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FRACTIONAL BROWNIAN MOTION WITH $H < \frac{1}{2}$ AS A LIMIT OF SCHEDULED TRAFFIC

VICTOR F. ARAMAN,* *American University of Beirut*
PETER W. GLYNN,** *Stanford University*

Abstract

In this paper we show that fractional Brownian motion with $H < \frac{1}{2}$ can arise as a limit of a simple class of traffic processes that we call 'scheduled traffic models'. To our knowledge, this paper provides the first simple traffic model leading to fractional Brownnian motion with $H < \frac{1}{2}$. We also discuss some immediate implications of this result for queues fed by scheduled traffic, including a heavy-traffic limit theorem.

Keywords: Fractional Brownian motion; scheduled traffic; heavy-tailed distribution; limit theorem

2010 Mathematics Subject Classification: Primary 60F17; 60J60; 60G99
Secondary 60G70; 90B30

1. Introduction

There is an extensive literature justifying the use of fractional Brownian motion (and, more generally, fractional Lévy motion) as a mathematical description of the complex aggregate traffic that is carried by data networks; see, for example, Kurtz (1996), Gurin *et al.* (1999), Mikosh *et al.* (2002), Pipiras *et al.* (2004), Kaj (2005), and Kaj and Taqqu (2008). One can support the use of such models either on the basis of statistical analysis, or on the basis of limit theorems that establish that such processes arise naturally as asymptotic descriptions of physically realistic models which characterize network traffic at less aggregated scales (say that of packets in the network). For example, Mikosh *et al.* (2002) showed that fractional Brownian motion can arise as a limit of a superposition of 'on–off' source models with appropriately heavy-tailed inputs. However, one common characteristic of these limit theorems is that the limit processes that arise always exhibit a nonnegative autocorrelation structure. In particular, the fractional Brownian motions that arise as such limits have associated Hurst parameters $H \geq \frac{1}{2}$.

In this paper we propose a simple traffic model that has the property that, when appropriately rescaled, convergence to a fractional Brownian motion (FBM) with $H < \frac{1}{2}$ ensues. Our main result (Theorem 1) provides a queueing level/point process level interpretation of such FBMs. The model that we consider is one that we call a 'scheduled traffic' model; its origin goes back at least as far as Cox and Smith (1961), in which such a point process is termed a 'regular arrival process with unpunctuality'. Customers are scheduled to arrive to the system at regular (say unit) intervals. So, customer j is scheduled to arrive at time j. However, because of random effects experienced along the path traveled to the system, customer j's actual arrival

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* Postal address: Olayan School of Business, American University of Beirut, Beirut 1107-2020, Lebanon.
Email address: va03@aub.edu.lb
** Postal address: Management Science and Engineering, Stanford University, Stanford, CA 94305-4121, USA.
Email address: glynn@stanford.edu

time is $j + \xi_j$. As a consequence, the number N_n of arrivals to the system in $(0, n]$ is given by

$$N_n = \sum_{j=-\infty}^{\infty} \mathbf{1}(\xi_j + j \in (0, n]),$$

where the customer index set is taken, for convenience, to be doubly infinite. Customers with ξ_j negative arrive 'early' and customers with 'perturbations' ξ_j that are positive arrive 'late'. Such traffic can be relevant, for instance, in modeling a doctor's office where patients are initially scheduled at regular intervals; the perturbations account for their early or late arrivals. In this paper we restrict our attention to nonnegative perturbations. We (reasonably) assume that the sequence of perturbations $(\xi_j : -\infty < j < \infty)$ is a family of independent and identically distributed (i.i.d.) random variables (RVs). Under this assumption, $(N_n : n \geq 0)$ has stationary increments (in discrete time), in the sense that $N_{n+m} - N_m \stackrel{\mathrm{D}}{=} N_n - N_0$ for $n, m \geq 0$ (where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution), and $\mathrm{E}\, N_1 = 1$.

We show elsewhere that there exists an RV Γ such that $N_n - n \Rightarrow \Gamma$ as $n \to \infty$ if (and only if) $\mathrm{E}\,|\xi_1| < \infty$ (where '$\Rightarrow$' denotes weak convergence on $D[0, \infty)$). In order that we obtain a functional limit theorem for $(N_n : n \geq 0)$ in which the limit process is an FBM, we will therefore consider heavy-tailed perturbations with $\mathrm{E}\,|\xi_1| = \infty$. In particular, we will assume that the perturbations are nonnegative and satisfy

$$\mathrm{P}(\xi_0 > x) \sim c x^{-\alpha} \tag{1}$$

as $x \to \infty$ for $0 < c < \infty$ and $0 < \alpha < 1$. In the case of the aircraft landing process, the delays can possibly be quite significant. In the presence of (1) we establish a Donsker-type functional limit theorem for the above scheduled traffic model in which the limit process is an FBM with $H = (1 - \alpha)/2$; see Section 2 for a full description of the result. Thus, such a scheduled traffic process exhibits a negative dependency structure. This is intuitively reasonable, as a scheduled traffic process has the characteristic that if one observes more arrivals than normal in one interval, this likely has occurred because either future customers have arrived early or because previously scheduled customers arrived late (thereby reducing the number of arrivals to either past or future intervals). We also note that $H \downarrow 0$ as $\alpha \uparrow 1$ (so that the level of negative dependence increases as the perturbations exhibit smaller fluctuations) and $H \uparrow \frac{1}{2}$ (the Brownian motion case) as $\alpha \downarrow 0$ (so that the perturbations are 'more random'). We end this section by giving another interpretation of the quantity N_n formulated above. Suppose that customer j arrives to an infinite server queue at time j. The service requirements are i.i.d.; the service requirement for customer j is the jth perturbation in the current model. The RV N_n can now be interpreted as the departure process for this infinite-server queue (i.e. N_n is the total number of departures in $[0, n]$), so our Theorem 1 is a Donsker-type theorem for the departure process from a D/G/∞ queue with the service time distribution having infinite mean.

This paper is organized as follows. In Section 2 we state and prove the main result of the paper (our functional limit theorem for scheduled traffic), while in Section 3 we describe the implications in the queueing context. Specifically, we study the workload process for a single-server queue fed by scheduled traffic in the 'heavy-traffic' setting.

2. The main result

For $t \geq 0$, let $X_n = (X_n(t) : t \geq 0)$ be defined via

$$X_n(t) = \frac{N_{\lfloor nt \rfloor} - \lfloor nt \rfloor}{n^{(1-\alpha)/2}}.$$

Also, for $H \in (0, 1)$, let $B_H = (B_H(t) \colon t \geq 0)$ be a mean-zero Gaussian process with covariance function given by

$$\operatorname{cov}(B_H(s), B_H(t)) = \tfrac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$$

for $s, t \geq 0$. The process B_H is a continuous path process with stationary increments for which $B_H(0) = 0$, and is the FBM (with zero mean and unit variance parameter) having Hurst parameter H.

Theorem 1. *Suppose that* $(\xi_j \colon -\infty < j < \infty)$ *is an i.i.d. sequence of positive RVs satisfying (1). Then*

$$X_n \Rightarrow \sqrt{2c(1-\alpha)^{-1}}\, B_H$$

as $n \to \infty$, *where* $H = (1-\alpha)/2$.

As is common in proving such results, we split the proof into two parts: the proof of the convergence of finite-dimensional distributions and verification of tightness.

Proposition 1. *Under the conditions of Theorem 1,*

$$X_n \xrightarrow{\text{FDD}} \sqrt{2c(1-\alpha)^{-1}}\, B_H$$

as $n \to \infty$, *where '*$\xrightarrow{\text{FDD}}$*' denotes weak convergence of finite-dimensional distributions.*

Proof. For notational simplicity, we prove convergence of finite-dimensional distributions for only two time epochs; the general case is essentially identical. We start by observing that, for $t \geq 0$,

$$\begin{aligned} X_n(t) &= n^{-H}\Bigg(\sum_{j=1}^{\lfloor nt\rfloor} \mathbf{1}(j+\xi_j \in (0, \lfloor nt\rfloor]) - \lfloor nt\rfloor + \sum_{j\leq 0} \mathbf{1}(j+\xi_j \in (0, \lfloor nt\rfloor])\Bigg) \\ &= n^{-H}\Bigg(-\sum_{j=1}^{\lfloor nt\rfloor} \mathbf{1}(j+\xi_j > \lfloor nt\rfloor) + \sum_{j\leq 0} \mathbf{1}(j+\xi_j \in (0, \lfloor nt\rfloor])\Bigg). \end{aligned}$$

For $0 \leq t_1 < t_2$ and $\theta_1, \theta_2 \in \mathbb{R}$, set $n_1 = \lfloor nt_1\rfloor$, $n_2 = \lfloor nt_2\rfloor$, $\tilde\theta_1 = n^{-H}\theta_1$, and $\tilde\theta_2 = n^{-H}\theta_2$. Then

$$\begin{aligned} \theta_1 X_n(t_1) + \theta_2 X_n(t_2) = {}& -(\tilde\theta_1 + \tilde\theta_2)\sum_{j=1}^{n_1} \mathbf{1}(\xi_j + j > n_2) - \tilde\theta_1 \sum_{j=1}^{n_1} \mathbf{1}(\xi_j + j \in (n_1, n_2]) \\ & - \tilde\theta_2 \sum_{j=n_1+1}^{n_2} \mathbf{1}(\xi_j + j > n_2) + (\tilde\theta_1 + \tilde\theta_2)\sum_{j\leq 0} \mathbf{1}(\xi_j + j \in (0, n_1]) \\ & + \tilde\theta_2 \sum_{j\leq 0} \mathbf{1}(\xi_j + j \in (n_1, n_2]). \end{aligned}$$

Let

$$\bar F(j) := \mathrm{P}(\xi_0 > j) \quad \text{for } j \geq 0.$$

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The i.i.d. structure of the ξ_js establishes that the log-moment generating function of $(X_n(t_1), X_n(t_2))$ (evaluated at (θ_1, θ_2)) is given by

$$\sum_{j=1}^{n_1} \log(1 + (\mathrm{e}^{-\tilde\theta_1-\tilde\theta_2} - 1)\bar F(n_2 - j) + (\mathrm{e}^{-\tilde\theta_1} - 1)(\bar F(n_1 - j) - \bar F(n_2 - j)))$$
$$+ \sum_{j=n_1+1}^{n_2} \log(1 + (\mathrm{e}^{-\tilde\theta_2} - 1)\bar F(n_2 - j))$$
$$+ \sum_{j\le 0} \log(1 + (\mathrm{e}^{\tilde\theta_1+\tilde\theta_2} - 1)(\bar F(-j) - \bar F(n_1 - j)) + (\mathrm{e}^{\tilde\theta_2} - 1)(\bar F(n_1 - j) - \bar F(n_2 - j))).$$

Because $\tilde\theta_i \to 0$ as $n \to \infty$ and $\log(1 + x) = x(1 + o(1))$ as $x \to 0$, it follows that

$$\begin{aligned}
&\log \mathrm{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
&\quad = \Bigg((\mathrm{e}^{-\tilde\theta_1-\tilde\theta_2} - \mathrm{e}^{-\tilde\theta_1}) \sum_{j=1}^{n_1} \bar F(n_2 - j) + (\mathrm{e}^{-\tilde\theta_1} - 1) \sum_{j=1}^{n_1} \bar F(n_1 - j) \\
&\qquad + (\mathrm{e}^{-\tilde\theta_2} - 1) \sum_{j=n_1+1}^{n_2} \bar F(n_2 - j) + (\mathrm{e}^{\tilde\theta_1+\tilde\theta_2} - 1) \sum_{j=0}^{\infty} (\bar F(j) - \bar F(n_1 + j)) \\
&\qquad + (\mathrm{e}^{\tilde\theta_2} - 1) \sum_{j=0}^{\infty} (\bar F(n_1 + j) - \bar F(n_2 + j)) \Bigg)(1 + o(1))
\end{aligned}$$

as $n \to \infty$. For $0 \le k_1 \le k_2$ and $r \ge k_2 - k_1$,

$$\sum_{j=0}^{r} (\bar F(k_1 + j) - \bar F(k_2 + j)) = \sum_{j=k_1}^{k_2-1} \bar F(j) - \sum_{j=k_1+r+1}^{k_2+r} \bar F(j).$$

Because $\sum_{j=k_1+r+1}^{k_2+r} \bar F(j) \le (k_2 - k_1)\bar F(k_1 + r + 1) \to 0$ as $r \to \infty$, clearly,

$$\sum_{j=0}^{\infty} (\bar F(k_1 + j) - \bar F(k_2 + j)) = \sum_{j=k_1}^{k_2-1} \bar F(j).$$

Consequently,

$$\begin{aligned}
&\log \mathrm{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
&\quad = \Bigg((\mathrm{e}^{-\tilde\theta_1-\tilde\theta_2} - \mathrm{e}^{-\tilde\theta_1}) \sum_{j=n_2-n_1}^{n_2-1} \bar F(j) + (\mathrm{e}^{-\tilde\theta_1} - 1) \sum_{j=0}^{n_1-1} \bar F(j) \\
&\qquad + (\mathrm{e}^{-\tilde\theta_2} - 1) \sum_{j=0}^{n_2-n_1-1} \bar F(j) + (\mathrm{e}^{\tilde\theta_1+\tilde\theta_2} - 1) \sum_{j=0}^{n_1-1} \bar F(j) \\
&\qquad + (\mathrm{e}^{\tilde\theta_2} - 1) \sum_{j=n_1}^{n_2-1} \bar F(j) \Bigg)(1 + o(1))
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\Bigg([(\theta_1+\theta_2)^2 - \theta_1^2 + O(n^{-H})] n^{-2H} \sum_{j=n_2-n_1}^{n_2-1} \bar{F}(j) \\
&\quad + [\theta_1^2 + O(n^{-H})] n^{-2H} \sum_{j=0}^{n_1-1} \bar{F}(j) + [\theta_2^2 + O(n^{-H})] n^{-2H} \sum_{j=0}^{n_2-n_1-1} \bar{F}(j) \\
&\quad + [(\theta_1+\theta_2)^2 + O(n^{-H})] n^{-2H} \sum_{j=0}^{n_1-1} \bar{F}(j) \\
&\quad + [\theta_2^2 + O(n^{-H})] n^{-2H} \sum_{j=n_1}^{n_2-1} \bar{F}(j) \Bigg)(1+o(1))
\end{aligned}
$$

as $n \to \infty$.

Choose $\delta \in (0, 1-\alpha)$, and observe that, for $t > 0$,

$$
\begin{aligned}
n^{-2H} \sum_{j=0}^{\lfloor nt \rfloor} \bar{F}(j) &= n^{\alpha-1} \sum_{j=0}^{\lfloor n^\delta \rfloor} \bar{F}(j) + n^{\alpha-1} \sum_{j=\lfloor n^\delta \rfloor + 1}^{\lfloor nt \rfloor} c j^{-\alpha}(1+o(1)) \\
&= o(1) + c \sum_{j=\lfloor n^\delta \rfloor + 1}^{\lfloor nt \rfloor} \left(\frac{1}{n}\right)\left(\frac{j}{n}\right)^{-\alpha} (1+o(1)) \\
&= o(1) + c \int_{n^{\delta-1}}^{t} x^{-\alpha}\,\mathrm{d}x (1+o(1)) \\
&= \frac{ct^{1-\alpha}}{1-\alpha} + o(1)
\end{aligned}
$$

as $n \to \infty$. Thus, we find that

$$
\begin{aligned}
&\log \mathrm{E} \exp(\theta_1 X_n(t_1) + \theta_2 X_n(t_2)) \\
&\quad \to \frac{c}{2(1-\alpha)} ([\theta_2^2 + 2\theta_1\theta_2](t_2^{2H} - (t_2 - t_1)^{2H}) + \theta_1^2 t_1^{2H} \\
&\qquad\qquad + \theta_2^2 (t_2 - t_1)^{2H} + [\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2] t_1^{2H} + \theta_2^2 (t_2^{2H} - t_1^{2H})) \\
&\quad = \frac{c}{1-\alpha} (\theta_1^2 t_1^{2H} + \theta_2^2 t_2^{2H} + \theta_1\theta_2 (t_2^{2H} + t_1^{2H} - |t_2 - t_1|^{2H}))
\end{aligned}
$$

as $n \to \infty$, which is precisely the joint log-moment generating function of the Gaussian finite-dimensional distribution of the limit process.

Proposition 2. *The sequence* $(X_n : n \geq 0)$ *is tight in* $D[0, \infty)$.

Proof. Note that because we established convergence of the moment generating functions in Proposition 1, it follows that $(|X_n(t)|^p : n \geq 0)$ is uniformly integrable for all $t \geq 0$ and $p > 0$. Hence, in view of Proposition 1, all the requisite conditions of Theorem 2.1 of Taqqu (1975) are satisfied, so that $((X_n(u) : 0 \leq u \leq t) : n \geq 0)$ is tight in $D[0, t]$ for each $t \geq 0$.

Propositions 1 and 2 together prove Theorem 1.

Remark 1. A very similar proof holds for a time-stationary scheduled arrival process formulated in continuous time. In particular, let $N(t)$ be the number of scheduled arrivals in $(0, t]$, so

$$
N(t) = \sum_{j=-\infty}^{\infty} \mathbf{1}(jh + Uh + \xi_j \in (0, t]),
$$

where customers are scheduled to arrive at times in $h\mathbb{Z}$ and U is a uniform [0,1] RV independent of $(\xi_j : j \in \mathbb{Z})$; the uniform RV U is introduced in order to induce time stationarity. If the distribution of ξ_0 satisfies (1) then

$$\frac{N(nt) - nt/h}{n^H} \Rightarrow \sqrt{\frac{2c}{1-\alpha}} B_H\left(\frac{t}{h}\right)$$

as $n \to \infty$ in $D[0, \infty)$.

3. Implications for queues

We now briefly describe the implications of our limit theorem for a queue that is fed by a scheduled arrival process with i.i.d. heavy-tailed perturbations $(\xi_n : n \in \mathbb{Z})$ satisfying (1). In particular, we consider such a queue in 'heavy traffic', in an environment in which the service times are deterministic. (We view this deterministic assumption as being realistic in this setting, given that a service provider would likely attempt only to schedule arrivals when the service times were of highly predictable duration.)

Specifically, we consider a family of queues, indexed by $\rho \in (0, 1)$, in which the number of arrivals in $(0, t]$ to the ρth system is given by $N(\rho t)$, where N satisfies the conditions of Theorem 1. If the ρth system starts off idle and the service times have unit duration, then the workload process $(W_\rho(t) : t \geq 0)$ for the ρth system is given by

$$W_\rho(t) = N(\rho t) - t - \min_{0 \leq s \leq t} [N(\rho s) - s].$$

Clearly, the utilization (or traffic intensity) of system ρ is ρ. Heavy traffic is therefore obtained by letting $\rho \uparrow 1$.

Theorem 2. *Under the same conditions as for Theorem 1,*

$$(1-\rho)^{(1-\alpha)/(1+\alpha)} W_\rho\left(\frac{\cdot}{(1-\rho)^{2/(1+\alpha)}}\right) \Rightarrow \sigma B_H(\cdot) - e(\cdot) - \min_{0 \leq s \leq e(\cdot)} [\sigma B_H(s) - s]$$

as $\rho \uparrow 1$ in $D[0, \infty)$, where $H = (1-\alpha)/2$, $\sigma^2 = 2c/(1-\alpha)$, and $e(t) = t$ for $t \geq 0$.

Proof. Note that

$$\begin{aligned}
&(1-\rho)^{(1-\alpha)/(1+\alpha)} W_\rho\left(\frac{t}{(1-\rho)^{2/(1+\alpha)}}\right) \\
&\quad = (1-\rho)^{H/(1-H)} W_\rho\left(\frac{t}{(1-\rho)^{1/(1-H)}}\right) \\
&\quad = (1-\rho)^{H/(1-H)} \Bigg[N\left(\frac{\rho t}{(1-\rho)^{1/(1-H)}}\right) - \frac{t}{(1-\rho)^{1/(1-H)}} \\
&\qquad\qquad - \min_{0 \leq s \leq t} \left(N\left(\frac{\rho s}{(1-\rho)^{1/(1-H)}}\right) - \frac{s}{(1-\rho)^{1/(1-H)}} \right) \Bigg] \\
&\quad = \rho^H \left(\frac{(1-\rho)^{1/(1-H)}}{\rho}\right)^H \\
&\qquad \times \Bigg[N\left(\frac{\rho t}{(1-\rho)^{1/(1-H)}}\right) - \frac{\rho t}{(1-\rho)^{1/(1-H)}} - (1-\rho)^{-H/(1-H)} t \\
&\qquad\quad - \min_{0 \leq s \leq t} \left(N\left(\frac{\rho s}{(1-\rho)^{1/(1-H)}}\right) - \frac{\rho s}{(1-\rho)^{1/(1-H)}} - (1-\rho)^{-H/(1-H)} s \right) \Bigg]
\end{aligned}$$

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$$
\begin{aligned}
&= \rho^H \bigg[\bigg(\frac{(1-\rho)^{1/(1-H)}}{\rho} \bigg)^H \bigg(N\bigg(\frac{\rho t}{(1-\rho)^{1/(1-H)}} \bigg) - \frac{\rho t}{(1-\rho)^{1/(1-H)}} \bigg) \\
&\qquad - t - (\rho^{-H} - 1)t \\
&\qquad - \min_{0 \le s \le t} \bigg(\bigg(\frac{(1-\rho)^{1/(1-H)}}{\rho} \bigg)^H \bigg(N\bigg(\frac{\rho s}{(1-\rho)^{1/(1-H)}} \bigg) - \frac{\rho s}{(1-\rho)^{1/(1-H)}} \bigg) \\
&\qquad\qquad - s - (\rho^{-H} - 1)s \bigg) \bigg].
\end{aligned}
$$

But, Theorem 1 implies that

$$
\bigg(\frac{(1-\rho)^{1/(1-H)}}{\rho} \bigg)^H \bigg(N\bigg(\frac{\rho\, \cdot}{(1-\rho)^{1/(1-H)}} \bigg) - \frac{\rho e(\cdot)}{(1-\rho)^{1/(1-H)}} \bigg) \Rightarrow \sigma B_H(\cdot)
$$

in $D[0,\infty)$ as $\rho \uparrow 1$. Since $(\rho^{-H} - 1)e(\cdot) \to 0$ uniformly on compact time intervals, as $\rho \uparrow 1$, the continuous mapping principle (see, for example, Billingsley (1999, p. 20)) implies the theorem.

Theorem 2 suggests the approximation

$$
W(t) \overset{\mathrm{D}}{\approx} (1-\rho)^{(\alpha-1)/(\alpha+1)} Z((1-\rho)^{2/(1+\alpha)} t)
$$

when $\rho := \mathrm{E}(N_1 - N_0)$ is close to 1, where '$\overset{\mathrm{D}}{\approx}$' denotes 'has approximately the same distribution as' (and has no rigorous meaning, other than that associated with Theorem 2 itself) and $Z = (Z(t)\colon t \ge 0)$ is the *regulated* FBM given by $Z(t) = \sigma B_H(t) - t - \min_{0 \le s \le t}[\sigma B_H(s) - s]$. One implication is that, when ρ is close to 1, the rough magnitude of $W(\cdot)$ is of the order $(1-\rho)^{(\alpha-1)/(\alpha+1)}$ (where $(\alpha-1)/(\alpha+1) \in (-2, 0)$) and the time scale over which $W(\cdot)$ fluctuates (in a relative sense) is of the order $(1-\rho)^{-2/(1+\alpha)}$. In particular, when α is close to 1 (so that N is almost deterministic), the magnitude of W is small and the fluctuations of W occur over time scales of the order $(1-\rho)^{-1}$.

On the other hand, if the service times $(V_n\colon n \in \mathbb{Z})$ associated with the scheduled arrival sequence are i.i.d. with unit mean and positive finite variance, then the corresponding heavy-traffic limit theorem for the workload process

$$
W_\rho(t) = \sum_{i=1}^{N(\rho t)} V_i - t - \min\bigg(\sum_{i=1}^{N(\rho s)} V_i - s \bigg)
$$

is easily shown to be

$$
(1-\rho) W_\rho\bigg(\frac{\cdot}{(1-\rho)^2} \bigg) \Rightarrow \eta B(\cdot) - e(\cdot) - \min_{0 \le s \le e(\cdot)} (\eta B(s) - s) \tag{2}
$$

as $\rho \uparrow 1$ in $D[0,\infty)$, where $B = (B(t)\colon t \ge 0)$ is standard Brownian motion with $B(0) = 0$ and $\eta^2 = \operatorname{var} V_1$. This limit theorem is identical to that obtained for a D/G/1 queue in heavy traffic, so in this asymptotic regime with positive variance service times, scheduled traffic behaves similarly to a deterministic arrival sequence. Furthermore, in this positive variance setting, the fluctuations of a scheduled queue in heavy traffic are larger (of the order $(1-\rho)^{-1}$) and occur over longer time scales (of the order $(1-\rho)^{-2}$) than in the context of deterministic service times (which are, as noted earlier, of the orders $(1-\rho)^{-(1-\alpha)/(1+\alpha)}$ and $(1-\rho)^{-2/(1+\alpha)}$, respectively).

A great deal is known about the behavior of the limiting regulated FBM process Z, and how its behavior contrasts with that of the regulated Brownian motion appearing in (2).

- As for the reflected Brownian motion in (2), $Z(t) \Rightarrow Z(\infty)$ as $t \to \infty$. However, in contrast to the Brownian case, $Z(\infty)$ has superexponential tails (so that the tails are lighter than the exponential tails that arise in the conventional heavy-traffic setting of (2)). In particular,

$$\mathrm{P}(Z(\infty) > x) \sim \frac{\mathcal{H}_{2H}\sqrt{\pi}\, D^{1/2H} A^{(2-H)/(2H)}}{B^{1/2} 2^{(1-H)/(2H)} \sigma^{(1-H)/H}} x^{(1-H)^2/H}\, \bar{\Phi}(A\,\sigma^{-1} x^{1-H})$$

  as $x \to \infty$, where $\bar{\Phi}$ is the tail of the standard normal distribution,

$$A = \left(\frac{H}{1-H}\right)^{-H} \frac{1}{1-H}, \qquad B = \left(\frac{H}{1-H}\right)^{-(H+2)} H, \qquad D = \left(\frac{H}{1-H}\right)^{-2H},$$

  and $\mathcal{H}_{2H}$ is the so-called Pickands constant; see Hüsler and Piterbarg (1999) for details.

- The convergence to equilibrium of $Z(t)$ to $Z(\infty)$ occurs roughly at a rate $\exp(-\theta^* t^{2-2H})$ (in 'logarithmic scale'), where θ^* involves solving a variational problem. This is faster than the roughly exponential rate to equilibrium associated with (2); see Mandjes *et al.* (2009) for details.

- The dynamics of the process Z, conditioned on an unusually long busy period of duration t, forces the process Z to make a large positive excursion (reaching a level of order t) during the busy period, whereas regulated Brownian motion (under the same conditioning) tends to exhibit much smaller positive fluctuations; see Mandjes *et al.* (2006) for details.

All these results point to the intuition that a scheduled arrival process (with deterministic service times) behaves much more predictably than does a queue fed by (for example) renewal input. This is in strong contrast to queues that can be approximated by regulated FBM with $H > \frac{1}{2}$, which generally have much worse behavior than conventional queues (i.e. equilibrium distributions with fatter than exponential tails, subexponential rates of convergence to equilibrium, etc.).

References

Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd edn. John Wiley, New York.

Cox, D. R. and Smith, W. L. (1961). *Queues*. John Wiley, New York.

Gurin, C. A. *et al.* (1999). Empirical testing of the infinite source Poisson data traffic model. Tech. Rep. 1257, School of Operations Research and Information Engineering, Cornell University.

Hüsler, J. and Piterbarg, V. (1999). Extremes of a certain class of Gaussian processes. *Stoch. Process. Appl.* **83,** 257–271.

Kaj, I. (2005). Limiting fractal random processes in heavy tailed systems. In *Fractals in Engineering, New Trends in Theory and Applications*, eds J. Lévy-Lehel and E. Lutton, Springer, London, pp. 199–218.

Kaj, I. and Taqqu, M. S. (2008). Convergence to fractional Brownian motion and to the telecom process: the integral representation approach. In *In and Out of Equilibrium 2* (Progress Prob. **60**), eds M. E. Vares and V. Sidoravicius, Birkhäuser, Basel, pp. 383–427.

Kurtz, T. G. (1996). Limit theorems for workload input models. In *Stochastic Networks: Theory and Applications*, eds S. Zachary, F. P. Kelly and I. Ziedins, Clarendon Press, Oxford, pp. 119–140.

Mandjes, M., Norros, I. and Glynn, P. (2009). On convergence to stationarity of fractional Brownian storage. *Ann. Appl. Prob.* **18,** 1385–1403.

Mandjes, M., Mannersalo, P., Norros, I. and van Uitert, M. (2006). Large deviations of infinite intersections of events in Gaussian processes. *Stoch. Process. Appl.* **116,** 1269–1293.

Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002). Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Prob.* **12,** 23–68.

Pipiras, V., Taqqu, M. S. and Levy, J. B. (2004). Slow, fast and arbitrary growth conditions for renewal-reward processes when both the renewals and the rewards are heavy-tailed. *Bernoulli* **10,** 121–163.

Taqqu, M. S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitsth.* **31,** 287–302.