$ in $\mathcal{ECA}_{\mathcal{O}_X}$, such that $\widehat{\mathcal{A}}^{(2)} = \mathcal{Q}+\widehat{\mathcal{A}}^{(1)}$.*

Proof. We leave the verification of the first claim to the reader.

Let $\mathcal{Q}$ denote the quotient of $\widehat{\mathcal{A}}^{(2)} \times_{\mathcal{A}} \widehat{\mathcal{A}}^{(1)}$ by the diagonally embedded copy of $\widehat{\mathfrak{g}}$. Then, $\mathcal{Q}$ is an extension of $\mathcal{T}$ by Ω^1_X.

The sheaf $\widehat{\mathcal{A}}^{(2)} \times_{\mathcal{A}} \widehat{\mathcal{A}}^{(1)}$ is a Leibniz algebra with respect to the bracket

$$[(q_2, q_1), (q_2', q_1')] = ([q_2, q_2'], [q_1, q_1'])$$

and carries the symmetric pairing

$$\langle (q_2, q_1), (q_2', q_1') \rangle = \langle q_2, q_2' \rangle - \langle q_1, q_1' \rangle,$$

where $q_i, q_i' \in \widehat{\mathcal{A}}^{(i)}$.

Since, for any Courant algebroid $\mathcal{P}$ the adjoint action of $\mathcal{P}$ on $\mathfrak{g}(\mathcal{P})$ and the pairing with the latter factor through $\overline{\mathcal{P}}$, it follows that the diagonally embedded copy of $\widehat{\mathfrak{g}}$ is a Leibniz ideal and the null space of the pairing on $\widehat{\mathcal{A}}^{(2)} \times_{\mathcal{A}} \widehat{\mathcal{A}}^{(1)}$. Therefore, the Leibniz bracket and the pairing descend to $\mathcal{Q}$.

The derivation $\partial : \mathcal{O}_X \to \mathcal{Q}$ is given by (the image in $\mathcal{Q}$ of)

$$\partial(f) = (\partial_2(f) - \partial_1(f)),$$

for $f \in \mathcal{O}_X$, where ∂_i is the derivation $\mathcal{O}_X \to \widehat{\mathcal{A}}^{(i)}$.

The Leibniz bracket, the symmetric pairing and the derivation as above are easily seen to define a structure of a (an exact) Courant algebroid on $\mathcal{Q}$. We claim that $\widehat{\mathcal{A}}^{(2)} = \mathcal{Q} + \widehat{\mathcal{A}}^{(1)}$.

To this end, note that the natural embedding $\widehat{\mathcal{A}}^{(2)} \times_{\mathcal{A}} \widehat{\mathcal{A}}^{(1)} \hookrightarrow \widehat{\mathcal{A}}^{(2)} \times_{\mathcal{T}_X} \widehat{\mathcal{A}}^{(1)}$ induces the embedding $\mathcal{Q} \hookrightarrow \widehat{\mathcal{A}}^{(2)} - \widehat{\mathcal{A}}^{(1)}$, where the latter is the Baer difference of extensions of $\mathcal{T}_X$ by $\widehat{\mathfrak{g}}$, hence the maps

$$\mathcal{Q} \times_{\mathcal{T}_X} \widehat{\mathcal{A}}^{(1)} \hookrightarrow (\widehat{\mathcal{A}}^{(2)} - \widehat{\mathcal{A}}^{(1)}) \times_{\mathcal{T}_X} \widehat{\mathcal{A}}^{(1)} \to (\widehat{\mathcal{A}}^{(2)} - \widehat{\mathcal{A}}^{(1)}) + \widehat{\mathcal{A}}^{(1)} \cong \widehat{\mathcal{A}}^{(2)},$$

where the last isomorphism is the canonical isomorphism of the Baer arithmetic of extensions of $\mathcal{T}_X$ by $\widehat{\mathfrak{g}}$. We leave it to the reader to check that the composition $\mathcal{Q} \times_{\mathcal{T}_X} \widehat{\mathcal{A}}^{(1)} \to \widehat{\mathcal{A}}^{(2)}$ induces a morphism $\mathcal{Q} + \widehat{\mathcal{A}}^{(1)} = \widehat{\mathcal{A}}^{(2)}$ of Courant extensions of $\mathcal{A}$. □

4.3 Cancellation

Suppose that $\widehat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. Let $\Delta : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}} \times_{\mathcal{A}} \widehat{\mathcal{A}}$ denote the diagonal embedding. By definition (see the proof of Lemma 4.1), $\widehat{\mathcal{A}} - \widehat{\mathcal{A}} = \widehat{\mathcal{A}} \times_{\mathcal{A}} \widehat{\mathcal{A}} / \Delta(\widehat{\mathfrak{g}})$. Therefore, the map $\widehat{\mathcal{A}} \to \widehat{\mathcal{A}} - \widehat{\mathcal{A}}$ (the composition of the diagonal with the projection) factors through the map

$$\widehat{\mathcal{A}}/\widehat{\mathfrak{g}} = \mathcal{T}_X \to \widehat{\mathcal{A}} - \widehat{\mathcal{A}}$$

which is easily seen to be a section of the projection $\widehat{\mathcal{A}} - \widehat{\mathcal{A}} \to \mathcal{T}_X$ and, in fact, a flat connection on the exact Courant algebroid $\widehat{\mathcal{A}} - \widehat{\mathcal{A}}$. Equivalently, there is a canonical isomorphism $\mathcal{Q}_0 \cong \widehat{\mathcal{A}} - \widehat{\mathcal{A}}$.

More generally, suppose that $\mathcal{Q}$ is an exact Courant algebroid and $\phi : \mathcal{Q} + \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}$ is a morphism of Courant extensions of $\mathcal{A}$. The (iso)morphism ϕ induces a flat connection on $\mathcal{Q}$, i.e. an isomorphism $\mathcal{Q} \cong \mathcal{Q}_0$ which is the composition

$$\mathcal{Q}_0 \cong \widehat{\mathcal{A}} - \widehat{\mathcal{A}} \xleftarrow{\phi - \mathrm{id}} (\mathcal{Q} + \widehat{\mathcal{A}}) - \widehat{\mathcal{A}} = \mathcal{Q} + (\widehat{\mathcal{A}} - \widehat{\mathcal{A}}) \cong \mathcal{Q} + \mathcal{Q}_0 = \mathcal{Q}.$$

4.4 The stack of Courant extensions

Suppose that $\mathcal{A}$ is a transitive Lie algebroid locally free of finite rank, $\mathfrak{g} := \mathfrak{g}(\mathcal{A})$, and $\langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}$. Thus, the Pontryagin class $\Pi(\mathcal{A}, \langle\ ,\ \rangle) \in H^4(X; \tau_{\leqslant 4}\Omega_X^{\geqslant 2})$ is defined (see § A.3 of Appendix A).

Lemma 4.2. *The stack $\mathcal{CExt}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ is locally non-empty if and only if, locally on X, $\mathcal{A}$ admits a connection with exact Pontryagin form.*

Proof. Suppose that $\widehat{\mathcal{A}}$ is a locally defined Courant extension of $\mathcal{A}$. By Lemma 2.5, $\widehat{\mathcal{A}}$ admits a (locally defined) connection, say, $\widehat{\nabla}$. Let ∇ denote the (locally defined) induced connection on $\mathcal{A}$. Then, according to Corollary 3.3, $\langle c(\nabla), c(\nabla) \rangle = 2dc_{\mathrm{rel}}(\widehat{\nabla})$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Conversely, suppose that, for a locally defined connection ∇ on $\mathcal{A}$, there exists a (locally defined) form $H \in \Omega^3_X$ such that $2dH = \langle c(\nabla), c(\nabla)\rangle$. Then, $\widehat{\mathcal{A}}_{\nabla,H}$ is a (locally defined) object of $\mathcal{CExT}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$. □

Corollary 4.1. *The transitive Lie algebroid $\mathcal{A}$ admits a connection with exact Pontryagin form locally on X if and only if $\mathcal{CExT}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ is a $\mathcal{ECA}_{\mathcal{O}_X}$-torsor.*

Proof. This follows from Lemma 4.1. □

The $\mathcal{ECA}_{\mathcal{O}_X}$-torsors are classified by $H^2(X; \Omega^2_X \to \Omega^{3,cl}_X)$.

Theorem 4.1. *Suppose that $\mathcal{A}$ admits a connection with exact Pontryagin form locally on X. Then, the image of the class of $\mathcal{CExT}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ under the natural map $H^2(X; \Omega^2_X \to \Omega^{3,cl}_X) \to H^4(X; \tau_{\leqslant 4}\Omega^{\geqslant 2}_X)$ is equal to $-\frac{1}{2}\Pi(\mathcal{A}, \langle\ ,\ \rangle)$, where $\Pi(\mathcal{A}, \langle\ ,\ \rangle)$ is the first Pontryagin class of $(\mathcal{A}, \langle\ ,\ \rangle)$.*

Proof. By assumption, there exists a cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X by open sets, connections ∇_i on $\mathcal{A}_i := \mathcal{A}|_{U_i}$ and forms $H_i \in \Omega^3_X(U_i)$ such that $dH_i = \mathcal{P}_i := \frac{1}{2}\langle c(\nabla_i) \wedge c(\nabla_i)\rangle \in \Omega^4_X(U_i)$. Let $A_{ij} = \nabla_j - \nabla_i \in \Omega^1_X(U_i \cap U_j) \otimes_{\mathcal{O}_X} \mathfrak{g}$.

Let $\widehat{\mathcal{A}}_i := \widehat{\mathcal{A}}_{\nabla_i,H_i}$ denote the corresponding Courant extensions with connections $\widehat{\nabla}_i$, so that $c_{\mathrm{rel}}(\widehat{\nabla}_i) = H_i$. The Courant extensions $\widehat{\mathcal{A}}_i$ form a system of local trivializations of the stack $\mathcal{CExT}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$.

The collection of forms $\mathcal{P}_i \in \Omega^4_X(U_i)$ (respectively, $H_i \in \Omega^3_X(U_i)$) constitutes the cochain $\mathcal{P}^{4,0} \in \check{C}^0(\mathcal{U}; \Omega^4_X)$ (respectively, $H \in \check{C}^0(\mathcal{U}; \Omega^3_X)$) which satisfies $dH = \mathcal{P}^{4,0}$.

Let $\mathcal{P}_{ij} = \mathcal{P}(\nabla_i, \nabla_j) \in \Omega^3_X(U_i \cap U_j)$ (defined by formula (3.6.1)), so that $d\mathcal{P}_{ij} = \mathcal{P}_j - \mathcal{P}_i$ (Lemma 3.9). Let $\widehat{\mathcal{P}}_{ij} = -H_j + H_i + \mathcal{P}_{ij}$. The forms $\widehat{\mathcal{P}}_{ij}$ are closed:

$$
\begin{aligned}
d\widehat{\mathcal{P}}_{ij} &= -dH_j + dH_i + d\mathcal{P}(\nabla_i, \nabla_j) = -dH_j + dH_i + \mathcal{P}_j - \mathcal{P}_i \\
&= -(dH_j - \mathcal{P}_j) + (dH_i - \mathcal{P}_i) = 0.
\end{aligned}
$$

The collection of forms $\mathcal{P}_{ij}$ (respectively, $\widehat{\mathcal{P}}_{ij}$) constitutes the cochain $\mathcal{P}^{3,1} \in \check{C}^1(\mathcal{U}; \Omega^3_X)$ (respectively, $\widehat{\mathcal{P}}^{3,1} \in \check{C}^1(\mathcal{U}; \Omega^3_X)$). These satisfy $\widehat{\mathcal{P}}^{3,1} = \mathcal{P}^{3,1} - \check{\partial}H$, $d\mathcal{P}^{3,1} = \check{\partial}\mathcal{P}^{4,0}$, $d\widehat{\mathcal{P}}^{3,1} = 0$.

Let $\mathcal{Q}_{ij} := \mathcal{Q}_{\widehat{\mathcal{P}}_{ij}}$ be the exact Courant algebroid with connection $\widehat{\nabla}_{ij}$ with $c(\widehat{\nabla}_{ij}) = \widehat{\mathcal{P}}_{ij}$. Since $H_j + \widehat{\mathcal{P}}_{ij} = H_i + \mathcal{P}_{ij}$, there are morphisms (on $U_i \cap U_j$)

$$
\mathcal{Q}_{ij} + \widehat{\mathcal{A}}_j \to \widehat{\mathcal{A}}_{\nabla_j, H_j + \mathcal{P}_{ij}} \to \widehat{\mathcal{A}}_i,
$$

of which the first one is defined by the formula $(\alpha + \xi, \beta + a + \xi) \mapsto ((\alpha + \beta) + a + \xi)$ (where $\alpha, \beta \in \Omega^1_X$, $a \in \mathfrak{g}$, $\xi \in \mathcal{T}_X$), while the second one is supplied by Corollary 3.3 (and given by the formula of Lemma 3.10). Let $\phi_{ij} : \mathcal{Q}_{ij} + \widehat{\mathcal{A}}_j \to \widehat{\mathcal{A}}_i$ denote the composition of the above maps. The composition $\phi_{ij} \circ (\mathrm{id} + \phi_{jk}) \circ (\mathrm{id} + \mathrm{id} + \phi_{ki})$:

$$
\begin{aligned}
(\mathcal{Q}_{ij} + \mathcal{Q}_{jk} + \mathcal{Q}_{ki}) + \widehat{\mathcal{A}}_i &= \mathcal{Q}_{ij} + (\mathcal{Q}_{jk} + (\mathcal{Q}_{ki} + \widehat{\mathcal{A}}_i)) \\
&\to \mathcal{Q}_{ij} + (\mathcal{Q}_{jk} + \widehat{\mathcal{A}}_k) \to \mathcal{Q}_{ij} + \widehat{\mathcal{A}}_j \to \widehat{\mathcal{A}}_i
\end{aligned}
$$

(defined on $U_i \cap U_j \cap U_k$) gives rise, according to § 4.3, to the morphism

$$
\mathcal{Q}_{ij} + \mathcal{Q}_{jk} + \mathcal{Q}_{ki} \to \mathcal{Q}_0,
$$

or, equivalently, to the flat connection ∇^0_{ijk} on $\mathcal{Q}_{ij} + \mathcal{Q}_{jk} + \mathcal{Q}_{ki}$.

On the other hand, $\mathcal{Q}_{ij} + \mathcal{Q}_{jk} + \mathcal{Q}_{ki}$ is canonically isomorphic to $\mathcal{Q}_{\widehat{\mathcal{P}}_{ij} + \widehat{\mathcal{P}}_{jk} + \widehat{\mathcal{P}}_{ki}}$, the exact Courant algebroid with connection ∇_{ijk} whose curvature is $\widehat{\mathcal{P}}_{ij} + \widehat{\mathcal{P}}_{jk} + \widehat{\mathcal{P}}_{ki} = \mathcal{P}_{ij} + \mathcal{P}_{jk} + \mathcal{P}_{ki}$.

The difference $\widehat{\mathcal{P}}_{ijk} := \nabla_{ijk} - \nabla^0_{ijk}$ is a 2-form on $U_i \cap U_j \cap U_k$ which satisfies $d\widehat{\mathcal{P}}_{ijk} = \mathcal{P}_{ij} + \mathcal{P}_{jk} + \mathcal{P}_{ki}$. The collection of forms $\widehat{\mathcal{P}}_{ijk} \in \Omega^2_X(U_i \cap U_j \cap U_k)$ forms the cochain $\widehat{\mathcal{P}}^{2,2} \in \check{C}^2(\mathcal{U}; \Omega^2_X)$ which satisfies $d\widehat{\mathcal{P}}^{2,2} = \check{\partial}\widehat{\mathcal{P}}^{3,1} = \check{\partial}\mathcal{P}^{3,1}$ and $\check{\partial}\widehat{\mathcal{P}}^{2,2} = 0$.

After the identification of the $\mathcal{O}_X$-modules underlying the Courant extensions $\widehat{\mathcal{A}}_i$, $\widehat{\mathcal{A}}_j$ and $\widehat{\mathcal{A}}_k$ with $\Omega^1_X \oplus \mathfrak{g} \oplus \mathcal{T}_X$, the composition $\phi_{ij} \circ (\mathrm{id} + \phi_{jk}) \circ (\mathrm{id} + \mathrm{id} + \phi_{ki})$ becomes $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \exp(-\frac{1}{2}\langle A_{ij} \wedge A_{jk}\rangle)$, the last equality due to Lemma 3.11. It follows from Remark 3.1 that $\widehat{\mathcal{P}}_{ijk} = -\frac{1}{2}\langle A_{ij} \wedge A_{jk}\rangle$.

Let $\mathcal{P} = \mathcal{P}^{4,0} - \mathcal{P}^{3,1} + \widehat{\mathcal{P}}^{2,2}$ and $\widehat{\mathcal{P}} = -\widehat{\mathcal{P}}^{3,1} + \widehat{\mathcal{P}}^{2,2}$. Let $d \pm \check{\partial}$ denote the differential in the total complex $\check{C}^\bullet(\mathcal{U}; \Omega^\bullet_X)$: for $B \in \check{C}^i(\mathcal{U}; \Omega^j_X)$, we have $(d \pm \check{\partial})B = dB + (-1)^j \check{\partial}B$. Then, $(d \pm \check{\partial})\mathcal{P} = (d \pm \check{\partial})\widehat{\mathcal{P}} = 0$ and $\widehat{\mathcal{P}} = \mathcal{P} + (d \pm \check{\partial})H$, i.e. $\widehat{\mathcal{P}}$ and $\mathcal{P}$ are cohomologous cycles in $\check{C}^\bullet(\mathcal{U}; \tau_{\leqslant 4}\Omega^{\geqslant 2}_X)$. A comparison of $\mathcal{P}$ with the first Pontryagin class $\pi(\mathcal{A}, \langle\ ,\ \rangle)$ (calculated in § A.3) reveals that $\mathcal{P} = \frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$.

By construction, the class of $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})$ is represented by $-\widehat{\mathcal{P}}$ viewed as a cocycle of total degree two in $\check{C}^\bullet(\mathcal{U}; \Omega^2_X \to \Omega^{3,cl})$ (because, by our definition, $\mathcal{Q}_{ij}$ represents $\widehat{\mathcal{A}}_i - \widehat{\mathcal{A}}_j$ as opposed to $\widehat{\mathcal{A}}_j - \widehat{\mathcal{A}}_i$) whose image (under the 'shift by two' isomorphism which is equal to the identity map) is $-\widehat{\mathcal{P}}$ viewed as a cocycle of total degree four in $\check{C}^\bullet(\mathcal{U}; \tau_{\leqslant 4}\Omega^{\geqslant 2}_X)$. Now $-\widehat{\mathcal{P}}$ is cohomologous to $-\mathcal{P} = -\frac{1}{2}\pi(\mathcal{A}, \langle\ ,\ \rangle)$, which finishes the proof. □

In view of (A.4.1), Theorem 4.1 can be restated in the following way for GL_n-torsors (equivalently, vector bundles). In this setting we will write $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A}_{\mathcal{E}})_{\mathrm{Tr}}$ for $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A}_{\mathcal{E}})_{\langle\ ,\ \rangle}$ as a reminder of the origins of the canonical pairing on the Lie algebra of endomorphisms of a vector bundle.

Corollary 4.2. *Suppose that $\mathcal{E}$ is a vector bundle on X. The class of the $\mathcal{ECA}_{\mathcal{O}_X}$-torsor $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A}_{\mathcal{E}})_{\mathrm{Tr}}$ is equal to $-\mathrm{ch}_2(\mathcal{E})$.*

5. Vertex algebroids

5.1 Vertex operator algebras

Throughout this section we follow the notation of [GMS04]. The following definitions are lifted from that work.

Definition 5.1. A $\mathbb{Z}_{\geqslant 0}$*-graded vertex algebra* is a $\mathbb{Z}_{\geqslant 0}$-graded k-module $V = \bigoplus V_i$, equipped with a distinguished vector $\mathbf{1} \in V_0$ (*vacuum vector*) and a family of bilinear operations

$$_{(n)} : \ V \times V \to V, \quad (a, b) \mapsto a_{(n)}b$$

of degree $-n-1$, $n \in \mathbb{Z}$, such that

$$\mathbf{1}_{(n)}a = \delta_{n,-1}a, \quad a_{(-1)}\mathbf{1} = a, \quad a_{(n)}\mathbf{1} = 0 \quad \text{if } n \geqslant 0,$$

and

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+l-j)}c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \{a_{(m+n-j)}b_{(l+j)}c - (-1)^n b_{(n+l-j)}a_{(m+j)}c\}$$

for all $a, b, c \in V$ and $m, n, l \in \mathbb{Z}$.

A morphism of vertex algebras is a map of graded k-modules (of degree zero) which maps the vacuum vector to the vacuum vector and commutes with all of the operations.

Let $\mathcal{V}ert$ denote the category of vertex algebras.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Let

$$\partial^{(j)}a := a_{(-1-j)}\mathbf{1}, \quad j \in \mathbb{Z}_{\geqslant 0}.$$

Then, $\partial^{(j)}$ is an endomorphism of V of degree j which satisfies (see [GMS04]):

(a) $\partial^{(j)}\mathbf{1} = \delta_{j,0}\mathbf{1}$;

(b) $\partial^{(0)} = \mathrm{Id}$;

(c) $\partial^{(i)} \cdot \partial^{(j)} = \binom{i+j}{i}\partial^{(i+j)}$;

(d) $(\partial^{(j)}a)_{(n)}b = (-1)^j\binom{n}{j}a_{(n-j)}b$;

(e) $\partial^{(j)}(a_{(n)}b) = \sum_{p=0}^{j} (\partial^{(p)}a)_{(n)}\partial^{(j-p)}b$;

for all $n \in \mathbb{Z}$.

The subject of the definition below is the restriction of the structure of a vertex algebra to the graded components of degrees zero and one.

DEFINITION 5.2. A 1-*truncated vertex algebra* is a septuple $v = (V_0, V_1, \mathbf{1}, \partial, {}_{(-1)}, {}_{(0)}, {}_{(1)})$ where:

(i) V_0, V_1 are k-modules;

(ii) $\mathbf{1}$ is an element of V_0 (*vacuum vector*);

(iii) $\partial : V_0 \to V_1$ is a k-linear map;

(iv) ${}_{(i)} : (V_0 \oplus V_1) \times (V_0 \oplus V_1) \to V_0 \oplus V_1$ (where $(i = -1, 0, 1)$) are k-bilinear operations of degree $-i-1$.

Elements of V_0 (respectively, V_1) will be denoted a, b, c (respectively, x, y, z). There are seven operations: $a_{(-1)}b, a_{(-1)}x, x_{(-1)}a, a_{(0)}x, x_{(0)}a, x_{(0)}y$ and $x_{(1)}y$. These are required to satisfy the following axioms.

Vacuum

$a_{(-1)}\mathbf{1} = a;\ x_{(-1)}\mathbf{1} = x;\ x_{(0)}\mathbf{1} = 0.$

Derivation

Deriv$_1$ $(\partial a)_{(0)}b = 0;\ (\partial a)_{(0)}x = 0;\ (\partial a)_{(1)}x = -a_{(0)}x.$

Deriv$_2$ $\partial(a_{(-1)}b) = (\partial a)_{(-1)}b + a_{(-1)}\partial b;\ \partial(x_{(0)}a) = x_{(0)}\partial a.$

Commutativity

Comm$_{-1}$ $a_{(-1)}b = b_{(-1)}a;\ a_{(-1)}x = x_{(-1)}a - \partial(x_{(0)}a).$

Comm$_0$ $x_{(0)}a = -a_{(0)}x;\ x_{(0)}y = -y_{(0)}x + \partial(y_{(1)}x).$

Comm$_1$ $x_{(1)}y = y_{(1)}x.$

Associativity

Assoc$_{-1}$ $(a_{(-1)}b)_{(-1)}c = a_{(-1)}b_{(-1)}c.$

Assoc$_0$ $\alpha_{(0)}\beta_{(i)}\gamma = (\alpha_{(0)}\beta)_{(i)}\gamma + \beta_{(i)}\alpha_{(0)}\gamma,\ (\alpha, \beta, \gamma \in V_0 \oplus V_1)$ whenever both sides are defined, i.e. the operation ${}_{(0)}$ is a derivation of all of the operations ${}_{(i)}$.

Assoc$_1$ $(a_{(-1)}x)_{(0)}b = a_{(-1)}x_{(0)}b.$

Assoc$_2$ $(a_{(-1)}b)_{(-1)}x = a_{(-1)}b_{(-1)}x + (\partial a)_{(-1)}b_{(0)}x + (\partial b)_{(-1)}a_{(0)}x.$

Assoc$_3$ $(a_{(-1)}x)_{(1)}y = a_{(-1)}x_{(1)}y - x_{(0)}y_{(0)}a.$

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

A *morphism* between two 1-truncated vertex algebras $f : v = (V_0, V_1, \dots) \to v' = (V'_0, V'_1, \dots)$ is a pair of maps of k-modules $f = (f_0, f_1)$, $f_i : V_i \to V'_i$ such that $f_0(\mathbf{1}) = \mathbf{1}'$, $f_1(\partial a) = \partial f_0(a)$ and $f(\alpha_{(i)}\beta) = f(\alpha)_{(i)} f(\beta)$, whenever both sides are defined.

Let $\mathcal{V}ert_{\leqslant 1}$ denote the category of 1-truncated vertex algebras. We have an obvious truncation functor

$$t : \mathcal{V}ert \to \mathcal{V}ert_{\leqslant 1} \tag{5.1.1}$$

which assigns to a vertex algebra $V = \bigoplus_i V_i$ the truncated algebra $tV := (V_0, V_1, \partial^{(1)}, \mathbf{1}, {}_{(-1)}, {}_{(0)}, {}_{(1)})$.

Remark 5.1. It follows easily that the operation ${}_{(-1)} : V_0 \times V_0 \to V_0$ endows V_0 with a structure of a commutative k-algebra.

5.2 Vertex algebroids

Suppose that X is a smooth variety over $\mathbb{C}$ (a complex manifold, a C^∞-manifold). In either case we will denote by $\mathcal{O}_X$ (respectively, $\mathcal{T}_X$, Ω^i_X) the corresponding structure sheaf (respectively, the sheaf of vector fields, the sheaf of differential i-forms).

A vertex $\mathcal{O}_X$-algebroid, as defined in this section, is, essentially, a sheaf of 1-truncated vertex algebras, whose degree zero component (which is a sheaf of algebras by § 5.1) is identified with $\mathcal{O}_X$.

Definition 5.3. A *vertex $\mathcal{O}_X$-algebroid* is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with a pairing

$$\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V},$$
$$f \otimes v \mapsto f * v,$$

such that $1 * v = v$ (i.e. a 'non-associative unital $\mathcal{O}_X$-module') equipped with:

(i) a structure of a Leibniz $\mathbb{C}$-algebra $[\ ,\ ] : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V}$;

(ii) a $\mathbb{C}$-linear map of Leibniz algebras $\pi : \mathcal{V} \to \mathcal{T}_X$ (the *anchor*);

(iii) a symmetric $\mathbb{C}$-bilinear pairing $\langle\ ,\ \rangle : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{O}_X$;

(iv) a $\mathbb{C}$-linear map $\partial : \mathcal{O}_X \to \mathcal{V}$ such that $\pi \circ \partial = 0$;

which satisfy

$$f * (g * v) - (fg) * v = \pi(v)(f) * \partial(g) + \pi(v)(g) * \partial(f), \tag{5.2.1}$$
$$[v_1, f * v_2] = \pi(v_1)(f) * v_2 + f * [v_1, v_2], \tag{5.2.2}$$
$$[v_1, v_2] + [v_2, v_1] = \partial(\langle v_1, v_2\rangle), \tag{5.2.3}$$
$$\pi(f * v) = f\pi(v), \tag{5.2.4}$$
$$\langle f * v_1, v_2\rangle = f\langle v_1, v_2\rangle - \pi(v_1)(\pi(v_2)(f)), \tag{5.2.5}$$
$$\pi(v)(\langle v_1, v_2\rangle) = \langle [v, v_1], v_2\rangle + \langle v_1, [v, v_2]\rangle, \tag{5.2.6}$$
$$\partial(fg) = f * \partial(g) + g * \partial(f), \tag{5.2.7}$$
$$[v, \partial(f)] = \partial(\pi(v)(f)), \tag{5.2.8}$$
$$\langle v, \partial(f)\rangle = \pi(v)(f), \tag{5.2.9}$$

for $v, v_1, v_2 \in \mathcal{V}$ and $f, g \in \mathcal{O}_X$.

A morphism of vertex $\mathcal{O}_X$-algebroids is a $\mathbb{C}$-linear map of sheaves which preserves all of the structures.

We denote the category of vertex $\mathcal{O}_X$-algebroids by $\mathcal{VA}_{\mathcal{O}_X}(X)$. It is clear that the notion of vertex $\mathcal{O}_X$-algebroid is local, i.e. vertex $\mathcal{O}_X$-algebroids form a stack which we denote by $\mathcal{VA}_{\mathcal{O}_X}$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

5.3 From vertex algebroids to 1-truncated vertex algebras

Suppose that $\mathcal{V}$ is a vertex $\mathcal{O}_X$-algebroid. For $f, g \in \mathcal{O}_X$ and $v, w \in \mathcal{V}$ let

$$f_{(-1)}g = fg, \quad f_{(-1)}v = f * v, \quad v_{(-1)}f = f * v - \partial\pi(v)(f), \tag{5.3.1}$$

$$v_{(0)}f = -f_{(0)}v = \pi(v)(f), \quad v_{(0)}w = [v, w], \tag{5.3.2}$$

$$v_{(1)}w = \langle v, w\rangle. \tag{5.3.3}$$

LEMMA 5.1. *The septuple $(\mathcal{O}_X, \mathcal{V}, 1, \partial, {}_{(-1)}, {}_{(0)}, {}_{(1)})$ is a sheaf of 1-truncated vertex operator algebras.*

Conversely, if the septuple $(\mathcal{O}_X, \mathcal{V}, 1, \partial, {}_{(-1)}, {}_{(0)}, {}_{(1)})$ is a sheaf of 1-truncated vertex operator algebras, then the formulas (5.3.1), (5.3.2) and (5.3.3) define a structure of a vertex $\mathcal{O}_X$-algebroid on $\mathcal{V}$.

5.4 Quantization of Courant algebroids

DEFINITION 5.4. We call a vertex algebroid *commutative* if the anchor map, the Leibniz bracket and the symmetric pairing are trivial.

Remark 5.2. Commutativity implies that the $*$-operation is associative, i.e. a commutative vertex algebroid is simply an $\mathcal{O}$-module $\mathcal{E}$ together with a derivation $\partial : \mathcal{O}_X \to \mathcal{E}$.

Suppose that $\mathcal{V}$ is a family of vertex $\mathcal{O}_X$-algebroids flat over $\mathbb{C}[[t]]$, such that the vertex algebroid $\mathcal{V}_0 = \mathcal{V}/t\mathcal{V}$ is commutative. Let $\mathcal{V} \to \mathcal{V}_0 : v \mapsto \overline{v}$ denote the 'reduction modulo t' map. For $f \in \mathcal{O}_X$, $v \in \mathcal{V}$, let $f\overline{v} = \overline{f * v}$. This operation endows $\mathcal{V}_0$ with a structure of a module over $\mathcal{O}_X$.

Since $\mathcal{V}_0$ is commutative, the Leibniz bracket, the symmetric pairing on $\mathcal{V}$ and the anchor map take values in $t\mathcal{V}$. For $f \in \mathcal{O}_X$ and $v, v_1, v_2 \in \mathcal{V}$ let

$$[\overline{v_1}, \overline{v_2}]_0 = \overline{\frac{1}{t}[v_1, v_2]}, \quad \langle\overline{v_1}, \overline{v_2}\rangle_0 = \overline{\frac{1}{t}\langle v_1, v_2\rangle}, \quad \pi_0(\overline{v}) = \overline{\frac{1}{t}\pi(v)}, \quad \partial_0(f) = \overline{\partial(f)}. \tag{5.4.1}$$

LEMMA 5.2. *The formulas (5.4.1) endow the $\mathcal{O}_X$-module $\mathcal{V}_0$ with the structure of a Courant algebroid with derivation ∂_0, anchor map π_0, Leibniz bracket $[\ ,\ ]_0$ and symmetric pairing $\langle\ ,\ \rangle_0$.*

5.5 The associated Lie algebroid

Suppose that $\mathcal{V}$ is a vertex $\mathcal{O}_X$-algebroid. Let

$$\Omega_{\mathcal{V}} \overset{\text{def}}{=} \mathcal{O}_X * \partial(\mathcal{O}_X) \subset \mathcal{V},$$

$$\overline{\mathcal{V}} \overset{\text{def}}{=} \mathcal{V}/\Omega_{\mathcal{V}}.$$

Note that the symmetrization of the Leibniz bracket takes values in $\Omega_{\mathcal{V}}$.

For $f, g, h \in \mathcal{O}_X$, we have

$$f * (g * \partial(h)) - (fg) * \partial(h) = \pi(\partial(h))(f) * \partial(g) + \pi(\partial(h))(g) * \partial(f) = 0,$$

because $\pi \circ \partial = 0$. Therefore, $\mathcal{O}_X * \Omega_{\mathcal{V}} = \Omega_{\mathcal{V}}$, and $\Omega_{\mathcal{V}}$ is an $\mathcal{O}_X$-module. The map $\partial : \mathcal{O}_X \to \Omega_{\mathcal{V}}$ is a derivation, hence induces the $\mathcal{O}_X$-linear map $\Omega^1_X \to \Omega_{\mathcal{V}}$.

Since the associator of the $\mathcal{O}_X$-action on $\mathcal{V}$ takes values in $\Omega_{\mathcal{V}}$, $\overline{\mathcal{V}}$ is an $\mathcal{O}_X$-module.

For $f, g, h \in \mathcal{O}_X$, we have

$$\pi(f\partial(g))(h) = f\pi(\partial(g))(h) = 0.$$

Therefore, π vanishes on $\Omega_{\mathcal{V}}$, hence factors through the map

$$\pi : \overline{\mathcal{V}} \to \mathcal{T}_X \tag{5.5.1}$$

of $\mathcal{O}_X$-modules.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

For $v \in \mathcal{V}$ and $f, g \in \mathcal{O}_X$, we have

$$\begin{aligned}[v, f\partial(g)] &= \pi(v)(f)\partial(g) + f[v, \partial(g)] \\ &= \pi(v)(f)\partial(g) + f\partial(\pi(v)(g)).\end{aligned}$$

Therefore, $[\mathcal{V}, \Omega_{\mathcal{V}}] \subseteq \Omega_{\mathcal{V}}$ and the Leibniz bracket on $\mathcal{V}$ descends to the operation

$$[\ ,\ ] : \overline{\mathcal{V}} \otimes_{\mathbb{C}} \overline{\mathcal{V}} \to \overline{\mathcal{V}}, \tag{5.5.2}$$

which is skew-symmetric because the symmetrization of the Leibniz bracket on $\mathcal{V}$ takes values in $\Omega_{\mathcal{V}}$ and satisfies the Jacobi identity because the Leibniz bracket on $\mathcal{V}$ does.

Lemma 5.3. *The $\mathcal{O}_X$-module $\overline{\mathcal{V}}$ with the bracket (5.5.2) and the anchor (5.5.1) is a Lie $\mathcal{O}_X$-algebroid.*

5.6 Transitive vertex algebroids

Definition 5.5. A vertex $\mathcal{O}_X$-algebroid is called *transitive* if the anchor map is surjective.

Remark 5.3. The vertex $\mathcal{O}_X$-algebroid $\mathcal{V}$ is called transitive if and only if the Lie $\mathcal{O}_X$-algebroid $\overline{\mathcal{V}}$ is.

Suppose that $\mathcal{V}$ is a transitive vertex $\mathcal{O}_X$-algebroid. The derivation ∂ induces the map

$$i : \Omega^1_X \to \mathcal{V}.$$

For $v \in \mathcal{V}$ and $f, g \in \mathcal{O}_X$, we have

$$\begin{aligned}\langle v, f\partial(g)\rangle &= f\langle v, \partial(g)\rangle - \pi(\partial(g))\pi(v)(f) \\ &= f\pi(v)(g) \\ &= \iota_{\pi(v)} fdg.\end{aligned}$$

If follows that the map i is adjoint to the anchor map π. The surjectivity of the latter implies the injectivity of the former. Since, in addition, $\pi \circ i = 0$, the sequence

$$0 \to \Omega^1_X \xrightarrow{i} \mathcal{V} \xrightarrow{\pi} \overline{\mathcal{V}} \to 0$$

is exact and i is isotropic.

5.7 Exact vertex algebroids

Definition 5.6. A vertex algebroid $\mathcal{V}$ is called *exact* if the map $\overline{\mathcal{V}} \to \mathcal{T}_X$ is an isomorphism.

Notation 5.1. We denote the stack of exact vertex $\mathcal{O}_X$-algebroids by $\mathcal{EVA}_X$.

A morphism of exact vertex algebroids induces a morphism of respective extensions of $\mathcal{T}_X$ by Ω^1_X, hence is an isomorphism of sheaves of $\mathbb{C}$-vector spaces. It is clear that the inverse isomorphism is a morphism of vertex $\mathcal{O}_X$-algebroids. Hence, $\mathcal{EVA}_X$ is a stack in groupoids.

Example 5.1. Suppose that $\mathcal{T}_X$ is freely generated as an $\mathcal{O}_X$-module by a locally constant subsheaf of Lie $\mathbb{C}$-subalgebras $\tau \subset \mathcal{T}_X$, i.e. the canonical map $\mathcal{O}_X \otimes_{\mathbb{C}} \tau \to \mathcal{T}_X$ is an isomorphism.

There is a unique structure of an exact vertex $\mathcal{O}_X$-algebroid on $\mathcal{V} = \Omega^1_X \oplus \mathcal{T}_X$ such that:

(a) $f * (1 \otimes t) = f \otimes t$ for $f \in \mathcal{O}_X$, $t \in \tau$;

(b) the anchor map is given by the projection $\mathcal{V} \to \mathcal{T}_X$;

(c) the map $\tau \to \mathcal{V}$ is a morphism of Leibniz algebras;

(d) the pairing on $\mathcal{V}$ restricts to the trivial pairing on τ;

(e) the derivation $\partial : \mathcal{O}_X \to \mathcal{V}$ is given by the composition $\mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \mathcal{V}$.

Indeed, the action of $\mathcal{O}_X$ is completely determined by (5.2.1): for $f, g \in \mathcal{O}_X$ and $t \in \tau$, we have

$$f * (g \otimes t) = f * (g * (1 \otimes t)) = fg \otimes t + t(f)dg + t(g)df.$$

In a similar fashion the bracket is completely determined by (5.2.2) and (5.2.3), and the pairing is determined by (5.2.5).

We leave the verification of the identities (5.2.1)–(5.2.9) to the reader.

5.8 Vertex extensions of Lie algebroids

Suppose that $\mathcal{A}$ is a Lie $\mathcal{O}_X$-algebroid.

Definition 5.7. A *vertex extension* of $\mathcal{A}$ is a vertex algebroid $\widehat{\mathcal{A}}$ together with an isomorphism $\overline{\widehat{\mathcal{A}}} = \mathcal{A}$ of Lie $\mathcal{O}_X$-algebroids.

A morphism of vertex extensions of $\mathcal{A}$ is a morphism of vertex algebroids which is compatible with the identifications.

Notation 5.2. We denote by $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{A})$ the stack of vertex extensions of $\mathcal{A}$.

5.9 Vertex extensions of transitive Lie algebroids

Assume from now on that $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$. Let $\mathfrak{g} = \mathfrak{g}(\mathcal{A})$.

Suppose that $\widehat{\mathcal{A}}$ is a vertex extension of $\mathcal{A}$. Then, the derivation $\partial : \mathcal{O}_X \to \widehat{\mathcal{A}}$ induces an isomorphism $\Omega^1_X \cong \Omega_{\widehat{\mathcal{A}}}$. The resulting exact (by the same argument as in Lemma 2.3) sequence

$$0 \to \Omega^1_X \to \widehat{\mathcal{A}} \to \mathcal{A} \to 0$$

is canonically associated to the vertex extension $\widehat{\mathcal{A}}$ of $\mathcal{A}$. Since a morphism of vertex extensions of $\mathcal{A}$ induces a morphism of associated extensions of $\mathcal{A}$ by Ω^1_X it is an isomorphism of the underlying sheaves. It is clear that the inverse isomorphism is a morphism of vertex extensions of $\mathcal{A}$.

Therefore, $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{A})$ is a stack in groupoids.

Remark 5.4. We see that $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{T}_X)$ is none other than $\mathcal{EVA}_{\mathcal{O}_X}$.

Suppose that $\widehat{\mathcal{A}}$ is a vertex extension of $\mathcal{A}$. Let $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{\mathcal{A}})$ denote the kernel of the anchor map (of $\widehat{\mathcal{A}}$). Thus, $\widehat{\mathfrak{g}}$ is a vertex (equivalently, Courant) extension of $\mathfrak{g}$.

Analysis similar to that of § 3.1 shows that:

(i) the symmetric pairing on $\widehat{\mathcal{A}}$ induces a symmetric $\mathcal{O}_X$-bilinear pairing on $\mathfrak{g}$ which is $\mathcal{A}$-invariant;

(ii) the vertex extension $\widehat{\mathfrak{g}}$ is obtained from the Lie algebroid $\mathcal{A}$ and the symmetric $\mathcal{A}$-invariant pairing on $\mathfrak{g}$ as in § 3.2.

5.10 The action of $\mathcal{ECA}_{\mathcal{O}_X}$

As before, $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$, $\mathfrak{g}$ denotes $\mathfrak{g}(\mathcal{A})$, $\langle\ ,\ \rangle$ is an $\mathcal{O}_X$-bilinear symmetric $\mathcal{A}$-invariant pairing on $\mathfrak{g}$, and $\widehat{\mathfrak{g}}$ is the Courant extension of $\mathfrak{g}$ constructed in § 3.2.

Notation 5.3. Let $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ denote the stack of vertex extensions of $\mathcal{A}$ which induce the given pairing $\langle\ ,\ \rangle$ on $\mathfrak{g}$.

Remark 5.5. Clearly, $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ is a stack in groupoids. Note that, if $\widehat{\mathcal{A}}$ is in $\mathcal{VE}xt_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$, then $\mathfrak{g}(\widehat{\mathcal{A}})$ is canonically isomorphic to $\widehat{\mathfrak{g}}$.

Suppose that $\mathcal{Q}$ is an exact Courant $\mathcal{O}_X$-algebroid and $\widehat{\mathcal{A}}$ is a vertex extension of $\mathcal{A}$. Let $\widehat{\mathcal{A}}+\mathcal{Q}$ denote the push-out of $\widehat{\mathcal{A}}\times_{\mathcal{T}_X}\mathcal{Q}$ by the addition map $\Omega^1_X\times\Omega^1_X\xrightarrow{+}\Omega^1_X$. Thus, a section of $\widehat{\mathcal{A}}+\mathcal{Q}$ is represented by a pair (a,q) with $a\in\widehat{\mathcal{A}}$ and $q\in\mathcal{Q}$ satisfying $\pi(a)=\pi(q)\in\mathcal{T}_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form $(\alpha,-\alpha)$ for some $\alpha\in\Omega^1_X$.

For $a\in\widehat{\mathcal{A}}$, $q\in\mathcal{Q}$ with $\pi(a)=\pi(q)$, $f\in\mathcal{O}_X$, let

$$f*(a,q)=(f*a,fq),\quad \partial(f)=\partial_{\widehat{\mathcal{A}}}(f)+\partial_{\mathcal{Q}}(f). \tag{5.10.1}$$

For $a_i\in\widehat{\mathcal{A}}$, $q_i\in\mathcal{Q}$ with $\pi(a_i)=\pi(q_i)$ let

$$[(a_1,q_1),(a_2,q_2)]=([a_1,a_2],[q_1,q_2]),\quad \langle(a_1,q_1),(a_2,q_2)\rangle=\langle a_1,a_2\rangle+\langle q_1,q_2\rangle. \tag{5.10.2}$$

These operations are easily seen to descend to $\widehat{\mathcal{A}}+\mathcal{Q}$.

The two maps $\Omega^1_X\to\widehat{\mathcal{A}}+\mathcal{Q}$ given by $\alpha\mapsto(\alpha,0)$ and $\alpha\mapsto(0,\alpha)$ coincide and the map ∂ of (5.10.1) factors through their common value as

$$\mathcal{O}_X\xrightarrow{d}\Omega^1_X\to\widehat{\mathcal{A}}+\mathcal{Q}. \tag{5.10.3}$$

Lemma 5.4. *The formulas (5.10.1) and (5.10.2) determine a structure of vertex extension of $\mathcal{A}$ on $\widehat{\mathcal{A}}+\mathcal{Q}$. Moreover, the map $\mathfrak{g}(\widehat{\mathcal{A}})\to\widehat{\mathcal{A}}+\mathcal{Q}$ defined by $a\mapsto(a,0)$ induces an isomorphism $\mathfrak{g}(\widehat{\mathcal{A}}+\mathcal{Q})\overset{\cong}{=}\mathfrak{g}(\widehat{\mathcal{A}})$ of vertex (equivalently, Courant) extensions of $\mathfrak{g}(\mathcal{A})$ (by Ω^1_X).*

Lemma 5.5. *Suppose that $\widehat{\mathcal{A}}^{(1)}$ and $\widehat{\mathcal{A}}^{(2)}$ are in $\mathcal{VE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$. Then, there exists a unique $\mathcal{Q}$ in $\mathcal{ECA}_{\mathcal{O}_X}$ such that $\widehat{\mathcal{A}}^{(2)}=\widehat{\mathcal{A}}^{(1)}+\mathcal{Q}$.*

Proof. Let $\mathcal{Q}$ denote the quotient of $\widehat{\mathcal{A}}^{(2)}\times_{\mathcal{A}}\widehat{\mathcal{A}}^{(1)}$ by the diagonally embedded copy of $\widehat{\mathfrak{g}}$. Then, $\mathcal{Q}$ is an extension of $\mathcal{T}$ by Ω^1_X. There is a unique structure of an exact Courant algebroid on $\mathcal{Q}$ defined as in Lemma 4.1, such that $\widehat{\mathcal{A}}^{(2)}=\widehat{\mathcal{A}}^{(1)}+\mathcal{Q}$. □

Proposition 5.1. *The stack of exact vertex $\mathcal{O}_X$-algebroids $\mathcal{EVA}_X$ is a torsor under $\mathcal{ECA}_X$.*

Proof. In view of Lemma 5.5 and the equality $\mathcal{EVA}_X=\mathcal{VE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{T}_X)$, it remains to show that $\mathcal{EVA}_X$ is locally non-empty.

In the analytic or C^∞ case, Example 5.1 provides a locally defined exact vertex algebroid. Indeed, locally on X there exists an (abelian) Lie $\mathbb{C}$-subalgebra τ such that $\mathcal{T}_X\cong\mathcal{O}_X\otimes_{\mathbb{C}}\tau$.

In the algebraic setting the same example shows that $\mathcal{EVA}_X$ is non-empty locally in étale topology. Since $H^2_{\text{ét}}(X,\Omega^2_X\to\Omega^{3,cl}_X)$ is canonically isomorphic to $H^2(X,\Omega^2_X\to\Omega^{3,cl}_X)$, it follows that $\mathcal{EVA}_X$ is non-empty Zariski-locally. □

5.11 Comparison of $\mathcal{ECA}_{\mathcal{O}_X}$-torsors

Suppose that $\widehat{\mathcal{A}}$ is a vertex extension of the Lie algebroid $\mathcal{A}$. Let $\langle\ ,\ \rangle$ denote the induced symmetric pairing on $\mathfrak{g}(\mathcal{A})$.

Suppose that $\mathcal{V}$ is an exact vertex algebroid. Let $\widehat{\mathcal{A}}-\mathcal{V}$ denote the push-out of $\widehat{\mathcal{A}}\times_{\mathcal{T}_X}\mathcal{V}$ by the difference map $\Omega^1_X\times\Omega^1_X\xrightarrow{-}\Omega^1_X$. Thus, a section of $\widehat{\mathcal{A}}-\mathcal{V}$ is represented by a pair (a,v) with $a\in\widehat{\mathcal{A}}$, $v\in\mathcal{V}$ satisfying $\pi(a)=\pi(v)\in\mathcal{T}_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form (α,α) for some $\alpha\in\Omega^1_X$.

For $a\in\widehat{\mathcal{A}}$, $v\in\mathcal{V}$ with $\pi(a)=\pi(v)$, $f\in\mathcal{O}_X$, let

$$f*(a,v)=(f*a,f*v),\quad \partial(f)=\partial_{\widehat{\mathcal{A}}}(f)-\partial_{\mathcal{V}}(f). \tag{5.11.1}$$

For $a_i\in\widehat{\mathcal{A}}$, $v_i\in\mathcal{V}$ with $\pi(a_i)=\pi(v_i)$ let

$$[(a_1,v_1),(a_2,v_2)]=([a_1,a_2],[v_1,v_2]),\quad \langle(a_1,v_1),(a_2,v_2)\rangle=\langle a_1,a_2\rangle-\langle v_1,v_2\rangle. \tag{5.11.2}$$

These operations are easily seen to descend to $\widehat{\mathcal{A}}-\mathcal{V}$.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

The two maps $\Omega^1_X \to \widehat{\mathcal{A}} - \mathcal{V}$ given by $\alpha \mapsto (\alpha, 0)$ and $\alpha \mapsto (0, -\alpha)$ coincide and the map ∂ of (5.11.1) factors through their common value as

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \widehat{\mathcal{A}} - \mathcal{V}. \tag{5.11.3}$$

LEMMA 5.6. *The formulas (5.11.1) and (5.11.2) determine a structure of a Courant extension of $\mathcal{A}$ on $\widehat{\mathcal{A}} - \mathcal{V}$. Moreover, the map $\mathfrak{g}(\widehat{\mathcal{A}}) \to \widehat{\mathcal{A}} - \mathcal{V}$ defined by $a \mapsto (a, 0)$ induces an isomorphism $\mathfrak{g}(\widehat{\mathcal{A}} - \mathcal{V}) \cong \mathfrak{g}(\widehat{\mathcal{A}})$ of Courant extensions of $\mathfrak{g}(\mathcal{A})$ (by Ω^1_X).*

Proof. This is left to the reader. □

PROPOSITION 5.2. *Suppose that $\mathcal{A}$ admits a vertex extension $\widehat{\mathcal{A}}$; let $\langle\ ,\ \rangle$ denote the induced invariant symmetric pairing on $\mathfrak{g}(\mathcal{A})$. Then, the assignment $\mathcal{V} \mapsto \widehat{\mathcal{A}} - \mathcal{V}$ extends to a functor*

$$\widehat{\mathcal{A}} - (\bullet) : \mathcal{EVA}_{\mathcal{O}_X} \to \mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle} \tag{5.11.4}$$

which anti-commutes with the respective actions of $\mathcal{ECA}_{\mathcal{O}_X}$ on $\mathcal{EVA}_{\mathcal{O}_X}$ and $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$. In particular, the functor (5.11.4) is an equivalence of stacks in groupoids. The isomorphism classes of the $\mathcal{ECA}_{\mathcal{O}_X}$-torsors $\mathcal{EVA}_{\mathcal{O}_X}$ and $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\langle\ ,\ \rangle}$ are opposite as elements of $H^2(X; \Omega^2_X \to \Omega^{3,cl}_X)$.

Proof. This is clear from the construction of § 5.11 and Lemma 5.6. □

THEOREM 5.1. *The class of $\mathcal{EVA}_{\mathcal{O}_X}$ in $H^2(X; \Omega^2_X \to \Omega^{3,cl}_X)$ is equal to $\mathrm{ch}_2(\Omega^1_X)$.*

Proof. According to Proposition 6.2, $\mathcal{A}_{\Omega^1_X}$ (the Atiyah algebra of Ω^1_X) admits a canonical vertex extension with the induced symmetric pairing on the Lie algebra $\underline{\mathrm{End}}_{\mathcal{O}_X}(\Omega^1_X) = \mathfrak{g}(\mathcal{A}_{\Omega^1_X})$ given by the trace of the product of endomorphisms. By Corollary 4.2, the class of $\mathcal{CE}x\mathcal{T}_{\mathcal{O}_X}(\mathcal{A}_{\Omega^1_X})_{\mathrm{Tr}}$ is equal to $-\mathrm{ch}_2(\Omega^1_X)$. The claim follows from Proposition 5.2. □

6. Algebroids over the de Rham complex

All of the notions of the preceding sections generalize in an obvious way to differential graded (DG) manifolds (i.e. manifolds whose structure sheaves are sheaves of commutative differential graded algebras).

For a manifold X let $X^\sharp$ denote the differential graded manifold with the underlying space X and the structure sheaf $\mathcal{O}_{X^\sharp}$ denote the de Rham complex $\Omega^\bullet_X$. In other words, $\mathcal{O}_{X^\sharp} = \bigoplus_i \Omega^i_X[-i]$ (as a sheaf of graded algebras). We will denote by $\partial_{\mathcal{O}_{X^\sharp}}$ the derivation given by the de Rham differential.

6.1 The structure of $\mathcal{T}_{X^\sharp}$

The tangent sheaf of $X^\sharp$ (of derivations of $\mathcal{O}_{X^\sharp}$), $\mathcal{T}_{X^\sharp}$, is a sheaf of differential graded Lie algebras with the differential $\partial_{\mathcal{T}_{X^\sharp}} = [\partial_{\mathcal{O}_{X^\sharp}},\ ]$ (note that $\partial_{\mathcal{O}_{X^\sharp}} \in \mathcal{T}^1_{X^\sharp}$).

6.1.1 Let $\widetilde{\mathcal{T}}_X$ denote the cone of the identity endomorphism of $\mathcal{T}_X$. That is, $\widetilde{\mathcal{T}_X}^i = \mathcal{T}_X$ for $i = -1, 0$ and zero otherwise. The only non-trivial differential is the identity map. The complex $\widetilde{\mathcal{T}}_X$ has the canonical structure of a sheaf of differential graded Lie algebras (DGLA).

The natural action of $\mathcal{T}_X$ (respectively, $\mathcal{T}_X[1]$) on $\mathcal{O}_{X^\sharp}$ by the Lie derivative (respectively, by, interior product) gives rise to the injective map of DGLA

$$\tau : \widetilde{\mathcal{T}}_X \to \mathcal{T}_{X^\sharp}. \tag{6.1.1}$$

The action τ extends in the canonical way to a structure of a Lie $\mathcal{O}_{X^\sharp}$-algebroid on $\mathcal{O}_{X^\sharp} \otimes_{\mathbb{C}} \widetilde{\mathcal{T}}_X$ with the anchor map

$$\tau_{\mathcal{O}_{X^\sharp}} : \mathcal{O}_{X^\sharp} \otimes_{\mathbb{C}} \widetilde{\mathcal{T}}_X \to \mathcal{T}_{X^\sharp} \tag{6.1.2}$$

the canonical extension τ to an $\mathcal{O}_{X^\sharp}$-linear map. Note that $\tau_{\mathcal{O}_{X^\sharp}}$ is surjective, i.e. the Lie $\mathcal{O}_{X^\sharp}$-algebroid $\mathcal{O}_{X^\sharp} \otimes_{\mathbb{C}} \widetilde{\mathcal{T}}_X$ is transitive. We denote this algebroid by $\widetilde{\mathcal{T}}_{X^\sharp}$.

Let $\mathcal{T}_{X^\sharp/X} \subset \mathcal{T}_{X^\sharp}$ denote the normalizer of $\mathcal{O}_X \subset \mathcal{O}_{X^\sharp}$. Since the action of $\mathcal{T}_X[1]$ is $\mathcal{O}_X$-linear, the map τ restricts to

$$\tau : \widetilde{\mathcal{T}_X}^{-1} = \mathcal{T}_X[1] \to \mathcal{T}_{X^\sharp/X}$$

and (the restriction of) $\tau_{\mathcal{O}_{X^\sharp}}$ factors through the map

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1] \to \mathcal{T}_{X^\sharp/X}, \tag{6.1.3}$$

which is easily seen to be an isomorphism.

Since the action τ is $\mathcal{O}_X$-linear modulo $\mathcal{T}_{X^\sharp/X}$, $\tau_{\mathcal{O}_{X^\sharp}}$ induces the map

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{T}_{X^\sharp}/\mathcal{T}_{X^\sharp/X}, \tag{6.1.4}$$

which is easily seen to be an isomorphism.

Therefore, there is an exact sequence of graded $\mathcal{O}_{X^\sharp}$-modules

$$0 \to \mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1] \to \mathcal{T}_{X^\sharp} \to \mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to 0. \tag{6.1.5}$$

The composition

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1] \to \mathcal{T}_{X^\sharp} \xrightarrow{\partial_{\mathcal{T}_{X^\sharp}}} \mathcal{T}_{X^\sharp}[1] \to \mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1]$$

is the identity map.

The natural action of $\mathcal{T}_{X^\sharp}$ on $\mathcal{O}_{X^\sharp} = \Omega^\bullet_X$ restricts to the action of ${\mathcal{T}_{X^\sharp}}^0$ on $\mathcal{O}_X$ and Ω^1_X. The action of ${\mathcal{T}_{X^\sharp}}^0$ on $\mathcal{O}_X$ gives rise to the map ${\mathcal{T}_{X^\sharp}}^0 \to \mathcal{T}_X$ which, together with the natural Lie bracket on ${\mathcal{T}_{X^\sharp}}^0$, endows the latter with a structure of a Lie $\mathcal{O}_X$-algebroid.

The action of ${\mathcal{T}_{X^\sharp}}^0$ on Ω^1_X gives rise to the map

$${\mathcal{T}_{X^\sharp}}^0 \to \mathcal{A}_{\Omega^1_X}, \tag{6.1.6}$$

where $\mathcal{A}_{\Omega^1_X}$ denotes the Atiyah algebra of Ω^1_X.

Lemma 6.1. *The map (6.1.6) is an isomorphism of Lie $\mathcal{O}_X$-algebroids.*

6.2 Exact Courant $\mathcal{O}_{X^\sharp}$-algebroids

Proposition 6.1. *Every exact Courant $\mathcal{O}_{X^\sharp}$-algebroid admits a unique flat connection.*

Proof. Consider an exact Courant $\mathcal{O}_{X^\sharp}$-algebroid

$$0 \to \Omega^1_{X^\sharp} \to \mathcal{Q} \xrightarrow{\pi} \mathcal{T}_{X^\sharp} \to 0.$$

Note that, since $\Omega^1_{X^\sharp}$ is concentrated in non-negative degrees, the map $\pi : \mathcal{Q}^{-1} \to {\mathcal{T}_{X^\sharp}}^{-1}$ is an isomorphism. Since ${\mathcal{T}_{X^\sharp}}^{-1}$ generates $\mathcal{T}_{X^\sharp}$ as a DG-module over $\mathcal{O}_{X^\sharp}$, the splitting is unique if it exists.

To establish the existence it is necessary and sufficient to show that the restriction of the anchor map to the DG-submodule of $\mathcal{Q}$ generated by $\mathcal{Q}^{-1}$ is an isomorphism.

Note that the map $\tau : \widetilde{\mathcal{T}}_X \to \mathcal{T}_{X^\sharp}$ lifts in a unique way to a morphism of complexes $\widetilde{\tau} : \widetilde{\mathcal{T}}_X \to \mathcal{Q}$. The map $\widetilde{\tau}$ is easily seen to be a morphism of DGLA. Let $\mathcal{Q}'$ denote the $\mathcal{O}_{X^\sharp}$-submodule of $\mathcal{Q}$ generated by the image of $\widetilde{\tau}$ (i.e. the DG $\mathcal{O}_{X^\sharp}$-submodule generated by $\mathcal{Q}^{-1}$).

Since

$$\widetilde{\tau} : \mathcal{T}_X[1] = \widetilde{\mathcal{T}}_X^{-1} \to \mathcal{Q}'$$

is $\mathcal{O}_X$-linear it extends to the map

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1] \to \mathcal{Q}' \tag{6.2.1}$$

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

such that the composition

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X[1] \to \mathcal{Q}' \xrightarrow{\pi} \mathcal{T}_{X^\sharp}$$

coincides with the composition of the isomorphism (6.1.3) with the inclusion into $\mathcal{T}_{X^\sharp}$. Therefore, (6.2.1) is a monomorphism whose image will be denoted $\mathcal{Q}''$, and π restricts to an isomorphism of $\mathcal{Q}''$ onto $\mathcal{T}_{X^\sharp/X}$.

Since

$$\widetilde{\tau} : \mathcal{T}_X = \widetilde{\mathcal{T}}^0_X \to \mathcal{Q}' \to \mathcal{Q}'/\mathcal{Q}''$$

is $\mathcal{O}_X$-linear it extends to the map

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{Q}'/\mathcal{Q}'' \tag{6.2.2}$$

which is surjective (since $\mathcal{Q}'/\mathcal{Q}''$ is generated as an $\mathcal{O}_{X^\sharp}$-module by the image of $\widetilde{\mathcal{T}}^0_X$ under $\widetilde{\tau}$), and such that the composition

$$\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{T}_X \to \mathcal{Q}'/\mathcal{Q}'' \xrightarrow{\pi} \mathcal{T}_{X^\sharp}/\mathcal{T}_{X^\sharp/X}$$

coincides with the isomorphism (6.1.4). Therefore, (6.2.2) is an isomorphism.

Now, the exact sequence (6.1.5) implies that π restricts to an isomorphism $\mathcal{Q}' \cong \mathcal{T}_{X^\sharp}$. The desired splitting is the inverse isomorphism. It is obviously compatible with brackets, hence is a flat connection. □

Corollary 6.1. *The stack $\mathcal{ECA}_{\mathcal{O}_{X^\sharp}}$ is equivalent to the (final) stack $X \supseteq U \mapsto [0]$, where $[0]$ is the category with one object and one morphism.*

Corollary 6.2. *An exact vertex $\mathcal{O}_{X^\sharp}$-algebroid exists and is unique up to canonical isomorphism.*

Proof. Since $\mathcal{EVA}_{\mathcal{O}_{X^\sharp}}$ is an affine space under $\mathcal{ECA}_{\mathcal{O}_{X^\sharp}}$ the uniqueness (local and global) follows from Corollary 6.1. Local existence and uniqueness implies global existence. □

6.3 The canonical vertex $\mathcal{O}_X$-algebroid

Let $\mathcal{V}^\bullet$ denote the unique exact vertex $\mathcal{O}_{X^\sharp}$-algebroid. The degree zero component of the exact sequence

$$0 \to \Omega^1_{X^\sharp} \to \mathcal{V}^\bullet \to \mathcal{T}_{X^\sharp} \to 0$$

is canonically isomorphic to

$$0 \to \Omega^1_X \to \mathcal{V}^0 \to \mathcal{A}_{\Omega^1_X} \to 0$$

using the canonical isomorphisms ${\Omega^1_{X^\sharp}}^0 \cong \Omega^1_X$ and (6.1.6).

Proposition 6.2. *The vertex $\mathcal{O}_{X^\sharp}$-algebroid structure on $\mathcal{V}$ restricts to a structure of a vertex extension of $\mathcal{A}_{\Omega^1_X}$ on $\mathcal{V}^0$. The induced symmetric pairing on $\underline{\mathrm{End}}_{\mathcal{O}_X}(\Omega^1_X)$ is given by the trace of the product of endomorphisms.*

Proof. The first statement is left to the reader.

The degree -1 component of the anchor map $\mathcal{V}^\bullet \to \mathcal{T}_{X^\sharp}$ is an isomorphism whose inverse gives the map $\mathcal{T}_X[1] \to \mathcal{V}^\bullet$. Combined with the $*$-multiplication by $\mathcal{O}^1_{X^\sharp} = \Omega^1_X$ it gives the map $\Omega^1_X \otimes_{\mathbb{C}} \mathcal{T}_X \to \mathcal{V}^0$ and the commutative diagram

$$\begin{array}{ccc} \Omega^1_X \otimes_{\mathbb{C}} \mathcal{T}_X & \xrightarrow{*} & \mathcal{V}^0 \\ \downarrow & & \downarrow \\ \underline{\mathrm{End}}_{\mathcal{O}_X}(\Omega^1_X) & \longrightarrow & \mathcal{A}_{\Omega^1_X} \end{array}$$

where the left vertical map is the canonical one.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

For $\phi, \psi \in \underline{\mathrm{End}}_{\mathcal{O}_X}(\Omega^1_X)$ represented, respectively, by $\alpha \otimes \xi$ and $\beta \otimes \eta$ with $\alpha, \beta \in \Omega^1_X[-1]$ and $\xi, \eta \in \mathcal{T}_X[1]$, the pairing $\langle \phi, \psi \rangle$ is calculated using (5.2.5):

$$\begin{aligned}\langle \phi, \psi \rangle &= \langle \alpha * \xi, \beta * \eta \rangle_{\mathcal{V}} = \alpha \langle \xi, \beta * \eta \rangle + \pi(\xi)(\pi(\beta * \eta)(\alpha)) \\ &= \pi(\xi)(\beta \pi(\eta)(\alpha)) = \pi(\xi)(\beta)\pi(\eta)(\alpha) - \beta \pi(\xi)(\pi(\eta)(\alpha)) = \iota_\xi \beta \cdot \iota_\eta \alpha = \mathrm{Tr}(\phi\psi)\end{aligned}$$

(with $\langle \xi, \beta * \eta \rangle = 0$ and $\pi(\xi)(\pi(\eta)(\alpha)) = 0$ since both have negative degrees). □

Acknowledgements

The author would like to thank Pavol Ševera for many inspiring discussions. Much of the present work was carried out during the author's visits to Université d'Angers and l'Institut des Hautes Études Scientifiques.

Appendix A. Characteristic classes of Lie algebroids

Below, we recall the basic definitions and facts regarding Lie algebroids as well as the construction of the higher Chern–Simons forms [CS74]. As an example we calculate explicitly the Čech–de Rham representative of the first Pontryagin class.

A.1 Lie algebroids

We refer the reader to [Mac87] for further details on Lie algebroids in the differential geometric context.

A.1.1 *Definitions.* Suppose that X is a manifold. A *Lie algebroid* on X is a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-modules equipped with an $\mathcal{O}_X$-linear map $\pi : \mathcal{A} \to \mathcal{T}_X$ called the *anchor map*, and a $\mathbb{C}$-linear pairing $[\ ,\ ] : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A}$ such that:

(i) the pairing $[\ ,\ ]$ is a Lie bracket (i.e. it is skew-symmetric and satisfies the Jacobi identity);

(ii) the map π is a morphism of Lie algebras (i.e. it commutes with the respective brackets);

(iii) the Leibniz rule holds, i.e. for $a_1, a_2 \in \mathcal{A}$, $f \in \mathcal{O}_X$, the identity $[a_1, fa_2] = \pi(a_1)(f)a_2 + f[a_1, a_2]$ is satisfied.

We denote by $\mathfrak{g}(\mathcal{A})$ the kernel of the anchor map π. This is an $\mathcal{O}_X$-Lie algebra.

A morphism of Lie algebroids is an $\mathcal{O}_X$-linear map of Lie algebras which commutes with the respective anchor maps.

A Lie algebroid is called *transitive* if the anchor map is surjective.

A.1.2 *Connections and curvature.* A *connection* on a (transitive) Lie algebroid $\mathcal{A}$ is an $\mathcal{O}_X$-linear splitting $\nabla : \mathcal{T}_X \to \mathcal{A}$ of the anchor map (i.e. $\pi \circ \nabla = \mathrm{id}$). A connection is called *flat* if it commutes with the respective brackets (i.e. if it is a morphism of Lie algebras).

For a connection ∇ the formula $c(\nabla)(\xi_1,\xi_2) := [\nabla(\xi_1),\nabla(\xi_2)] - \nabla([\xi_1,\xi_2])$ defines the $\mathcal{O}_X$-linear map $c(\nabla) : \wedge^2\mathcal{T} \to \mathfrak{g}(\mathcal{A})$ called the *curvature* of the connection ∇. Clearly, ∇ is flat if and only if $c(\nabla) = 0$.

A.1.3 *Examples.* The tangent sheaf $\mathcal{T}_X$ (the anchor map being the identity) is the final object in the category of Lie algebroids on X.

An algebroid with the trivial anchor map is the same thing as a sheaf of $\mathcal{O}_X$-Lie algebras.

An action of a Lie algebra $\mathfrak{g}$ on X, the action given by the morphism of Lie algebras $\alpha : \mathfrak{g} \to \Gamma(X;\mathcal{T}_X)$, gives rise to a structure of a Lie algebroid on $\mathcal{O}_X \otimes \mathfrak{g}$ in a natural way: the anchor map is given by $f \otimes a \mapsto f\alpha(a)$ and the bracket is defined by

$$[f_1 \otimes a_1, f_2 \otimes a_2] = f_1 f_2 \otimes [a_1,a_2] + f_1\alpha(a_1)(f_2) \otimes a_2 - f_2\alpha(a_2)(f_1) \otimes a_1.$$

A.1.4 *Atiyah algebras.* An important class of examples of transitive Lie algebroids is Atiyah algebras. Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$ and $p : P \to X$ is a principal G bundle. The *Atiyah algebra* of P, denoted $\mathcal{A}_P$, is the sheaf whose local sections are pairs $(\tilde{\xi},\xi)$, where ξ is a (locally defined) vector field on X and $\tilde{\xi}$ is a G-invariant vector field on P which lifts ξ. Thus, $\mathcal{A}_P = (p_*\mathcal{T}_P)^G$ with the induced bracket and the anchor map is given by

$$\mathcal{A}_{\mathcal{E}} = (p_*\mathcal{T}_{\mathcal{E}})^G \xrightarrow{dp} (p_*p^*\mathcal{T}_X)^G = \mathcal{T}_X.$$

In terms of the preceding description the anchor is given by the projection $(\tilde{\xi},\xi) \mapsto \xi$. In this case $\mathfrak{g}(\mathcal{A})$ is the $\mathcal{E}$-twist of $\mathcal{O}_X \otimes \mathfrak{g}$.

The Atiyah algebra of a vector bundle, which is just the Atiyah algebra of the corresponding GL-bundle, admits another description. Namely, for a vector bundle $\mathcal{F}$ let $\mathrm{Diff}^{\leqslant n}(\mathcal{F},\mathcal{F})$ denote the sheaf of differential operators of order n acting on $\mathcal{F}$. Then $\mathcal{A}_{\mathcal{F}}$ is determined by the following pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{F}) & \longrightarrow & \mathcal{A}_{\mathcal{F}} & \xrightarrow{\pi} & \mathcal{T}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathbf{1}} & & \\
0 & \longrightarrow & \mathrm{Diff}^{\leqslant 0}(\mathcal{F},\mathcal{F}) & \longrightarrow & \mathrm{Diff}^{\leqslant 1}(\mathcal{F},\mathcal{F}) & \longrightarrow & \mathcal{T}_X \otimes \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{F}) & \longrightarrow & 0
\end{array}$$

as the sheaf of differential operators of order one with scalar principal symbol.

A.1.5 *Pull-back of algebroids.* Recall that, for a Lie algebroid $\mathcal{A}$ on X and a map $\phi : Y \to X$, the pull-back $\phi^+\mathcal{A}$ is defined by the following Cartesian diagram.

$$\begin{array}{ccc}
\phi^+\mathcal{A} & \longrightarrow & \phi^*\mathcal{A} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\pi} \\
\mathcal{T}_Y & \xrightarrow{d\phi} & \phi^*\mathcal{T}_X
\end{array}$$

Thus, a section of $\phi^+\mathcal{A}$ is a pair $(f \otimes a, \xi)$, where $f \in \mathcal{O}_Y$, $a \in \mathcal{A}$, $\xi \in \mathcal{T}_Y$, so that

$$f \otimes a \in \phi^*\mathcal{A} = \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{A},$$

and $f \otimes \pi(a) = d\phi(\xi)$. The anchor map is given by the projection $(f \otimes a, \xi) \mapsto \xi$.

The bracket on $\phi^+\mathcal{A}$ is the unique one which restricts to the bracket on $\phi^{-1}\mathcal{A}$ (induced by that on $\mathcal{A}$) and obeys the Leibniz rule. Note that $\mathfrak{g}(\phi^+\mathcal{A}) \stackrel{\sim}{=} \phi^*\mathfrak{g}(\mathcal{A})$ as sheaves of $\mathcal{O}_Y$-Lie algebras.

A.2 Higher Chern–Simons forms

Suppose that $\mathcal{U}$ is a cover of X by open subsets. Let $X_0 = \coprod_{U\in\mathcal{U}} U$, $\epsilon : X_0 \to X$ be the map induced by the inclusions $i_U : U \hookrightarrow X$. Let $X_i = X_0 \times_X \cdots \times_X X_0$ denote the $(i+1)$-fold product which will be indexed by $\{0,\dots,i\}$. For $j = 0,\dots,i$ let $\mathrm{pr}_j : X_i \to X_0$ (respectively, $s_j : X_i \to X_{i-1}$) denote the projection onto (respectively, along) the jth factor.

The collection of all X_i together with the projections s_j and the diagonal maps is a simplicial manifold denoted $X_\bullet$, and ϵ extends to the map $\epsilon : X_\bullet \to X$, where the latter is regarded as a constant simplicial object.

A sheaf F on X gives rise to a simplicial sheaf ϵ^*F on $X_\bullet$ so that the complex (associated to the simplicial abelian group) $\Gamma(X_\bullet; F)$ is the complex $\check{C}(\mathcal{U}; F)$ of F-valued cochains on the cover $\mathcal{U}$. We will denote by F_i the restriction of ϵ^*F to X_i. Note that there is a canonical isomorphism $F_i \overset{\sim}{=} \mathrm{pr}_j^* F_0$.

Let Δ^i denote the standard i-dimensional simplex:

$$\Delta^i = \left\{\vec{t} = (t_0,\dots,t_i) \,\middle|\, t_j \in \mathbb{R},\ \sum_{j=0}^{i} t_j \leqslant 1\right\}.$$

Integration over Δ^i gives rise to the map

$$\int_{\Delta_i} : \Gamma(X_i \times \Delta^i; \Omega^a_{X_i\times\Delta^i}) \to \Gamma(X_i; \Omega^{a-i}_{X_i}),$$

which satisfies the Stokes formula

$$\int_{\Delta_i} d\alpha = d\int_{\Delta_i} \alpha + \sum_{j=0}^{i}(-1)^j \int_{\Delta^{i-1}} (\mathrm{id}\times\partial_j)^*\alpha,$$

where $\partial_j : \Delta^{i-1} \to \Delta_i$ is the map $(t_0,\dots,t_{i-1}) \mapsto (t_0,\dots,t_{j-1},0,t_j,\dots,t_{i-1})$.

The forms on the product $X_i \times \Delta^i$ are naturally bi-graded:

$$\Omega^a_{X_i\times\Delta^i} = \Omega^{a-i,i}_{X_i\times\Delta^i} + \Omega^{a-i+1,i-1}_{X_i\times\Delta^i} + \cdots + \Omega^{a,0}_{X_i\times\Delta^i}.$$

The 'integration over Δ^i' map is non-trivial only on the first summand (of maximal Δ^i-degree) which, on the other hand, lies in the kernel of the restriction maps ∂_j^*. Hence, for $\alpha = \sum_{k=0}^{i} \alpha^{a-k,k}$ with $\alpha^{a-k,k} \in \Gamma(X_i \times \Delta^i; \Omega^{a-k,k}_{X_i\times\Delta^i})$ the Stokes formula takes the form

$$\int_{\Delta_i} d\alpha = d\int_{\Delta_i} \alpha^{a-i,i} + \sum_{j=0}^{i}(-1)^j \int_{\Delta^{i-1}} (\mathrm{id}\times\partial_j)^*\alpha^{a-i+1,i-1}. \tag{A.2.1}$$

Suppose that $\mathcal{A}$ is a Lie algebroid on X, locally free of finite rank over $\mathcal{O}_X$, and let $\mathfrak{g} := \mathfrak{g}(\mathcal{A})$. Let $\widetilde{\mathcal{A}}_i = \mathrm{pr}^+_{X_i} \mathcal{A}_i$, where $\mathrm{pr}_{X_i} : X_i \times \Delta^i \to X_i$ denotes the projection, so that

$$\widetilde{\mathcal{A}}_i = (\mathcal{A}_i \boxtimes \mathcal{O}_{\Delta^i} \oplus \mathcal{O}_{X_i} \boxtimes \mathcal{T}_{\Delta^i}) \otimes_{\mathcal{O}_{X_i}\boxtimes\mathcal{O}_{\Delta^i}} \mathcal{O}_{X_i\times\Delta^i}$$

as an $\mathcal{O}_{X_i\times\Delta^i}$-module with the obvious Lie algebroid structure.

For a connection ∇_0 on $\mathcal{A}_0$ let $\widetilde{\nabla}_i$ denote the connection on $\widetilde{\mathcal{A}}_i$ given by the formula

$$\widetilde{\nabla}_i = \sum_{j=0}^{i} \mathrm{pr}_j^* \nabla_0 \boxtimes t_j + 1 \boxtimes \mathrm{id}.$$

The curvature of $\widetilde{\nabla}_i$, $c(\widetilde{\nabla}_i) \in \Omega^2_{X_i\times\Delta^i} \otimes \widetilde{\mathfrak{g}}_i$, decomposes as $c(\widetilde{\nabla}_i) = c(\widetilde{\nabla}_i)^{2,0} + c(\widetilde{\nabla}_i)^{1,1} + c(\widetilde{\nabla}_i)^{0,2}$ with $c(\widetilde{\nabla}_i)^{2-k,k} \in \Omega^{2-k,k}_{X_i\times\Delta^i} \otimes \widetilde{\mathfrak{g}}_i$. Since $\widetilde{\nabla}_i$ is flat along Δ^i, the $(0,2)$-component of the curvature form vanishes.

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

Suppose that $P : \mathfrak{g}^{\otimes_{\mathcal{O}_X} d} \to \mathcal{O}_X$ is an $\mathcal{A}$-invariant map. Then, the map

$$\widetilde{P}_i : \widetilde{\mathfrak{g}}_i^{\otimes_{\mathcal{O}_{X_i\times\Delta^i}} d} \to \mathcal{O}_{X_i\times\Delta^i}$$

is $\widetilde{\mathcal{A}}_i$-invariant. The invariance of $\widetilde{P}_i$ implies that the differential form

$$\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d}) \in \Gamma(X_i \times \Delta^i; \Omega^{2d}_{X_i\times\Delta^i})$$

is closed. In view of the remarks above, we have

$$\begin{aligned}\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d}) &= \widetilde{P}_i((c(\widetilde{\nabla}_i)^{2,0} + c(\widetilde{\nabla}_i)^{1,1})^{\wedge d}) \\ &= \sum_{k=0}^{d} \binom{d}{k} \widetilde{P}_i((c(\widetilde{\nabla}_i)^{2,0})^{\wedge(d-k)} \wedge (c(\widetilde{\nabla}_i)^{1,1})^{\wedge k})\end{aligned}$$

with the kth summand, denoted $\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d})^{2d-k,k}$ below, homogeneous of bi-degree $(2d-k,k)$.

Recall that $s_j : X_i \to X_{i-1}$ denotes the projection along the jth factor. The Lie algebroids $(\mathrm{id}\times\partial_j)^{+}\widetilde{\mathcal{A}}_i$ and $(s_j\times\mathrm{id})^{+}\widetilde{\mathcal{A}}_{i-1}$ are canonically isomorphic in a way compatible with the connections (induced by) $\widetilde{\nabla}_i$ on the former and $\widetilde{\nabla}_{i-1}$ on the latter. It follows that

$$(\mathrm{id}\times\partial_j)^{*}\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d})^{2d-k,k} = (s_j\times\mathrm{id})^{*}\widetilde{P}_{i-1}(c(\widetilde{\nabla}_{i-1})^{\wedge d})^{2d-k,k}.$$

Using the following:

(i)
$$\begin{aligned}&\int_{\Delta^{i-1}} (\mathrm{id}\times\partial_j)^{*}\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d})^{2d-i+1,i-1} \\ &\quad= \int_{\Delta^{i-1}} (s_j\times\mathrm{id})^{*}\widetilde{P}_{i-1}(c(\widetilde{\nabla}_{i-1})^{\wedge d})^{2d-i+1,i-1} \\ &\quad= s_j^{*}\int_{\Delta^{i-1}} \widetilde{P}_{i-1}(c(\widetilde{\nabla}_{i-1})^{\wedge d})^{2d-i+1,i-1};\end{aligned}$$

(ii) the fact that the differential in the Čech complex is given by $\check{\partial} = \sum_j (-1)^j s_j^{*}$;

(iii) the Stokes formula (A.2.1);

(iv) and noticing that the left-hand side of the latter vanishes because $\widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d})$ is closed;

one sees that

$$\sum_i \int_{\Delta^i} \widetilde{P}_i(c(\widetilde{\nabla}_i)^{\wedge d})^{d-i,i} \in \check{C}^{\bullet}(\mathcal{U}; \tau_{\leqslant 2d}\Omega_X^{\geqslant d})$$

is a cocycle of total degree $2d$ in the Čech–de Rham complex which extends

$$P(c(\nabla_0)^{\wedge d}) \in \check{C}^0(\mathcal{U}; \Omega_X^{2d,cl}),$$

and whose class in $H^{2d}(X; \tau_{\leqslant 2d}\Omega_X^{\geqslant d})$ is independent of the choices (of the covering $\mathcal{U}$ and the connection ∇_0) made. (In fact, independence of the choice of the connection follows from the above construction applied to the simplicial manifold $X_\bullet$ in place of X, the algebroid $\mathcal{A}_\bullet = \epsilon^{*}\mathcal{A}$ and the connection $\mathrm{pr}_\bullet^{*}\nabla$ induced by a choice of a connection ∇ on $\mathcal{A}_0$.)

A.3 The first Pontryagin class

Below we will carry out the above calculation in the situation when $P = \langle\ ,\ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $\mathfrak{g}$. The corresponding class in $H^4(X; \tau_{\leqslant 4}\Omega_X^{\geqslant 2})$ is called the (first) Pontryagin class of the pair $(\mathcal{A}, \langle\ ,\ \rangle)$ and will be denoted $\Pi(\mathcal{A}, \langle\ ,\ \rangle)$. It follows from the discussion in § A.2 that $\Pi(\mathcal{A}, \langle\ ,\ \rangle)$ is represented, in terms of a covering $\mathcal{U}$ and a connection ∇_0, by the cocycle $\Pi^{4,0} + \Pi^{3,1} + \Pi^{2,2}$ with $\Pi^{i,4-i} \in \check{C}^{4-i}(\mathcal{U}; \Omega_X^i)$ (where we have suppressed the dependence on the data).

Clearly, $\Pi^{4,0} = \langle c(\nabla_0) \wedge c(\nabla_0)\rangle$.

A.3.1 *Calculation of* $\Pi^{3,1}$. Let $A = \mathrm{pr}_1^*\nabla_0 - \mathrm{pr}_0^*\nabla_0$, $\nabla := \mathrm{pr}_0^*\nabla_0$, $t := t_1$. With this notation we have $\widetilde{\nabla}_1 = \nabla + At + d_t$, and

$$c(\widetilde{\nabla}_1) = \tfrac{1}{2}[\widetilde{\nabla}_1, \widetilde{\nabla}_1] = \tfrac{1}{2}[\nabla,\nabla] + [\nabla, A]t + \tfrac{1}{2}[A,A]t^2 - Adt,$$

so that

$$(c(\widetilde{\nabla}_1)\wedge c(\widetilde{\nabla}_1))^{3,1} = -([\nabla,\nabla] + 2[\nabla,A]t + [A,A]t^2)\wedge Adt,$$

which, upon integration, gives

$$\begin{aligned}\Pi^{3,1} &= -(\langle[\nabla,\nabla]\wedge A\rangle + \langle[\nabla,A]\wedge A\rangle + \tfrac{1}{3}\langle[A,A]\wedge A\rangle)\\ &= -2(\langle c(\nabla)\wedge A\rangle + \tfrac{1}{2}\langle[\nabla,A]\wedge A\rangle + \tfrac{1}{6}\langle[A,A]\wedge A\rangle).\end{aligned}$$

A.3.2 *Calculation of* $\Pi^{2,2}$. Let $A_{ij} = \mathrm{pr}_j^*\nabla_0 - \mathrm{pr}_i^*\nabla_0$, $\nabla := \mathrm{pr}_1^*\nabla_0$. With this notation we have

$$\begin{aligned}\widetilde{\nabla}_2 &= \sum_{j=0}^{2}\mathrm{pr}_j^*\nabla_0\cdot t_j + d_t = \sum_{j=0}^{2}(\mathrm{pr}_1^*\nabla_0 + A_{j1})\cdot t_j + d_t\\ &= \nabla + A_{10}t_0 + A_{12}t_2 + d_t = \nabla - A_{01}t_0 + A_{12}t_2 + d_t,\end{aligned}$$

$$c(\widetilde{\nabla}_2) = \tfrac{1}{2}[\widetilde{\nabla}_2, \widetilde{\nabla}_2] = \cdots + A_{01}dt_0 - A_{12}dt_2$$

(ignoring the terms which do not contain dt_0 and dt_2), so that

$$(c(\widetilde{\nabla}_2)\wedge c(\widetilde{\nabla}_2))^{2,2} = 2A_{01}\wedge A_{12}\wedge dt_0\wedge dt_2,$$

which, upon integration, gives

$$\Pi^{2,2} = -\langle A_{01}\wedge A_{12}\rangle.$$

A.4 Pontryagin class for vector bundles

Suppose that $\mathcal{E}$ is a GL_n-torsor on X. Let $\mathcal{A}_{\mathcal{E}}$ denote the Atiyah algebra of $\mathcal{E}$. Thus, $\mathcal{A}_{\mathcal{E}}$ is a transitive Lie algebroid with $\mathfrak{g}(\mathcal{A}_{\mathcal{E}}) = \mathfrak{gl}_n^{\mathcal{E}}$. The Lie algebra $\mathfrak{gl}_n$ carries the canonical invariant symmetric pairing given by the trace of the product of endomorphisms which induces the canonical $\mathcal{A}_{\mathcal{E}}$-invariant symmetric pairing on $\mathfrak{gl}_n^{\mathcal{E}}$. We will denote this pairing by $\langle\ ,\ \rangle_{\mathrm{can}}$ and write $\Pi(\mathcal{E})$ for $\Pi(\mathcal{A}_{\mathcal{E}}, \langle\ ,\ \rangle_{\mathrm{can}})$.

Recall that the Chern character form of a connection ∇ on $\mathcal{E}$, denoted $\mathrm{ch}(\mathcal{E},\nabla) = \sum_n \mathrm{ch}_n(\mathcal{E},\nabla)$, where $\mathrm{ch}_n(\mathcal{E},\nabla)$ is a form of degree $2n$, is defined by

$$\mathrm{ch}(\mathcal{E},\nabla) = \mathrm{Tr}(\exp(c(\nabla))) = \sum \frac{1}{n!}\mathrm{Tr}(c(\nabla)^{\wedge n}).$$

In particular, the characteristic class ch_2 is related to the Pontryagin class by

$$\Pi(\mathcal{E}) = 2\,\mathrm{ch}_2(\mathcal{E}). \tag{A.4.1}$$

References

BD04 A. Beilinson and V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51 (American Mathematical Society, Providence, RI, 2004); MR2058353 (2005d:17007).

Blo81 S. Bloch, *The dilogarithm and extensions of Lie algebras*, in *Proc. Conf. on Algebraic K-theory*, Northwestern University, Evanston, IL, 1980, Lecture Notes in Mathematics, vol. 854 (Springer, Berlin, 1981), 1–23; MR618298 (83b:17010).

Bre94 L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque **225** (1994); MR1301844 (95m:18006).

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press

CS74 S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974), 48–69; MR0353327 (50 #5811).

Cou90 T. J. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), 631–661; MR998124 (90m:58065).

GMS03 V. Gorbounov, F. Malikov and V. Schechtman, *Gerbes of chiral differential operators. III*, in *The orbit method in geometry and physics*, Marseille, 2000, Progress in Mathematics, vol. 213 (Birkhäuser, Boston, 2003), 73–100; MR1995376 (2005a:17028).

GMS04 V. Gorbounov, F. Malikov and V. Schechtman, *Gerbes of chiral differential operators. II. Vertex algebroids*, Invent. Math. **155** (2004), 605–680; MR2038198 (2005e:17047).

LWX97 Z.-J. Liu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), 547–574; MR1472888 (98f:58203).

Mac87 K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, vol. 124 (Cambridge University Press, Cambridge, 1987); MR896907 (89g:58225).

MSV99 F. Malikov, V. Schechtman and A. Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys. **204** (1999), 439–473; MR1704283 (2000j:17035a).

Paul Bressler bressler@math.ias.edu
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

https://doi.org/10.1112/S0010437X07002710 Published online by Cambridge University Press