PROBLEMS FOR SOLUTION

P. 146. (i) Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers such that $\sigma(n_i) - n_i$ is a constant, where $\sigma(n)$ is the sum of the divisors of n. Prove that each n_i is prime.

(ii) For each $k \geq 1$, show that there exist integers $n_1 < n_2 < \ldots < n_k$, none of which is a prime, such that $\sigma(n_i) - n_i$ is constant.

P. Erdős

P. 147. Let p be a prime with $p \equiv 1 (\text{mod } 3)$. Prove that $(x + 1)^p - x^p - 1 \equiv 0 \pmod{p^3}$ has at least two solutions in the range $1 \leq x \leq p - 1$.

H.A. Heilbronn, University of Toronto

P. 148. Let X be a locally separable connected metric space. Prove that X is separable. Is this true if X is not metric?

J. Marsden, University of California, Berkeley

SOLUTIONS

P. 136. Find a topological space X which is T_o and such that Y' fails to be closed for at least one subset Y of X. (Here Y' denotes the set of all accumulation points of Y.)

P.A. Pittas, Dalhousie University

Solution by J. Marsden, University of California, Berkeley

Let $X = \{x_1, x_2, \ldots\} \cup \{x\}$ with topology $\{U_n = \{x_k : k \geq n\} \cup \{x\}\}$. This space is T_o but not T_1. Let $Y = \{x\}$. Then $Y' = \{x_1, x_2, \ldots\}$ which is not a closed set.

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Also solved by J.B. Wilker and the proposer. Both Marsden and the proposer pointed out that such an X is necessarily infinite.

P. 137. If X is a complete metric space and T is a contraction in X, then T has a unique fixed point. This fails to hold if T has only the property $d(Tx, Ty) < d(x, y)$.

K. L. Singh, Memorial University

Solution by P. Ewer, St. Mary's University, Halifax

The subspace $X = [1, \infty)$ of the real line is complete. Let T be defined by $T_x = x + \frac{1}{x}$. Then certainly T leaves no point of X fixed. Suppose $x, y \in X$ and $x < y$. Then

$$d(T_x, T_y) = (y - x) - (\frac{1}{x} - \frac{1}{y}) < d(x, y).$$

Also solved by S. Aalto, J.A. Baker, D. Lind, J. Marsden, J.B. Wilker and the proposer.

In general it is clear that such a T has at most one fixed point, and J. Marsden points out that a fixed point does exist when one assumes that X is compact. For a new proof that a contractive T has a fixed point see the note, Another Proof of the Contraction Mapping Principle by Boyd and Wong to appear in this section of the Bulletin.

P. 138. Prove that the set S is finite if and only if there is a permutation π of S such that no proper non-empty subset S' has the property $\pi(S') \not\subseteq S'$.

J. Marcia, University of Calgary

Solution by D. Lind, Cambridge University

If S is finite, a cyclic permutation π of S has the property that $\pi(S') \subset S'$ for all non-empty proper subsets S' of S. Conversely, suppose S is infinite, and let $\chi \in S$. For a permutation π of S, put $S' = \bigcup_{n=0}^{\infty} \pi^n(\chi)$. Then $\pi(S') \subset S'$. If $S' = S$, then $\pi^{-1}(\chi) = \pi^k(\chi)$ for some $k \geq 0$, so S is finite, a contradiction. Hence S' is a non-empty proper subset of S.

Also solved by W.D. Jackson, J. Schaer, J.B. Wilker and the proposer.

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P. 139. Prove $S(a,b) = (a-b)^{n-1}[aS(1,0) - bS(0,-1)]$ where $S(a,b)$ = determinant of a matrix of order n in which each element is either a or b.

K. Schmidt, University of Manitoba

Solution by S. Spital, California State College

Let the subtraction of the first row of $S(a, b)$ from the remaining rows be indicated by

$$S(a,b) = T\begin{pmatrix} a, & b \\ a-b, \; 0, & b-a \end{pmatrix}$$

where the upper two arguments identify the entries in the first row, and the lower three the entries in the remaining $n-1$ by n block. The required result now follows from well known properties of determinants:

$$T\begin{pmatrix} a, & b \\ a-b, \; 0, & b-a \end{pmatrix} = (a-b)^{n-1}\; T\begin{pmatrix} a, & b \\ 1, \; 0, & -1 \end{pmatrix},$$

$$a\,S(1,0) - b\,S(0,-1) = T\begin{pmatrix} a, & 0 \\ 1, \; 0, & -1 \end{pmatrix} + T\begin{pmatrix} 0, & b \\ 1, \; 0, & -1 \end{pmatrix} = T\begin{pmatrix} a, & b \\ 1, \; 0, & -1 \end{pmatrix}.$$

Also solved by L. Carlitz, R.C. Mullin and E. Nemeth (jointly), J.B. Wilker and the proposer.

P. 140. Every integral two by two matrix is a sum of three squares; and the number three is best possible.

I. Connell, McGill University

Solution by L. Carlitz, Duke University

1. Put $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider

$$A - \begin{bmatrix} x & b \\ c & 1-x \end{bmatrix}^2 = A - \begin{bmatrix} x^2+bc & b \\ c & (1-x)^2+bc \end{bmatrix} = \begin{bmatrix} a-x^2-bc & 0 \\ 0 & d-(1-x)^2-bc \end{bmatrix}.$$

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We can choose x so that

$$a - x^2 - bc = d - (1 - x)^2 - bc,$$

that is $2x = a - d + 1$, provided $a \equiv d + 1 \pmod 2$. Since

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix}^2,$$

it follows that A is a sum of two squares when $a - d$ is odd. If $a - d$ is even, we have

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 + \begin{bmatrix} a - 1 & b \\ c & d \end{bmatrix},$$

which is evidently a sum of three squares. Thus every A is a sum of at most three squares.

2. We show now that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod 2$$

and $a \equiv d + 2 \pmod 4$, A is not a sum of two squares. Assume

(1) $$A = \begin{bmatrix} x & y \\ z & x' \end{bmatrix}^2 + \begin{bmatrix} t & u \\ v & t' \end{bmatrix}^2,$$

so that

$$\begin{aligned} x^2 + yz + t^2 + uv &= a \\ y(x + x') + u(t + t') &= b \\ z(x + x') + v(t + t') &= c \\ x'^2 + yz + t'^2 + uv &= d. \end{aligned}$$

Subtracting the fourth equation from the first we get

$$x^2 + t^2 = a - d + x'^2 + t'^2.$$

Since $a - d \equiv 2 \pmod 4$ it follows that either

(i) $x \equiv t \equiv 1$, $x' \equiv t' \equiv 0$ or (ii) $x \equiv t \equiv 0$, $x' \equiv t' \equiv 1 \pmod 2$.

In either case we have

$$x + x' \equiv t + t' \equiv 1 \pmod 2.$$

It follows that

(2) $$y + u \equiv b \equiv 0, \; z + v \equiv c \equiv 0 \pmod 2.$$

On the other hand

$$x^2 + yz + t^2 + uv \equiv yz + uv \equiv a \pmod 2,$$

so that

(3) $$yz + uv \equiv 1 \pmod 2.$$

But, by (2), $u \equiv y$, $v \equiv z \pmod 2$, which contradicts (3). Hence (1) is impossible.

3. We shall now show that in all other cases A is a sum of two squares. We consider first the case

$$a - d \equiv 2 \pmod 4$$

and either b or c (or both) odd.

Put

(4) $$a - d + 2 = 4e.$$

We show that

(5) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & y \\ z & 1-e \end{bmatrix}^2 + \begin{bmatrix} e & u \\ v & 1-e \end{bmatrix}^2,$$

that is

$$\begin{aligned} 2e^2 + yz + uv &= a \\ y + u &= b \\ z + v &= c \\ 2(1-e)^2 + yz + uv &= d\,. \end{aligned}$$

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Subtracting the fourth equation from the first, we get $4e - 2 = a - d$, in agreement with (4). Thus the fourth equation can be ignored. Eliminating u and v, we get

$$yz + (b - y)(c - z) + 2e^2 = a$$

or

$$(2y - b)(2z - c) + bc + 4e^2 = 2a .$$

Now assume c odd and take $z = (c + 1)/2$.

Then

$$2y + b(c - 1) + 4e^2 = 2a ,$$

so that y is determined.

4. Finally, we take $a \equiv d \pmod 4$. If $b \equiv c \equiv 0 \pmod 2$, consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} x & b/2 \\ c/2 & 2-x \end{bmatrix}^2 = \begin{bmatrix} a - x^2 - bc/4 & 0 \\ 0 & d - (2-x)^2 - bc/4 \end{bmatrix}$$

This is of the form

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix}^2$$

if $4x = a - d + 4$.

Hence assume b or c odd. Put $a - d = 4e$. Take

$$x + x' = t + t' = 1 .$$

Consider

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & y \\ z & 1-e \end{bmatrix}^2 + \begin{bmatrix} e+1 & u \\ v & -e \end{bmatrix}^2 ,$$

that is

$$\begin{aligned} e^2 + (e+1)^2 + yz + uv &= a \\ y + u &= b \\ z + v &= c \\ e^2 + (1-e)^2 + yz + uv &= d \end{aligned}$$

Subtracting the fourth from the first we get $4e = a - d$. Eliminating u, v we get

$$e^2 + (e+1)^2 + yz + (b-y)(c-z) = a,$$

or

$$(2y-b)(2z-c) + bc + 2e^2 + 2(e+1)^2 = 2a .$$

If c is odd, take $2z = c + 1$, so that

$$2y + b(c-1) + 2e^2 + 2(e+1)^2 = 2a,$$

thus determining $\underline{y}$.

To sum up, every $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a sum of at most three squares. Two squares will suffice unless

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod 2$$

and $a \equiv d + 2 \pmod 4$.

Editor's comment: One case is omitted in the above analysis: each of a, b, c, d is even, say $a = 2a_1$, etc., and $a - d \equiv 2 \bmod 4$; but this can be dealt with as in paragraph 3 above. We arrive at the equation $(2y-b)(2z-c) + bc + 4e^2 = 2a$, or, $(y-b_1)(z-c_1) + b_1c_1 + e^2 = a_1$, and we may take $z = c_1 + 1$, thus obtaining y.

Also solved, but not completely as in the above solution, by J.B. Wilker and the proposer.

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