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SOLVING FINITE TIME HORIZON DYNKIN GAMES BY OPTIMAL SWITCHING

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Abstract

This paper uses recent results on continuous-time finite-horizon optimal switching problems with negative switching costs to prove the existence of a saddle point in an optimal stopping (Dynkin) game. Sufficient conditions for the game's value to be continuous with respect to the time horizon are obtained using recent results on norm estimates for doubly reflected backward stochastic differential equations. This theory is then demonstrated numerically for the special cases of cancellable call and put options in a Black–Scholes market.

Keywords: Optimal stopping game; Nash equilibrium; saddle point; optimal stopping; Snell envelope; optimal switching

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Secondary 91A05; 60G40; 91G80

1. Introduction

Recent papers such as [7], [9], and [25] have shown a connection between Dynkin games and optimal switching problems with two modes. In particular, letting $0 < T < \infty$ denote the horizon, the results of [7] and [9] show that the value process $(V_t)_{0\leq t\leq T}$ of a Dynkin game in continuous time (see Section 2.1, below) exists and satisfies $V_t = Y_t^1 - Y_t^0$, where $Y^1 = (Y_t^1)_{0\leq t\leq T}$ and $Y^0 = (Y_t^0)_{0\leq t\leq T}$ are the respective value processes for the optimal switching problem with initial modes 1 and 0. Separately, it has been shown (see [2] and [13]) how to construct two nonnegative supermartingales that solve a Dynkin game on a finite time horizon. Furthermore, appropriate debut times of these supermartingales can be used to form a saddle point strategy for the game.

It is therefore apparent that *classical two-player* Dynkin games and two-mode optimal switching problems are strongly coupled in the following sense: starting with either the Dynkin game or the optimal switching problem, one can use its parameters and solution to formulate and solve the other problem. This paper complements these findings by proving, under appropriate conditions, that the solution to a two-mode optimal switching problem furnishes the existence of a saddle point for the corresponding Dynkin game. This is accomplished by the method of Snell envelopes which appeared in [1] for optimal switching problems on one hand, and in [2] and [13] for Dynkin games on the other hand. In the process, we relate the solution pair for the two-mode optimal switching problem to a pair of supermartingales which lie between the early exit values of the game. This condition is referred to in some contexts as *Mokobodski's hypothesis*.

The content of this paper is as follows. Section 2 introduces the Dynkin game and its auxiliary optimal switching problem. Section 3 then outlines some notation and standing assumptions.

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The main result on the existence of equilibria in the Dynkin game is presented in Section 4. Additional results on the dependence of the game's solution on the time horizon are discussed in Section 5. Numerics which showcase this theory can be found in Section 6, followed by the conclusion, acknowledgements, and references.

2. Preliminaries

2.1. The Dynkin game

Optimal stopping games, also referred to as stochastic games of timing or Dynkin games, were introduced by Eugene Dynkin during the 1960s. These games have been studied extensively since then and have garnered renewed interest due to the introduction of game contingent claims (also known as Israeli options) in [11]. The particular variant of the Dynkin game which is described below was studied in recent papers such as [2], [7], and [8].

We work on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le \infty}$ satisfying $\mathcal{F} = \mathcal{F}_\infty := \vee_t \mathcal{F}_t$ and the usual conditions of right-continuity and completeness. We use $\mathbf{1}_A$ to represent the indicator function of a set (event) A. The shorthand notation a.s. means 'almost surely'. For $0 \le T \le \infty$ set $\mathbb{F}_T = (\mathcal{F}_t)_{0 \le t \le T}$, and for each $t \in [0, T]$ let $\mathcal{T}_{t,T}$ denote the set of $\mathbb{F}_T$-stopping times ν which satisfy $t \le \nu \le T$ $\mathbb{P}$-a.s. For a given $S \in \mathcal{T}_{0,T}$, we write $\mathcal{T}_{S,T} = \{\nu \in \mathcal{T}_{0,T} : \nu \ge S,\ \mathbb{P}\text{-a.s.}\}$. Let $\mathbb{E}$ denote the corresponding expectation operator. For notational convenience the dependence on $\omega \in \Omega$ is often suppressed. A horizon $T \in (0, \infty)$ is fixed for the discussion which follows and for the majority of this paper. However, we often emphasise the dependence on T since the horizon is varied below in Section 5.

Let $t \in [0, T]$ be given and associate with two players MIN and MAX the stopping times $\sigma \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{t,T}$. The game between MIN and MAX is played from time t until $\sigma \wedge \tau$, where $x \wedge y := \min(x, y)$. During this time MIN pays MAX at a (random) rate of $\psi(\cdot)$ per unit time. If MIN exits the game prior to T and either before or at the same time that MAX exits, $\sigma < T$ and $\sigma \le \tau$, MIN pays MAX the amount $\gamma_-(\sigma)$. Alternatively, if MAX exits the game first, $\tau < \sigma$, then MAX pays to MIN the amount $\gamma_+(\tau)$. If neither player exits the game before time T, we set $\sigma = \tau = T$ and MIN pays MAX the amount Γ. We define this payoff for the Dynkin game on $[t, T]$ in terms of the conditional expected cost to player MIN:

$$\begin{aligned} D_{t,T}(\sigma, \tau) = \mathbb{E}\bigg[\int_t^{\sigma\wedge\tau} \psi(s)\,\mathrm{d}s + \gamma_-(\sigma)\,\mathbf{1}_{\{\sigma \le \tau\}}\,\mathbf{1}_{\{\sigma < T\}} - \gamma_+(\tau)\,\mathbf{1}_{\{\tau<\sigma\}} \\ + \Gamma\,\mathbf{1}_{\{\sigma=\tau=T\}} \,\bigg|\, \mathcal{F}_t\bigg], \qquad \sigma, \tau \in \mathcal{T}_{t,T}. \end{aligned} \tag{1}$$

This is a zero-sum game since costs (gains) for MIN are the gains (costs) for MAX. For a given $t \in [0, T]$, player MIN chooses the strategy $\sigma \in \mathcal{T}_{t,T}$ to minimise $D_{t,T}(\sigma, \tau)$ whereas MAX plays the strategy $\tau \in \mathcal{T}_{t,T}$ to maximise it. This leads to upper and lower values for the game on $[t, T]$, which are denoted by V_t^+ and V_t^- respectively:

$$V_t^+ = \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau), \qquad V_t^- = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau).$$

Definition 1. (*Game value.*) The Dynkin game on $[t, T]$ is said to be 'fair' if there is equality between the time-t upper and lower values, i.e.

$$\operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau) = V_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau).$$

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The common value, denoted by V_t, is also referred to as the solution or value of the game on $[t, T]$.

When studying Dynkin games, the first course of action is to verify that the game is fair. Afterwards, one searches for strategies for the players which give the game's value or approximates it closely. This leads to the concept of a *Nash equilibrium*.

Definition 2. (*Nash equilibrium.*) A pair of stopping times $(\sigma^*, \tau^*) \in \mathcal{T}_{t,T} \times \mathcal{T}_{t,T}$ is said to constitute a *Nash equilibrium* or a *saddle point* for the game on $[t, T]$ if, for any $\sigma, \tau \in \mathcal{T}_{t,T}$,

$$D_{t,T}(\sigma^*, \tau) \leq D_{t,T}(\sigma^*, \tau^*) \leq D_{t,T}(\sigma, \tau^*).$$

It is not difficult to verify that the existence of a saddle point $(\sigma^*, \tau^*) \in \mathcal{T}_{t,T} \times \mathcal{T}_{t,T}$ implies the game on $[t, T]$ is fair and its value is given by

$$\operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau) = D_{t,T}(\sigma^*, \tau^*) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\sigma \in \mathcal{T}_{t,T}} D_{t,T}(\sigma, \tau).$$

Under quite mild integrability and regularity assumptions on ψ and $\gamma_\pm$, it is known (see, for example, [3]) that there exists a càdlàg $\mathbb{F}_T$-adapted process $(V_t)_{0 \leq t \leq T}$ such that for each t the random variable V_t gives the fair value of the Dynkin game on $[t, T]$. Furthermore, if the stopping costs $\gamma_\pm$ are sufficiently regular then the debut times D_t^+ and D_t^- defined by

$$D_t^+ := \inf\{s \geq t \colon V_s = -\gamma_+(s)\} \wedge T, \qquad D_t^- := \inf\{s \geq t \colon V_s = \gamma_-(s)\} \wedge T$$

form a saddle point (D_t^-, D_t^+) for the Dynkin game on $[t, T]$. We arrive at a similar conclusion in this paper using two-mode optimal switching.

2.2. Two-mode optimal switching

The two-mode optimal switching or 'starting and stopping' problem has been studied in a variety of contexts as the papers [7] and [9] and the references therein can attest. Following convention, we denote the two modes by 0 and 1. For $i \in \{0, 1\}$ there is a random profit rate $\psi_i \colon \Omega \times [0, T] \to \mathbb{R}$ and time T reward $\Gamma_i \colon \Omega \to \mathbb{R}$. For each $(i, j) \in \{0, 1\} \times \{0, 1\}$ there is a cost for switching from i to j determined by the mapping $\gamma_{i,j} \colon \Omega \times [0, T] \to \mathbb{R}$.

Definition 3. (*Auxiliary two-mode switching problem parameters.*) Define parameters for the optimal switching problem from payoff (1) of the Dynkin game as follows.

Switching costs. For $i \in \{0, 1\}$, set $\gamma_{ii}(\cdot) = 0$, $\gamma_{i,1-i}(t) := \gamma_-(t)\,\mathbf{1}_{\{i=0\}} + \gamma_+(t)\,\mathbf{1}_{\{i=1\}}$.

Profit rate. Set $\psi_1(\cdot) \equiv \psi(\cdot)$ and $\psi_0(\cdot) \equiv 0$.

Terminal reward. Set $\Gamma_1 \equiv \Gamma$ and $\Gamma_0 \equiv 0$.

Definition 4. (*Admissible switching controls.*) For a fixed time $t \in [0, T]$ and initial mode $i \in \{0, 1\}$, an admissible switching control $\alpha = (\tau_n, \iota_n)_{n \geq 0}$ consists of the following parts.

- A nondecreasing sequence $\{\tau_n\}_{n \geq 0} \subset \mathcal{T}_{t,T}$ with $\tau_0 = t$ $\mathbb{P}$-a.s.
- A sequence $\{\iota_n\}_{n \geq 0}$, where $\iota_0 = i$ is the fixed initial value, $\iota_n \colon \Omega \to \{0, 1\}$ is $\mathcal{F}_{\tau_n}$-measurable and satisfies $\iota_{2n} = i$, and $\iota_{2n+1} = 1 - i$ for $n \geq 0$.
- The stopping times $\{\tau_n\}_{n \geq 0}$ are finite in the following sense:

$$\mathbb{P}(\{\tau_n < T, \text{ for all } n \geq 0\}) = 0.$$

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- The (double) sequence α satisfies

$$\mathbb{E}\Big[\sup_n |C_n^\alpha|\Big] < \infty,$$

where C_n^α is the total cost of the first $n \geq 1$ switches under α:

$$C_n^\alpha := \sum_{k=1}^{n} \gamma_{\iota_{k-1},\iota_k}(\tau_k)\,\mathbf{1}_{\{\tau_k < T\}}, \qquad n \geq 1.$$

Let $\mathcal{A}_{t,i}$ denote the set of admissible switching controls. We write $\mathcal{A}_i$ when $t = 0$ and drop the superscript i when the initial mode is not important for the discussion.

Associated with each $\alpha \in \mathcal{A}$ is a (random) function $\boldsymbol{u}\colon \Omega \times [0, T] \to \{0, 1\}$ referred to as the mode indicator function:

$$\boldsymbol{u}_t := \iota_0\,\mathbf{1}_{[\tau_0,\tau_1]}(t) + \sum_{n\geq 1} \iota_n\,\mathbf{1}_{(\tau_n,\tau_{n+1}]}(t), \qquad t \in [0, T].$$

The objective function for the switching control problem associated with the Dynkin game on $[t, T]$ is given by

$$J(\alpha; t, i) = \mathbb{E}\bigg[\int_t^T \psi_{\boldsymbol{u}_s}(s)\,\mathrm{d}s + \Gamma_{\boldsymbol{u}_T} - \sum_{n\geq 1} \gamma_{\iota_{n-1},\iota_n}(\tau_n)\,\mathbf{1}_{\{\tau_n < T\}} \;\bigg|\; \mathcal{F}_t\bigg], \qquad \alpha \in \mathcal{A}_{t,i}. \tag{2}$$

Together with appropriate integrability assumptions on ψ and Γ, the objective function is well defined for any $\alpha \in \mathcal{A}$. For $(t, i) \in [0, T] \times \{0, 1\}$ given and fixed, the goal is to find a control $\alpha^* \in \mathcal{A}_{t,i}$ that maximises the performance index:

$$J(\alpha^*; t, i) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,i}} J(\alpha; t, i).$$

Remark 1. Processes or functions with superscripts or subscripts in terms of the random mode indicators ι_n are interpreted in the following way:

$$Y^{\iota_n} = \sum_{j\in\{0,1\}} \mathbf{1}_{\{\iota_n = j\}}\,Y^j, \qquad n \geq 0,$$

$$\gamma_{\iota_{n-1},\iota_n}(\cdot) = \sum_{j\in\{0,1\}} \sum_{k\in\{0,1\}} \mathbf{1}_{\{\iota_{n-1}=j\}}\,\mathbf{1}_{\{\iota_n=k\}}\,\gamma_{j,k}(\cdot), \qquad n \geq 1.$$

3. Notation and assumptions

3.1. Notation

In this paper we frequently refer to concepts such as 'predictable' and 'quasi-left-continuous' from the general theory of the stochastic processes. The reader may consult reference texts such as [10] and [24] for further details. We note that we follow the conventions of [23] and [24] for predictable times and processes (defined on the parameter set $(0, \infty)$).

- For $p \geq 1$, let L^p denote the set of random variables Z satisfying $\mathbb{E}[|Z|^p] < \infty$.

- For $p \geq 1$, let $\mathcal{M}^p$ denote the set of $\mathbb{F}$-progressively measurable, real-valued processes $X = (X_t)_{t\geq 0}$ satisfying
$$\mathbb{E}\bigg[\int_0^\infty |X_t|^p \,\mathrm{d}t\bigg] < \infty.$$
- For $p \geq 1$, let $\mathcal{S}^p$ denote the set of $\mathbb{F}$-progressively measurable processes X satisfying
$$\mathbb{E}\Big[\Big(\sup_{t\geq 0} |X_t|\Big)^p\Big] < \infty.$$
- Let $\mathcal{Q}$ denote the set of $\mathbb{F}$-adapted, càdlàg processes which are quasi-left-continuous (left-continuous over stopping times).

For a given $0 < T < \infty$ we use the analogous notation $\mathcal{M}_T^p$, $\mathcal{S}_T^p$, and $\mathcal{Q}_T$ for the finite time horizon $[0, T]$.

3.2. Assumptions

In this subsection $T \in (0, \infty)$ is arbitrary.

Assumption 1. *We impose the following integrability, measurability, and regularity assumptions:*

- *the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and is quasi-left-continuous,*
- *the instantaneous payoff rate satisfies $\psi \in \mathcal{M}_T^2$,*
- *the early exit stopping costs for the game satisfy $\gamma_-, \gamma_+ \in \mathcal{S}_T^2 \cap \mathcal{Q}_T$,*
- *the terminal payoff satisfies $\Gamma \in L^2$ and is $\mathcal{F}_T$-measurable.*

Assumption 2. *We have the following stopping costs assumptions:*

$$-\gamma_+(T) \leq \Gamma \leq \gamma_-(T), \qquad \mathbb{P}\text{-a.s.}, \tag{3}$$
$$\text{for all } t \in [0, T]: \qquad \gamma_-(t) + \gamma_+(t) > 0, \qquad \mathbb{P}\text{-a.s.} \tag{4}$$

Condition (3) is standard in the literature on Dynkin games [3] whilst condition (4) is typical of optimal switching problems [7].

4. Existence of a Nash equilibrium via optimal switching

In this section we use martingale methods to prove for every $t \in [0, T]$ that there exists a saddle point (σ_t^*, τ_t^*) for the Dynkin game on $[t, T]$ with payoff (1).

4.1. The Snell envelope

Remember that an $\mathbb{F}_T$-progressively measurable process X is said to belong to class $[D]$ if the set of random variables $\{X_\tau,\ \tau \in \mathcal{T}_{0,T}\}$ is uniformly integrable.

Proposition 1. *Let $G = (G_t)_{0\leq t\leq T}$ be an adapted, $\mathbb{R}$-valued, càdlàg process that belongs to class $[D]$. Then there exists a unique (up to indistinguishability), adapted $\mathbb{R}$-valued, càdlàg process $Z = (Z_t)_{0\leq t\leq T}$ such that Z is the smallest supermartingale which dominates G. The process Z is called the Snell envelope of G and it has the following properties.*

(i) *For any $\theta \in \mathcal{T}_{0,T}$ we have*
$$Z_\theta = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\theta,T}} \mathbb{E}[G_\tau \mid \mathcal{F}_\theta], \quad \text{and therefore } Z_T = U_T.$$

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(ii) *(Meyer decomposition) There exist a uniformly integrable càdlàg martingale M and a predictable integrable increasing process A such that, for all $0 \le t \le T$,*

$$Z_t = M_t - A_t, \qquad A_0 = 0. \tag{5}$$

(iii) *Let $\theta \in \mathcal{T}_{0,T}$ be given and $\{\tau_n\}_{n\ge 0} \subset \mathcal{T}_{\theta,T}$ be an increasing sequence of stopping times tending to a limit $\tau \in \mathcal{T}_{\theta,T}$ and such that $\mathbb{E}[G^-_{\tau_n}] < \infty$ for $n \ge 0$. Suppose that the following condition is satisfied for any such sequence:*

$$\limsup_{n\to\infty} G_{\tau_n} \le G_\tau,$$

then $\tau^*_\theta \in \mathcal{T}_{\theta,T}$ defined by

$$\tau^*_\theta = \inf\{t \ge \theta \colon Z_t = G_t\} \wedge T \tag{6}$$

is optimal after θ in the sense that

$$Z_\theta = \mathbb{E}[Z_{\tau^*_\theta} \mid \mathcal{F}_\theta] = \mathbb{E}[G_{\tau^*_\theta} \mid \mathcal{F}_\theta] = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\theta,T}} \mathbb{E}[G_\tau \mid \mathcal{F}_\theta].$$

(iv) *For every $\theta \in \mathcal{T}_{0,T}$, if τ^*_θ is the stopping time defined in (6) then the stopped process $(Z_{t\wedge\tau^*_\theta})_{\theta\le t\le T}$ is a (uniformly integrable) càdlàg martingale.*

Proofs of these properties can be found in [5], [18], or [21] for instance.

4.2. The martingale approach to optimal switching problems

Under Assumptions 1 and 2, we can prove that there exists a unique pair of processes $(Y^0_t, Y^1_t)_{0\le t\le T}$ such that, for $i \in \{0, 1\}$, Y^i solves the optimal switching problem in a probabilistic sense. This can be accomplished using the theory of Snell envelopes and the details can be found in [17].

Theorem 1. *There exists a unique pair of processes $(Y^0_t, Y^1_t)_{0\le t\le T}$ belonging to $\mathcal{S}^2_T \cap \mathcal{Q}_T$ satisfying $\mathbb{P}$-a.s.,*

$$Y^i_t = \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\biggl[\int_t^\theta \psi_i(s)\,\mathrm{d}s + \Gamma_i\,\mathbf{1}_{\{\theta=T\}} + \{Y^{1-i}_\theta - \gamma_{i,1-i}(\theta)\}\,\mathbf{1}_{\{\theta<T\}} \biggm| \mathcal{F}_t\biggr], \qquad Y^i_T = \Gamma_i, \tag{7}$$

where $i \in \{0, 1\}$ and $0 \le t \le T$. Furthermore, for every $(t, i) \in [0, T] \times \{0, 1\}$, there exists a control $\alpha^* \in \mathcal{A}_{t,i}$ such that

$$Y^i_t = J(\alpha^*; t, i) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,i}} J(\alpha; t, i).$$

4.3. Existence of a Nash equilibrium

Let Y^0 and Y^1 be the processes in Theorem 1 and define $G^i = (G^i_t)_{0\le t\le T}$, $i \in \{0, 1\}$, by

$$G^i_t = \Gamma_i\,\mathbf{1}_{\{t=T\}} + \{Y^{1-i}_t - \gamma_{i,1-i}(t)\}\,\mathbf{1}_{\{t<T\}}. \tag{8}$$

The process $(G^i_t + \int_0^t \psi_i(s)\,\mathrm{d}s)_{0\le t\le T}$ is càdlàg and in $\mathcal{S}^2_T$. By Proposition 1, it follows that $(Y^i_t + \int_0^t \psi_i(s)\,\mathrm{d}s)_{0\le t\le T}$ is the Snell envelope of $(G^i_t + \int_0^t \psi_i(s)\,\mathrm{d}s)_{0\le t\le T}$. By Assumptions 1 and 2, and as $Y^i \in \mathcal{S}^2_T \cap \mathcal{Q}_T$ for $i \in \mathbb{I}$, G^i is quasi-left-continuous on $[0, T)$ with a possible positive jump at T. We can therefore apply Proposition 1(iii) to verify that for any $t \in [0, T]$, the stopping time $\rho_t^{i,*}$, defined by

$$\rho_t^{i,*} = \inf\{s \geq t \colon Y_s^i = Y_s^{1-i} - \gamma_{i,1-i}(s)\} \wedge T, \tag{9}$$

is the optimal first switching time on $[t, T]$ when starting in mode $i \in \{0, 1\}$. For each $t \in [0, T]$, use (9) to define a pair of stopping times (σ_t^*, τ_t^*) by

$$\sigma_t^* = \rho_t^{0,*}, \qquad \tau_t^* = \rho_t^{1,*}. \tag{10}$$

We will prove that (σ_t^*, τ_t^*) is a saddle point for the Dynkin game on $[t, T]$. In order to do so, we first establish the following lemma which relates the pair (Y^0, Y^1) to *Mokobodski's hypothesis*.

Lemma 1. *The processes Y^0 and Y^1 of Theorem 1 satisfy the condition, for all $\tau \in \mathcal{T}_{0,T}$,*

$$-\gamma_+(\tau) \leq Y_\tau^1 - Y_\tau^0 \leq \gamma_-(\tau), \qquad \mathbb{P}\text{-a.s.} \tag{11}$$

Proof. For $i \in \{0, 1\}$, let $G^i = (G_t^i)_{0 \leq t \leq T}$ be defined as in (8). Remember that $Y_t^i + \int_0^t \psi_i(s)\,\mathrm{d}s$ is the Snell envelope of $G_t^i + \int_0^t \psi_i(s)\,\mathrm{d}s$ on $0 \leq t \leq T$. Let $\tau \in \mathcal{T}_{0,T}$ be arbitrary. By the dominating property of the (right-continuous) Snell envelope, $Y_\tau^i \geq G_\tau^i$ holds $\mathbb{P}$-a.s. and this shows that

$$0 \leq Y_\tau^i - G_\tau^i = Y_\tau^i + \gamma_{i,1-i}(\tau) - Y_\tau^{1-i}, \quad \text{a.s. on } \{\tau < T\}.$$

From this we obtain

$$-\gamma_+(\tau) \leq Y_\tau^1 - Y_\tau^0 \leq \gamma_-(\tau), \quad \text{a.s. on } \{\tau < T\}.$$

On the other hand, we have $Y_\tau^1 - Y_\tau^0 = \Gamma$, $\mathbb{P}$-a.s. on the event $\{\tau = T\}$. Using this with (3) gives

$$-\gamma_+(\tau) \leq Y_\tau^1 - Y_\tau^0 \leq \gamma_-(\tau), \quad \text{a.s on } \{\tau = T\},$$

and claim (11) holds. □

Theorem 2. *Let Y^0 and Y^1 be the processes in Theorem 1. Then for every $t \in [0, T]$, (σ_t^*, τ_t^*) defined in (10) satisfies*

$$Y_t^1 - Y_t^0 = D_{t,T}(\sigma_t^*, \tau_t^*), \qquad \mathbb{P}\text{-a.s.}, \tag{12}$$

where $D_{t,T}(\cdot, \cdot)$ is payoff (1). Furthermore, for any $\sigma, \tau \in \mathcal{T}_{t,T}$,

$$D_{t,T}(\sigma_t^*, \tau) \leq D_{t,T}(\sigma_t^*, \tau_t^*) \leq D_{t,T}(\sigma, \tau_t^*). \tag{13}$$

Proof. The claim is trivially satisfied for $t = T$, so, henceforth, let $t \in [0, T)$ be a given but arbitrary time. For $i \in \{0, 1\}$, let $G^i = (G_s^i)_{0 \leq s \leq T}$ be defined as in (8). Define a process $\hat{Y}^1 = (\hat{Y}_s^1)_{0 \leq s \leq T}$ by $\hat{Y}_s^1 := Y_s^1 + \int_0^s \psi(r)\,\mathrm{d}r$. By [23, Theorem II.77.4], a stopped supermartingale is also a supermartingale. For every $\sigma, \tau \in \mathcal{T}_{t,T}$ the stopped Snell envelopes $(Y^0_{s \wedge (\sigma \wedge \tau_t^*)})_{t \leq s \leq T}$ and $(\hat{Y}^1_{s \wedge (\sigma_t^* \wedge \tau)})_{t \leq s \leq T}$ are therefore supermartingales. Additionally, using the martingale property of the stopped Snell envelope in Proposition 1, we see that $\hat{Y}^1 - Y^0$ satisfies the following:

- $(\hat{Y}_s^1 - Y_s^0)_{t \leq s \leq (\sigma_t^* \wedge \tau_t^*)}$ is a martingale,

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- for any $\sigma, \tau \in \mathcal{T}_{t,T}$, $(\hat{Y}_s^1 - Y_s^0)_{t \le s \le (\sigma_t^* \wedge \tau)}$ is a supermartingale,
- for any $\sigma, \tau \in \mathcal{T}_{t,T}$, $(\hat{Y}_s^1 - Y_s^0)_{t \le s \le (\sigma \wedge \tau_t^*)}$ is a submartingale.

This characterisation enables us to prove both (12) and (13). The arguments used to establish (13) are essentially the same as we used to show (12), modulo straightforward changes from equalities to inequalities based on Assumption 2 and Lemma 1. We therefore only prove (12).

The martingale property of $\hat{Y}^1 - Y^0$ on $[t, \sigma_t^* \wedge \tau_t^*]$ allows us to deduce the following:

$$Y_t^1 - Y_t^0 = \mathbb{E}\bigg[\int_t^{\sigma_t^* \wedge \tau_t^*} \psi(r)\,\mathrm{d}r + Y^1_{\sigma_t^* \wedge \tau_t^*} - Y^0_{\sigma_t^* \wedge \tau_t^*} \,\bigg|\, \mathcal{F}_t\bigg]. \tag{14}$$

The term involving the pair (Y^0, Y^1) inside of the conditional expectation may be written as

$$\begin{aligned}\mathbb{E}[Y^1_{\sigma_t^* \wedge \tau_t^*} - Y^0_{\sigma_t^* \wedge \tau_t^*} \mid \mathcal{F}_t] &= \mathbb{E}[(Y^1_{\sigma_t^*} - Y^0_{\sigma_t^*})\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}} \mid \mathcal{F}_t] \\ &\quad + \mathbb{E}[(Y^1_{\tau_t^*} - Y^0_{\tau_t^*})\,\mathbf{1}_{\{\tau_t^* < \sigma_t^*\}} \mid \mathcal{F}_t].\end{aligned} \tag{15}$$

By (10) and conditional on the event $\{\tau_t^* < T\}$, optimality of the stopping time τ_t^* gives the following:

$$Y^1_{\tau_t^*}\,\mathbf{1}_{\{\tau_t^* < T\}} = [-\gamma_+(\tau_t^*) + Y^0_{\tau_t^*}]\,\mathbf{1}_{\{\tau_t^* < T\}}\,. \tag{16}$$

Furthermore, $\mathbf{1}_{\{\sigma_t^* > \tau_t^*\}} = \mathbf{1}_{\{\sigma_t^* > \tau_t^*\}}\,\mathbf{1}_{\{\tau_t^* \le T\}} = \mathbf{1}_{\{\sigma_t^* > \tau_t^*\}}\,\mathbf{1}_{\{\tau_t^* < T\}}$ since $\tau_t^* \le T$ and $\sigma_t^* \le T$, $\mathbb{P}$-a.s., and we can use (16) to verify the following, $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathbb{E}[(Y^1_{\tau_t^*} - Y^0_{\tau_t^*})\,\mathbf{1}_{\{\tau_t^* < \sigma_t^*\}} \mid \mathcal{F}_t] &= \mathbb{E}[(Y^1_{\tau_t^*} - Y^0_{\tau_t^*})\,\mathbf{1}_{\{\tau_t^* < \sigma_t^*\}}\,\mathbf{1}_{\{\tau_t^* < T\}} \mid \mathcal{F}_t] \\ &= \mathbb{E}[(-\gamma_+(\tau_t^*))\,\mathbf{1}_{\{\tau_t^* < \sigma_t^*\}} \mid \mathcal{F}_t].\end{aligned} \tag{17}$$

By (10) and conditional on the event $\{\sigma_t^* < T\}$, optimality of the stopping time σ_t^* gives

$$Y^0_{\sigma_t^*}\,\mathbf{1}_{\{\sigma_t^* < T\}} = [-\gamma_-(\sigma_t^*) + Y^1_{\sigma_t^*}]\,\mathbf{1}_{\{\sigma_t^* < T\}},$$

which is used to deduce

$$\mathbb{E}[(Y^1_{\sigma_t^*} - Y^0_{\sigma_t^*})\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}\,\mathbf{1}_{\{\sigma_t^* < T\}} \mid \mathcal{F}_t] = \mathbb{E}[\gamma_-(\sigma_t^*)\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}\,\mathbf{1}_{\{\sigma_t^* < T\}} \mid \mathcal{F}_t]. \tag{18}$$

Since $\tau_t^* \le T$ $\mathbb{P}$-a.s. we have $\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}\,\mathbf{1}_{\{\sigma_t^* = T\}} = \mathbf{1}_{\{\sigma_t^* = \tau_t^* = T\}}$, and using $Y_T^1 = \Gamma$ and $Y_T^0 = 0$ a.s., we obtain

$$\mathbb{E}[(Y^1_{\sigma_t^*} - Y^0_{\sigma_t^*})\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}\,\mathbf{1}_{\{\sigma_t^* = T\}} \mid \mathcal{F}_t] = \mathbb{E}[\Gamma\,\mathbf{1}_{\{\sigma_t^* = \tau_t^* = T\}} \mid \mathcal{F}_t]. \tag{19}$$

Again, since $\sigma_t^* \le T$ $\mathbb{P}$-a.s., we can use (18) and (19) to assert

$$\begin{aligned}\mathbb{E}[(Y^1_{\sigma_t^*} - Y^0_{\sigma_t^*})\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}} \mid \mathcal{F}_t] &= \mathbb{E}[(Y^1_{\sigma_t^*} - Y^0_{\sigma_t^*})\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}(\mathbf{1}_{\{\sigma_t^* < T\}} + \mathbf{1}_{\{\sigma_t^* = T\}}) \mid \mathcal{F}_t] \\ &= \mathbb{E}[\gamma_-(\sigma_t^*)\,\mathbf{1}_{\{\sigma_t^* \le \tau_t^*\}}\,\mathbf{1}_{\{\sigma_t^* < T\}} \mid \mathcal{F}_t] \\ &\quad + \mathbb{E}[\Gamma\,\mathbf{1}_{\{\sigma_t^* = \tau_t^* = T\}} \mid \mathcal{F}_t].\end{aligned} \tag{20}$$

We then prove claim (12) by using (15), (17), and (20) substituted into (14). □

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Remark 2. The results of Theorem 2 were obtained in a similar fashion to results in several other papers in the literature which have used probabilistic approaches. For instance, [19] (particularly Theorem 1) which used martingale methods for Dynkin games, [20] (particularly Theorem 2.1) which has a semiharmonic characterisation of the value function for the Dynkin game in a Markovian setting, and [2] and [8] which used the concept of doubly reflected backward stochastic differential equations (DRBSDEs).

Remark 3. Although we started with a Dynkin game and subsequently formulated an optimal switching problem, we could have derived these results by doing the reverse. More precisely, take any two-mode optimal switching problem (satisfying the assumptions in Section 3) with terminal reward data Γ_1, Γ_0, and instantaneous profit processes ψ_1, ψ_0. We then formulate the corresponding Dynkin game by setting $\Gamma := \Gamma_1 - \Gamma_0$ and $\psi := \psi_1 - \psi_0$ and using the switching cost function to identify the stopping costs for the game as in Definition 3.

5. Dependence of the game's solution on the time horizon

We suppose in this section and the next that there exists a standard Brownian motion $B = (B_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and furthermore that $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the completed natural filtration of B. It is well known that in this case all $\mathbb{F}$-stopping times are predictable. Therefore, all $\mathbb{F}$-adapted processes belonging to $\mathcal{Q}$ have paths which are $\mathbb{P}$-a.s. continuous.

Suppose that ψ and $\gamma_\pm$ of Section 2.1 are defined on all of $[0,\infty)$ with $\psi \in \mathcal{M}^2$ and $\gamma_\pm \in \mathcal{S}^2 \cap \mathcal{Q}$ ($\gamma_\pm$ still satisfying Assumption 2). Additionally, for simplicity and ease of notation in what follows, we suppose that $\psi \equiv 0$ and define two processes $L = (L_t)_{t\geq 0}$ and $U = (U_t)_{t\geq 0}$ by $L_t = -\gamma_+(t)$ and $U_t = \gamma_-(t)$.

For $0 < T \leq \infty$ and $t \in [0,T]$, we define the following payoff for a Dynkin game. For $\sigma, \tau \in \mathcal{T}_{t,T}$,

$$D_{t,T}(\sigma,\tau) = \mathbb{E}[U_\sigma \mathbf{1}_{\{\sigma\leq\tau\}} \mathbf{1}_{\{\sigma<T\}} + L_\tau \mathbf{1}_{\{\tau<\sigma\}} + \Gamma^T \mathbf{1}_{\{\sigma=\tau=T\}} \mid \mathcal{F}_t], \tag{21}$$

where $\Gamma^T \in L^2$ is $\mathcal{F}_T$-measurable. In the case $T = \infty$ we assume $\liminf_t U_t \leq \limsup_t L_t$ and Γ^∞ satisfies either $\Gamma^\infty := \limsup_t L_t$ or $\Gamma^\infty := \liminf_t U_t$ as appropriate.

Under appropriate conditions in both finite- and infinite-horizon settings, it is known (see, for example, [3], or this paper for the finite-horizon case) that there is a càdlàg $\mathbb{F}_T$-adapted process V^T such that the random variable V_t^T is the value of the game with payoff, (21). In this section we prove that the deterministic (since $\mathcal{F}_0$ is trivial) mapping $T \mapsto V_0^T$ is continuous on $(0,\infty)$. This will be obtained as a straightforward consequence of recent results in [22] on norm estimates for DRBSDEs.

5.1. DRBSDEs

In order to motivate the discussion on DRBSDEs we make the following observations. By Theorem 1, we know that for each $T \in (0,\infty)$ given and fixed that there exist processes $Y^{0,T}$ and $Y^{1,T}$ belonging to $\mathcal{S}_T^2 \cap \mathcal{Q}_T$ satisfying (7). Moreover, since $\psi \equiv 0$ it is also true that $Y^{0,T}$ and $Y^{1,T}$ are Snell envelopes of appropriate processes and are therefore supermartingales. Let $(M^{i,T}, A^{i,T})$ denote the Meyer decomposition for $Y^{i,T}$, $i \in \{0,1\}$ (cf. (5)). We note that both $M^{i,T}$ and $A^{i,T}$ belong to $\mathcal{S}_T^2$ since $Y^{i,T} \in \mathcal{S}_T^2$ and the filtration $\mathbb{F}_T$ is quasi-left-continuous. Using this decomposition, $Y_T^{i,T} = \Gamma_i$, and Brownian martingale representation for $M^{i,T}$, we have, for all $t \in [0,T]$,

$$Y_t^{i,T} = \Gamma^{i,T} - \int_t^T \zeta_s^{i,T}\,\mathrm{d}B_s + A_T^{i,T} - A_t^{i,T}, \qquad \mathbb{P}\text{-a.s.}, \tag{22}$$

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where $\zeta^{i,T} \in \mathcal{M}_T^2$ is predictable. Furthermore, one can also show (see, for example, [12, Proposition B.11]) that

$$\int_0^T [Y_t^{i,T} - (Y_t^{1-i,T} - \gamma_{i,1-1}(t))]\,\mathrm{d}A_t^{i,T} = 0, \qquad \mathbb{P}\text{-a.s.} \tag{23}$$

Recall from Theorem 2 that the process $V^T = (V_t^T)_{0\le t\le T}$ defined by $V_t^T = Y_t^{1,T} - Y_t^{0,T}$ solves the Dynkin game with payoff (21). Recalling Definition 3 and Lemma 1 and using (22) and (23), we see that on $[0, T]$ the process V^T satisfies

$$\begin{gathered} V_t^T = \Gamma^T - \int_t^T \zeta_s^T\,\mathrm{d}B_s + K_T^T - K_t^T, \\ [V_t^T - L_t]\,\mathrm{d}A_t^{1,T} = [U_t - V_t^T]\,\mathrm{d}A_t^{0,T} = 0, \qquad L \le V^T \le U, \end{gathered} \tag{24}$$

where $\zeta^T := \zeta^{1,T} - \zeta^{0,T}$ and $K^T := A^{1,T} - A^{0,T}$.

We now introduce some notation and recall some results from [22]. For $0 < T < \infty$ and $\mathbb{F}_T$-adapted càdlàg processes X and X':

- $\|X\|_{\mathcal{S}_T^2} := (\mathbb{E}[(\sup_{0\le t\le T}|X_t|)^2])^{1/2}$,
- for $0 \le t_1 < t_2 \le T$, $\vee_{t_1}^{t_2} X$ denotes the total variation of X over $(t_1, t_2]$,
- $\|(X, X')\|_{\mathcal{S}_T^2} := (\|X^+\|_{\mathcal{S}_T^2}^2 + \|(X')^-\|_{\mathcal{S}_T^2}^2)^{1/2}$, where X^+ is the positive part of X and $(X')^-$ is the negative part of X',
- letting $\hat{X}_t = \max(X_t, X_{t-})$, $\check{X}'_t = \min(X'_t, X'_{t-})$,

$$\begin{aligned} \|(X, X')\|_T^2 := \sup_\pi \mathbb{E}\biggl[\biggl(\sum_{i=0}^{n-1}([\mathbb{E}[\hat{X}_{\tau_{i+1}} \mid \mathcal{F}_{\tau_i}] - \check{X}'_{\tau_i}]^+ + [\hat{X}_{\tau_i} - \mathbb{E}[\check{X}'_{\tau_{i+1}} \mid \mathcal{F}_{\tau_i}]]^+)\biggr)^2\biggr] \\ + \|(X, X')\|_{\mathcal{S}_T^2}^2, \end{aligned}$$

where the supremum is taken over all stopping time partitions $\pi\colon 0 = \tau_0 \le \cdots \le \tau_n = T$.

Definition 5. Following [22, p. 10], a (global) solution to the DRBSDE associated with a coefficient (or driver) $f(\omega, t, v, z)\colon \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, an $\mathcal{F}_T$-measurable terminal value Γ^T, and respective lower and upper barriers, L and U, is a triple (V, ζ, K) of $\mathbb{F}_T$-progressively measurable processes satisfying

$$\begin{gathered} V_t = \Gamma^T + \int_t^T f(s, V_s, \zeta_s)\,\mathrm{d}s - \int_t^T \zeta_s\,\mathrm{d}B_s + K_T - K_t, \\ [V_{t-} - L_{t-}]\,\mathrm{d}A_t^+ = [U_{t-} - V_{t-}]\,\mathrm{d}A_t^- = 0, \qquad L \le V \le U, \end{gathered} \tag{25}$$

where V is càdlàg, K is a process of finite variation with orthogonal decomposition $K := A^+ - A^-$, and

$$\|(V, \zeta, K)\|_T^2 := \mathbb{E}\biggl[\biggl(\sup_{0\le t\le T}|V_t|\biggr)^2 + \int_0^T |\zeta_t|^2\,\mathrm{d}t + \biggl(\bigvee_0^T K\biggr)^2\biggr] < \infty.$$

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Recalling (24) and the properties of (V^T, ζ^T, K^T), we see that the triple (V^T, ζ^T, K^T) is a solution to the DRBSDE (24) in the sense of Definition 5. Moreover, using Lemma 1 (Mokobodski's hypothesis) and [22, Theorem 3.4], for instance, we also know that (V^T, ζ^T, K^T) is, modulo indistinguishability, the unique solution to (24) in this instance.

5.2. Dependence of solutions to DRBSDEs on the time horizon

Henceforth, we consider solutions to the DRBSDE (25) only with $f \equiv 0$. Let us fix $T \in (0, \infty)$ and let $\{T_n\}_{n\geq 0} \subset (0, \infty)$ be any sequence monotonically decreasing to T: $T_n \downarrow T$. We extend the unique solution (V, ζ, K) to (25) on $[0, T]$ to (V^T, ζ^T, K^T) defined on $[0, T_0]$ in the following way. For each $t \in [0, T_0]$,

$$V_t^T := V_{t\wedge T}, \qquad \zeta_t^T := \zeta_{t\wedge T}\,\mathbf{1}_{\{t\leq T\}}, \qquad K_t^T \equiv A_t^{+,T} - A_t^{-,T} \quad \text{with } A_t^{\pm,T} := A_{t\wedge T}^{\pm}. \tag{26}$$

Defining the respective lower and upper barriers L^T and U^T on $[0, T_0]$ by $L_t^T := L_{t\wedge T}$ and $U_t^T := U_{t\wedge T}$, it is straightforward to check that (V^T, ζ^T, K^T) is the unique solution on $[0, T_0]$ to the DRBSDE

$$\begin{gathered} V_t^T = \Gamma^T - \int_t^{T_0} \zeta_s^T \,\mathrm{d}B_s + K_{T_0}^T - K_t^T, \\ [V_{t-}^T - L_{t-}^T]\,\mathrm{d}A_t^{+,T} = [U_{t-}^T - V_{t-}^T]\,\mathrm{d}A_t^{-,T} = 0, \qquad L^T \leq V^T \leq U^T, \end{gathered} \tag{27}$$

in the sense of Definition 5.

Assumption 3. *Suppose that we are given a sequence $\{\Gamma^{T_n}\}_{n\geq 0}$ of random variables satisfying:*

- *each Γ^{T_n} is $\mathbb{F}_{T_n}$-measurable,*
- $L_{T_n} \leq \Gamma^{T_n} \leq U_{T_n}$,
- $\Gamma^{T_n} \to \Gamma^T$ *a.s. as* $n \to \infty$,
- $\sup_{n\geq 0} |\Gamma^{T_n}| \in L^2$.

Note that the last two conditions imply $\Gamma^{T_n} \to \Gamma^T$ in L^2 as $n \to \infty$. Let $(V^{T_n}, \zeta^{T_n}, K^{T_n})$ denote the unique solution on $[0, T_n]$ to the DRBSDE (25). We then extend these solutions to $[0, T_0]$ in the same way as before (see (26)–(27)), with respective lower and upper barriers L^{T_n} and U^{T_n}. We continue writing $(V^{T_n}, \zeta^{T_n}, K^{T_n})$ to denote these extensions to avoid excessive notation.

Define $\delta^{(n)}V := (V^{T_n} - V^T)$ and similarly for other cases. Pham and Zhang [22, Theorem 3.5] proved the following estimate:

$$\begin{aligned} &\mathbb{E}\bigg[\sup_{0\leq t\leq T_0} [|\delta^{(n)}V_t|^2 + |\delta^{(n)}K_t|^2] + \int_0^{T_0} |\delta^{(n)}\zeta_t|^2 \,\mathrm{d}t\bigg] \\ &\quad \leq C\mathbb{E}[|\delta^{(n)}\Gamma|^2] + C(\mathbb{E}[|\Gamma^T|^2 + |\Gamma^{T_n}|^2] + \|(L^{T_n}, U^{T_n})\|_{T_0} + \|(L^T, U^T)\|_{T_0}) \\ &\qquad \times \bigg(\mathbb{E}\bigg[\sup_{0\leq t\leq T_0} [|\delta^{(n)}L_t|^2 + |\delta^{(n)}U_t|^2]\bigg]\bigg)^{1/2}, \end{aligned} \tag{28}$$

where C is a positive constant.

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5.3. Dependence of the value of the Dynkin game on the time horizon

We now return to the theme of this section, which is to show that $T \mapsto V_0^T$ is continuous on $(0, \infty)$. For this it suffices to show that for every $T \in (0, \infty)$ and arbitrary sequence $\{T_n\}_{n\geq 0} \subset (0, \infty)$ satisfying $T_n \to T$, that $V_0^{T_n} \to V_0^T$ with V^{T_n} (respectively, V^T) denoting the unique solution to (25) with $f \equiv 0$ and time horizon $[0, T_n]$ (respectively, $[0, T]$), and where convergence takes place in the usual Euclidean sense. We argue by showing that $T \mapsto V_0^T$ is right-continuous and left-continuous at each point in $(0, \infty)$, noting further that it is sufficient to prove this sequential convergence for monotone sequences $\{T_n\}_{n\geq 0} \subset (0, \infty)$. We only show $T \mapsto V_0^T$ is right-continuous since the other case follows by similar reasoning.

Theorem 3. *Let $T \in (0, \infty)$ be arbitrary and $\{T_n\}_{n\geq 0} \subset (0, \infty)$ be any sequence satisfying $T_n \downarrow T$. Let $D_{0,T}(\cdot, \cdot)$ (respectively, $D_{0,T_n}(\cdot, \cdot)$) be payoff (21) for the Dynkin game with horizon $[0, T]$ (respectively, $[0, T_n]$). Suppose that the terminal values Γ^T and $\{\Gamma^{T_n}\}_{n\geq 0}$ in these respective payoffs satisfy Assumption 3. Then, letting V_0^T and $\{V_0^{T_n}\}_{n\geq 0}$ denote the values for these games (which exist by Theorem 2), we have*

$$\lim_{n\to\infty} |V_0^{T_n} - V_0^T|^2 = 0$$

and the map $T \mapsto V_0^T$ is therefore right-continuous on $(0, \infty)$.

Proof. From the discussion in Section 5.2, we can assert that there exists a positive constant C such that (cf. (28))

$$|V_0^{T_n} - V_0^T|^2 \leq C\mathbb{E}[|\delta^{(n)}\Gamma|^2] + C(\mathbb{E}[|\Gamma^T|^2 + |\Gamma^{T_n}|^2] + \|(L^{T_n}, U^{T_n})\|_{T_0} + \|(L^T, U^T)\|_{T_0}) \times \bigg(\mathbb{E}\bigg[\sup_{0\leq t\leq T_0} [|\delta^{(n)} L_t|^2 + |\delta^{(n)} U_t|^2]\bigg]\bigg)^{1/2}. \tag{29}$$

Note that $\mathbb{E}[|\Gamma^{T_n}|^2]$ is uniformly bounded in n since $\sup_{n\geq 0} |\Gamma^{T_n}| \in L^2$ by Assumption 3. Pham and Zhang [22, Theorem 3.4] verified that the norm $\|(L, U)\|_{T_0}$ is finite, and it is not difficult to see that $\|(L^T, U^T)\|_{T_0} \leq \|(L^{T_n}, U^{T_n})\|_{T_0} \leq \|(L, U)\|_{T_0}$ for every n. Using this in (29) shows that we have

$$|V_0^{T_n} - V_0^T|^2 \leq C\mathbb{E}[|\delta^{(n)}\Gamma|^2] + C\bigg(\mathbb{E}\bigg[|\Gamma^T|^2 + \sup_{k\geq 0} |\Gamma^{T_k}|^2\bigg] + 2\|(L, U)\|_{T_0}\bigg) \times \bigg(\mathbb{E}\bigg[\sup_{0\leq t\leq T_0} [|\delta^{(n)} L_t|^2 + |\delta^{(n)} U_t|^2]\bigg]\bigg)^{1/2}, \tag{30}$$

and the right-hand side of (30) is finite for all $n \geq 0$. We have

$$\sup_{0\leq t\leq T_0} [|\delta^{(n)} L_t|^2 + |\delta^{(n)} U_t|^2] = \sup_{T\leq t\leq T_n} [|L_t - L_T|^2 + |U_t - U_T|^2],$$

which decreases monotonically to 0 a.s. as $n \to \infty$. By making use of the monotone convergence theorem and $\lim_{n\to\infty} \mathbb{E}[|\delta^{(n)}\Gamma|^2] = 0$ by Assumption 3, passing to the limit $n \to \infty$ in (30) gives

$$0 \leq \liminf_{n\to\infty} |V_0^{T_n} - V_0^T|^2 \leq \limsup_{n\to\infty} |V_0^{T_n} - V_0^T|^2 \leq 0,$$

and the claim follows. □

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6. Numerical examples

6.1. Cancellable call and put options

In this section we use the same probabilistic setup as in Section 5. We assume a Black–Scholes market with constant risk-free rate of interest $r > 0$ and risky asset price process $S = (S_t)_{t\geq 0}$ which satisfies

$$S_t = S_0 \exp\biggl(\biggl(r - \frac{\rho^2}{2}\biggr)t + \rho B_t\biggr), \qquad t \geq 0, \tag{31}$$

where $S_0 > 0$ and $\rho > 0$ are constants. A call (respectively, put) option on the underlying asset S with finite expiration $T > 0$ is a contingent claim that gives the holder the right, but not the obligation, to buy (respectively, sell) the asset S at a predetermined strike price K by time T. If this option is of 'American' style then the holder can exercise this right at any time $\tau \in [0, T]$. The payoff $G(S_\tau)$ of the option when exercised at time $\tau \in [0, T]$ is given by

$$G(S_\tau) = \begin{cases} (S_\tau - K)^+ & \text{for a call option,} \\ (K - S_\tau)^+ & \text{for a put option.} \end{cases}$$

A cancellable (game) version of the option grants the writer the ability to cancel it at a premature time $0 \leq \sigma < T$. If the writer decides to exercise this right then the option holder receives the payoff of the standard option plus an additional amount $\delta > 0$, which is a penalty imposed on the writer for terminating the contract early. The expected value of the cash flow from the writer to the seller at time 0 is given by

$$D_{0,T}(\sigma, \tau) = \mathbb{E}[\mathrm{e}^{-r\sigma}(G(S_\sigma) + \delta)\,\mathbf{1}_{\{\sigma<\tau\}}\,\mathbf{1}_{\{\sigma<T\}} + \mathrm{e}^{-r\tau}G(S_\tau)\,\mathbf{1}_{\{\tau\leq\sigma\}}]. \tag{32}$$

The holder of the contract would like to choose the exercise time τ to maximise the payoff. On the other hand, the writer would like to minimise this payoff by choosing the appropriate cancellation time σ. We assume that σ and τ are chosen from the set $\mathcal{T}_{0,T}$ of stopping times.

Equation (32) is the payoff for a Dynkin game between the option writer and holder (albeit slightly different from (1)). The assumptions listed in Section 3 can be verified for this game, and an inspection of the proof of Theorem 2 shows that its conclusion remains valid for payoff (32). The cancellable call/put option can therefore be valued using optimal switching.

6.2. Approximation procedure

Suppose that we are additionally given an integer $0 < M < \infty$ and an increasing sequence of times $\{t_m\}_{m=0}^M \subset [0, T]$ satisfying $t_0 = 0$ and $t_M = T$. Set $\hat{\mathbb{F}} = \{\mathcal{F}_{t_m}\}_{m=0}^M$ and for each t_m and $i \in \{0, 1\}$, let $\hat{\mathcal{A}}_{t_m,i}^{(M)} \subset \mathcal{A}_{t_m,i}$ be the subclass of controls $\alpha = (\tau_n, \iota_n)_{\geq 0}$ where each τ_n takes values in $\{t_m, \dots, t_M\}$ and satisfies $\mathbb{P}(\{\tau_n < T\} \cap \{\tau_n = \tau_{n+1}\}) = 0$ for $n \geq 1$. Our discrete-time approximation to the auxiliary optimal switching problem starting in mode $i \in \{0, 1\}$ at time t_m takes a similar form as (2) (with $\psi_1 = \psi_0 = 0$ for simplicity): $\alpha \in \hat{\mathcal{A}}_{t_m,i}^{(M)}$,

$$\hat{J}^{(M)}(\alpha; t_m, i) = \mathbb{E}\biggl[\Gamma_{\iota_{N(\alpha)}} - \sum_{n\geq 1} \gamma_{\iota_{n-1},\iota_n}(\tau_n)\,\mathbf{1}_{\{\tau_n<T\}} \biggm| \mathcal{F}_{t_m}\biggr],$$

where $\iota_{N(\alpha)}$ is the last mode switched to before T under the control α. The results of [16] show that there exist $\hat{\mathbb{F}}$-adapted sequences $\hat{Y}^{(M),i} = \{\hat{Y}_m^{(M),i}\}_{m=0}^M$, $i \in \{0, 1\}$, defined by

$$\hat{Y}_M^{(M),i} = \Gamma_i, \qquad \hat{Y}_m^{(M),i} = \max_{j\in\{0,1\}}\{-\gamma_{i,j}(t_m) + \mathbb{E}[\hat{Y}_{m+1}^{(M),j} \mid \mathcal{F}_{t_m}]\} \quad \text{for } m = M-1, \dots, 0, \tag{33}$$

such that $\max_{m\in\{0,\dots,M\}} |\hat{Y}_m^{(M),i}| \in L^2$ and $\hat{Y}_m^{(M),i} = \operatorname{ess\,sup}_{\alpha\in\hat{\mathcal{A}}_{t_m,i}^{(M)}} \hat{J}^{(M)}(\alpha; t_m, i)$, $\mathbb{P}$-a.s.

For each $M = 1, 2, \dots$, define $\hat{V}^{(M)} = \{\hat{V}_m^{(M)}\}_{m=0}^{M}$ by $\hat{V}_m^{(M)} := \hat{Y}_m^{(M),1} - \hat{Y}_m^{(M),0}$ and recall the particular parametrisation given in Definition 3. Recalling Theorem 2, we see that the random variable $\hat{V}_m^{(M)}$ can be used to approximate the value of the continuous-time Dynkin game with payoff $D_{t_m,T}(\cdot,\cdot)$ (cf. (1)). There is, however, a more efficient backward induction formula for $\hat{V}^{(M)}$. For $m = M-1, \dots, 0$ and $i \in \{0,1\}$ define events $\mathcal{C}_m^i$ and $\mathcal{D}_m^i$ as follows:

$$\begin{aligned} \mathcal{C}_m^i &:= \{\hat{Y}_m^{(M),i} = \mathbb{E}[\hat{Y}_{m+1}^{(M),i} \mid \mathcal{F}_{t_m}]\}, \\ \mathcal{D}_m^i &:= \{\hat{Y}_m^{(M),i} = -\gamma_{i,1-i}(t_m) + \mathbb{E}[\hat{Y}_{m+1}^{(M),1-i} \mid \mathcal{F}_{t_m}]\}. \end{aligned} \tag{34}$$

Note that $\mathbb{P}(\mathcal{C}_m^i \cup \mathcal{D}_m^i) = 1$ for every $i \in \{0,1\}$ and $m = M-1, \dots, 0$. It is not difficult to verify (using Assumption 2 and optimality arguments, see [16]) that $\mathbb{P}(\mathcal{D}_m^0 \cap \mathcal{D}_m^1) = 0$ for $m = M-1, \dots, 0$ and this leads to, $\mathbb{P}$-a.s.,

$$\hat{Y}_m^{(M),i}\, \mathbf{1}_{\mathcal{D}_m^{1-i}} = \mathbb{E}[\hat{Y}_{m+1}^{(M),i} \mid \mathcal{F}_{t_m}]\, \mathbf{1}_{\mathcal{D}_m^{1-i}}, \tag{35}$$

$$\hat{Y}_m^{(M),1} - \hat{Y}_m^{(M),0} = (\hat{Y}_m^{(M),1} - \hat{Y}_m^{(M),0})\left(\sum_{i=0}^{1} \mathbf{1}_{\mathcal{D}_m^i \cap \mathcal{C}_m^{1-i}} + \mathbf{1}_{\mathcal{C}_m^0 \cap \mathcal{C}_m^1}\right). \tag{36}$$

Using $\hat{V}_m^{(M)} = \hat{Y}_m^{(M),1} - \hat{Y}_m^{(M),0}$, (35) and (36), definition (34) for the events $\mathcal{C}_m^i$ and $\mathcal{D}_m^i$, and the backward induction formula (33), one can show that $\hat{V}^{(M)}$ satisfies, $\mathbb{P}$-a.s.,

$$\hat{V}_M^{(M)} = \Gamma, \qquad \hat{V}_m^{(M)} = \min(\gamma_-(t_m), \max(-\gamma_+(t_m), \mathbb{E}[\hat{V}_{m+1}^{(M)} \mid \mathcal{F}_{t_m}])) \quad \text{for } m = M-1, \dots, 0.$$

In order to account for exponential discounting, assuming that the rewards and costs have not already been discounted, the backward induction formula should be written as

$$\hat{V}_M^{(M)} = \Gamma, \qquad \hat{V}_m^{(M)} = \min(\gamma_-(t_m), \max(-\gamma_+(t_m), \mathbb{E}[\mathrm{e}^{-r(t_{m+1}-t_m)}\hat{V}_{m+1}^{(M)} \mid \mathcal{F}_{t_m}])) \quad \text{for } m = M-1, \dots, 0. \tag{37}$$

The reader can compare the backward induction formula (37) to the one appearing in [11, Theorem 2.1]. In a Markovian setting, the least-squares Monte Carlo (LSMC) regression method ([6, Chapter 8, Section 6]) can be used to numerically approximate the conditional expectation in (37).

6.3. Numerical results for the cancellable call and put options

We now present numerical results for the cancellable call and put options. The backward induction formula (37) with the LSMC algorithm was used to this effect, with simple monomials of degree 2 used to approximate the conditional expectations. For each run of the algorithm, 10 000 sample paths $\{\hat{S}_m\}_{m=0}^{M}$ of the geometric Brownian motion (31) were simulated using antithetic sampling and the relation

$$\hat{S}_0 = S_0, \qquad \hat{S}_{m+1} = \hat{S}_m \exp\left(\left[r - \frac{\rho^2}{2}\right]h + \rho\sqrt{h}\xi_{m+1}\right), \qquad m = 0, \dots, M-1,$$

where $h = T/M$ is the step size and $\{\xi_m\}_{m=1}^{M}$ is a sequence of independent and identically distributed standard normal random variables. The option's value was set to the empirical average of the results from 100 runs of the algorithm.

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The same model parameters were used to value the cancellable call and put options. These parameters were obtained from [14, p. 128] and are as follows: $r = 0.06$, $\rho = 0.4$, $K = 100$, and $\delta = 5$. We computed option values on a finite time horizon with $T = 0.5 \times 2^q$, $q = 0, \dots, 8$, initial spot price $S_0 \in \{60, 140\}$, and $M = 1000$ time steps.

6.3.1. *Numerical results for the cancellable call option.* In Figure 1 we show numerical results for the option values for $S_0 \in \{60, 140\}$. The solid line shows finite-horizon option values whilst the dashed line is the perpetual option's value. The latter was calculated using the following formula obtained from [4]:

$$V_0^\infty = \begin{cases} \dfrac{\delta S_0}{K} & \text{if } S_0 \in [0, K], \\ S_0 - K + \delta & \text{if } S_0 \in (K, \infty). \end{cases}$$

For both cases shown in Figure 1, the finite-horizon option values appear to be continuous with respect to the time horizon T. Furthermore, in Figure 1(b), the option values apparently converge to the perpetual option's value as $T \to \infty$.

6.3.2. *Numerical results for the cancellable put option.* In Figure 2 we provide the analogous illustrations for the cancellable put option. The perpetual option's value in this case was calculated using the following formula obtained from [15]:

$$\delta \geq \delta^*\colon V_0^\infty = V^{\mathrm{AP}}(S_0),$$

$$\delta < \delta^*\colon V_0^\infty = \begin{cases} K - S_0 & \text{if } S_0 \in (0, k^*], \\ (K - k^*)\left(\dfrac{S_0}{k^*}\right)^{-(\gamma-1)} \left\{ \left(\left(\dfrac{S_0}{K}\right)^{\gamma} - \left(\dfrac{S_0}{K}\right)^{-\gamma} \right) \times \left(\left(\dfrac{k^*}{K}\right)^{\gamma} - \left(\dfrac{k^*}{K}\right)^{-\gamma} \right)^{-1} \right\} & \\ \quad + \delta \left(\dfrac{S_0}{K}\right)^{-(\gamma-1)} \left\{ \left(\left(\dfrac{S_0}{k^*}\right)^{-\gamma} - \left(\dfrac{S_0}{k^*}\right)^{\gamma} \right) \times \left(\left(\dfrac{k^*}{K}\right)^{\gamma} - \left(\dfrac{k^*}{K}\right)^{-\gamma} \right)^{-1} \right\} & \text{if } S_0 \in (k^*, K), \\ \delta \left(\dfrac{S_0}{K}\right)^{-(2\gamma-1)} & \text{if } S_0 \in [K, \infty), \end{cases}$$

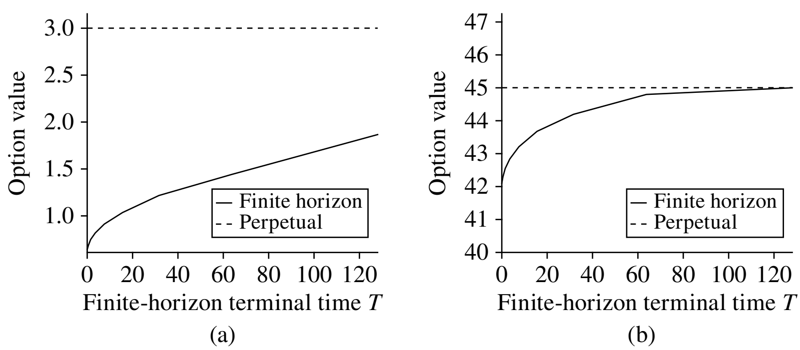

FIGURE 1: Finite- and infinite-horizon cancellable call option values for $S_0 \in \{60, 140\}$. Part (a) shows $S_0 = 60$; part (b) shows $S_0 = 140$.

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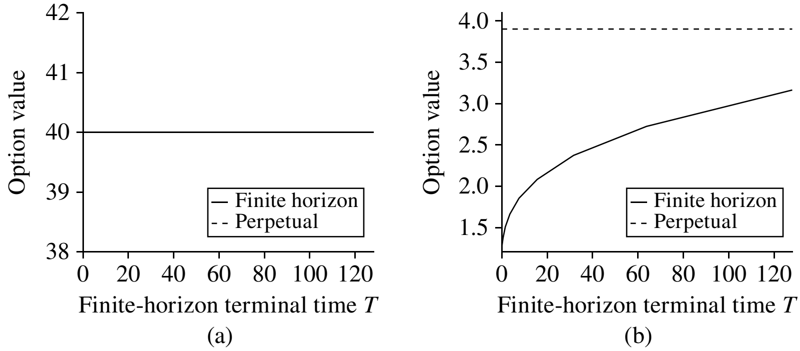

FIGURE 2: Finite- and infinite-horizon cancellable put option values for $S_0 \in \{60, 140\}$. Part (a) shows $S_0 = 60$; part (b) shows $S_0 = 140$.

where $\gamma = r/\rho^2 + \frac{1}{2}$, $S \mapsto V^{\mathrm{AP}}(S)$ is the time 0 value for the perpetual American put option as a function of the initial asset price, $\delta^* = V^{\mathrm{AP}}(K)$, and k^*/K is the solution in $(0, 1)$ to the following equation:

$$y^{2\gamma} + 2\gamma - 1 = 2\gamma\left(1 + \frac{\delta}{K}\right)y.$$

For the interested reader, we note that $\delta^* = V^{\mathrm{AP}}(100) \approxeq 30.3$ and $k^* \approxeq 69.9$ to one decimal place. This means $V_0^\infty = K - S_0$ when $S_0 = 60$ and $V_0^\infty = \delta(S_0/K)^{-(2\gamma-1)}$ when $S_0 = 140$. In terms of the continuity of $T \mapsto V_0^T$ and possible convergence to the perpetual option value, from Figure 2 one draws similar conclusions to those for the cancellable call option.

7. Conclusion

This paper showed how the solution to a two-mode optimal switching problem can be used to derive the solution to a Dynkin game in continuous time and on a finite time horizon $[0, T]$. Under certain hypotheses, the value V_t of the Dynkin game starting from $t \geq 0$ exists and satisfies $V_t = Y_t^1 - Y_t^0$, where Y_t^1 and Y_t^0 are the respective optimal values for the optimal switching problem with initial modes 1 and 0. Furthermore, $(Y_t^1)_{0\leq t\leq T}$ and $(Y_t^0)_{0\leq t\leq T}$ (and therefore $V = (V_t)_{0\leq t\leq T}$) are right-continuous processes, and a Nash equilibrium solution to the Dynkin game can be constructed using appropriate debut times of V. Results on DRBSDEs were used to prove that the value of the game is a continuous function of the time horizon parameter T. This result was confirmed via numerical experiments for cancellable call and put options.

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