J. Appl. Prob. **45,** 1–15 (2008)
Printed in England

ON THE STRUCTURE OF A SWING CONTRACT'S OPTIMAL VALUE AND OPTIMAL STRATEGY

SHELDON M. ROSS,* *University of Southern California*
ZEGANG ZHU,** *University of California, Berkeley*

Abstract

Consider a sales contract, called a swing contract, between a seller and a buyer concerning some underlying commodity, with the contract specifying that during some future time interval the buyer will purchase an amount of the commodity between some specified minimum and maximum values. The purchase price and capacity at each time point is also prespecified in the contract. Assuming a random market price process and ignoring the possibility of storage, we look for the maximal expected net gain for the buyer of such a contract, and the strategy that achieves this maximal expected net gain. We study the effects that various contract constraints and market price processes have on the optimal strategy and on the contract value. We show how we can reduce the general swing contract to a multiple exercising of American (Bermudan) style options. Also, in important special cases, we give explicit expressions for the optimal contract value function and the optimal strategy.

Keywords: Swing contract; American call option; optimal strategy; supermodular; submodular

2000 Mathematics Subject Classification: Primary 90B50
Secondary 90C40

1. Introduction

Swing contracts are widely used in the commodity markets, especially in the gas and electricity markets. Under a swing contract, the buyer is obliged to buy a specified minimum total amount of commodity, but has the freedom to buy up to a specified maximal total amount, during a specified time period. However, the purchase pattern over the period is at the buyer's choice, with minimum and maximum limits at any single time. Also, the commodity purchase prices (called strike prices) under the contract, specified at contract origination, may vary over the period.

This type of contract provides both the buyer and seller partial security about the future demand, supply, and pricing of the commodity. It also provides flexibility to the buyer on the timing and the amount of purchase over the period.

Throughout the paper, we assume no storage of commodity. This assumption is realistic for some commodities, such as electricity which has a high cost of storage. Also, we assume a highly liquid market for the commodity, so that market price is readily available and realizable.

Received 27 February 2006; revision received 30 October 2007.
* Postal address: Daniel J. Epstein Department of Industrial and Systems Engineering, University of Southern California, 3715 McClintock Avenue, GER 240, Los Angeles, CA 90089-0193, USA. Email address: smross@usc.edu
Supported by the National Science Foundation, grant ECS-0224779, the University of California, and by a grant from OpenLink Financial Inc.
** Current address: 100 W 93rd St., Apt. 27E, New York, NY 10025, USA. Email address: zhuzgang@yahoo.com
Supported by the National Science Foundation, grant ECS-0224779, and the University of California.

The buyer's net monetary gain from purchasing a unit amount of commodity at any time is taken to be the spot market price minus the strike price at that time. In this paper we assume that the buyer's objective is to maximize the expected total (discounted) monetary gains and to find the corresponding purchasing strategy that maximizes this expected total gain.

Barbieri and Garman [2], [3] gave descriptions of several variants of the swing contract, without details on how to value them. Deng [6], Schwartz [12], and Schwartz and James [13] provided some stochastic models of the spot market price processes. Carmona and Touzi [4], and Carmona and Dayanic [5] obtained some structural results of the optimal strategy and the value function for put swings by using a stopping time approach. Ali *et al.* [1] and Jaillet *et al.* [7] considered swing contracts that allow the buyer to add or reduce purchase amounts from a base amount, and gave numerical approximations of the maximal contract value by first using the recombining trinomial or binomial tree approach to approximate the underlying commodity market price process, and then applying brute force dynamic programming to calculate the contract value. Pilipovic and Wengler [9] also discussed special cases of swing contracts which can be solved with simple procedures. Keppo [8] gave a mathematical formulation for finding the optimal strategy and the contract value function, assuming that the commodity could be traded continuously; he also provided some upper and lower bounds for the contract value function. In this paper we adopt Keppo's definition of swing contracts, except now we assume that the commodity is purchased at discrete-time grids. Algorithms to calculate the contract value by utilizing the structural results in this paper, as well as techniques for approximating continuous-time exercising contracts by their discrete-time counterparts, are contained in a separate paper by the authors [11].

The paper is organized as follows. In Section 2 we give a mathematical description of the swing contract, as well as the dynamic programming formulation for the contract value function and the optimal strategy. In Section 3 we give some definitions that will be used throughout the paper. In Section 4 we give structural results on how the contract constraints affect the contract value and optimal strategy. In Section 5 we discuss how the spot market price affects both the contract value and the corresponding optimal strategy. Section 6 combines the results of Sections 4 and 5 to give a detailed description of the structure of the optimal strategy, as well as explicit expressions in special cases. This section also includes applications and extensions of results to the American call option. Section 7 includes the concluding remarks, possible extensions, and areas of future research. Some technical proofs of the theorems are deferred to Appendix A.

2. Model description and formulation

A swing contract normally specifies that over some future time grids $0 = t_0 < t_1 < \cdots < t_n$ (where $t_n < T$), the buyer will buy a total amount of the commodity that is between u and $u + v$. We call u the obligatory amount and v the bonus amount. It also specifies the price K_i (called the strike price) for a unit amount of purchase at time t_i, $i = 0, \dots, n$, where K_i may be different from K_j if $i \neq j$. The contract also specifies the single purchasing limit M, i.e. the amount of purchase at any time grid t_i must be between 0 and M. (While we assume that the commodities are in continuous form, we show that all results also apply to commodities with discrete counts as well.) Other than these constraints, the buyer can choose any purchasing amount at any time grid. The contract may also specify a lower purchasing limit at each time, but this constraint can be eliminated simply by considering the extra purchasing amount beyond this lower limit at each time.

We will use $C_i^h(u, v)$ to denote the subcontract that has the same parameters as the original contract, except now the purchasing opportunities are limited to the time grids $t_i, \ldots, t_h$ only (so $C_0^n(u, v)$ is the original contract). We will omit the superscript h when $h = n$. So, after the buyer purchases q units of commodity at time t_i for subcontract $C_i^h(u, v)$, the remaining subcontract is $C_{i+1}^h((u - q)^+, v - (u - q)^-)$, where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.

For contract $C_i(u, v)$, the amount of purchase at time t_i must be in the feasible set $U_i(u, v) = [(u - (n - i)M)^+, M \wedge (u + v)]$, where $x \wedge y = \min(x, y)$. We use $R_i = [0, (n - i + 1)M]$ to denote the range of the total amount that the buyer can possibly buy during $[t_i, t_n]$, and use D_i to denote the total amounts that the buyer can possibly buy after time t_i, i.e. $D_i = R_{i+1}$.

The buyer's net gain by purchasing one unit of commodity from the seller at time t_i is $S_i - K_i$, where S_i is the spot market price and K_i is the strike price. We will assume that the spot market price process $\{S(t) : t \geq 0\}$ is a single-factor Markov process. This assumption is satisfied by most price process models in the papers mentioned previously.

From the above discussions we see that the state of the system at time t_i can be summarized by the triplet (u, v, s), where s is the spot market price for the commodity at that time and the remaining contract is $C_i(u, v)$. Let r be the risk-free interest rate that, for simplicity, is assumed to be constant throughout the paper.

Assuming that the buyer's objective is to maximize his/her expected total (discounted) net gain over the contract lifetime, let $H_i(u, v, s)$ be the maximal expected total remaining net gain (discounted to t_i) when the state at time t_i is (u, v, s) (so $H_0(u, v, s)$ is the contract value function at contract origination). Define $\mathbf{1}_A = 1$ if statement A is true, and $\mathbf{1}_A = 0$ otherwise. Then, at contract maturity, we have

$$H_n(u, v, s) = u(s - K_n)\,\mathbf{1}_{(s \leq K_n)} + ((u + v) \wedge M)(s - K_n)\,\mathbf{1}_{(s > K_n)}\,.$$

For $j \geq i$, let $S_j^{i,s} = [S(t_j) \mid S(t_i) = s]$, where $[X \mid A]$ denotes the random variable X given that event A is true. Now, for $i = 0, \ldots, n - 1$, we define $\theta_{i+1} \equiv t_{i+1} - t_i$ and

$$V_i(u, v, s) = \mathrm{E} \exp(-r\theta_{i+1}) H_{i+1}(u, v, S_{i+1}^{i,s}).$$

Then the contract value function at time t_i is

$$H_i(u, v, s) = \max_{q \in U_i(u,v)} \{q(s - K_i) + V_i((u - q)^+, v - (u - q)^-, s)\}.$$

We will use $q_i(u, v, s)$ to denote the maximizer in the above iteration formula. If there are multiple solutions, the maximizer is defined to be the smallest solution.

3. Definitions and notation

We will use $f(x) \uparrow_x$ to mean that f is nondecreasing in x, where x may be single or multidimensional, and use $x \leq y$ to mean that each component of x is smaller than or equal to the corresponding component of y. We use $f(x) \uparrow_x a$ to mean that f is nondecreasing in x (one-dimensional) and has limit a as x tends to ∞. We define $f(x) \downarrow_x$ and $f(x) \downarrow_x a$ similarly.

We will use A^{c} to denote the complement of event A.

Definition. For a bivariate function $f(x, y)$, define

1. $f(x, y)$ to be submodular if

$$f(x + h_1, y + h_2) + f(x, y) \leq f(x + h_1, y) + f(x, y + h_2) \quad \text{for all } h_1, h_2 \geq 0,$$

and supermodular if

$$f(x+h_1, y+h_2) + f(x, y) \geq f(x+h_1, y) + f(x, y+h_2) \quad \text{for all } h_1, h_2 \geq 0;$$

2. $f \in \mathbb{F}_1$ if
$$f(x+h, y) \geq f(x, y+h) \quad \text{for all } h \geq 0,$$
and $f \in \mathbb{F}_2$ if
$$f(x+h, y) \leq f(x, y+h) \quad \text{for all } h \geq 0;$$

3. $f \in \mathbb{F}_{12}$ if
$$f(x+h, y) - f(x, y) \in \mathbb{F}_2 \quad \text{for all } h \geq 0,$$
and $f \in \mathbb{F}_{21}$ if
$$f(x, y+h) - f(x, y) \in \mathbb{F}_1 \quad \text{for all } h \geq 0.$$

If f is twice differentiable then f being submodular resembles the condition $\partial^2 f(x, y)/\partial x \partial y \leq 0$ and $f \in \mathbb{F}_{12}$ resembles the condition $\partial^2 f(x, y)/\partial x^2 \leq \partial^2 f(x, y)/\partial x \partial y$. Similarly, $f \in \mathbb{F}_{21}$ resembles the condition $\partial^2 f(x, y)/\partial x \partial y \leq \partial^2 f(x, y)/\partial y^2$.

4. Properties on contract constraints

4.1. Monotonicity

The results in this subsection are quite intuitive, however, they build the foundation for the proof of linearity results that follow this subsection.

Theorem 1. *For all $i \leq n$ and $u \in R_i$,*

$$H_i(u, v, s) \in \mathbb{F}_{12} \bigcap \mathbb{F}_{21} \text{ and is submodular in } (u, v),$$
$$H_i(u, v, s) \text{ is concave in } (u, v) \text{ componentwise},$$
$$H_i(u, r-u, s) \text{ is concave in } u \in [0, r],$$
$$q_i(u, v, s) \uparrow_{(u,v)} \text{ and } q_i(u, v, s) \in \mathbb{F}_1 \text{ as a function of } (u, v).$$

From Theorem 1 we obtain bounds on the general contract value function in terms of $H_i(u, 0, s)$ and $H_i(0, v, s)$ for all feasible u and v.

Corollary 1. *If $u + v \in R_i$ then*

$$\begin{aligned}
&\frac{u}{u+v} H_i(u+v, 0, s) + \frac{v}{u+v} H_i(0, u+v, s) \\
&\quad \leq H_i(u, v, s) \\
&\quad \leq \min(H_i(0, u+v, s), H_i(0, v, s) + H_i(u, 0, s)).
\end{aligned}$$

Proof. The left-hand side inequality is immediate from the concavity of $H_i(u, r-u, s)$ in u. Obviously, $H_i(u, v, s) \leq H_i(0, u+v, s)$.

Now consider that a portfolio consists of one contract $C_i(u, 0)$ and one contract $C_i(0, v)$. Since we can always choose the same purchase pattern for this portfolio as the one for $C_i(u, v)$, then $H_i(u, v, s) \leq H_i(u, 0, s) + H_i(0, v, s)$.

Suppose that there are two swing contracts, $C_{1,i}(\cdot, \cdot)$ (contract 1) and $C_{2,i}(\cdot, \cdot)$ (contract 2). The only difference between the two contracts is that, for contract 1 and some $I \in \{0, \ldots, n\}$, there is an additional purchasing opportunity at time $\hat{T} \in (t_I, t_{I+1})$ (assuming that $t_{n+1} = T$).

Let the value functions for contracts 1 and 2 be H_i^1 and H_i^2, respectively. Then we have the following result.

Theorem 2. *For $i = 0, \ldots, I$, $j = 1, 2$, and $u \in R_i$,*

$$H_i^j(u, v, s) \text{ is submodular in } (u, j), \text{ and in } (v, j) \text{ when } u = 0,$$
$$q_i^1(u, v, s) \leq q_i^2(u, v, s).$$

Proof. See Appendix A.

A Markov process is homogeneous if $[S(t) \mid S(0) = x]$ has the same distribution as $[S(t_0 + t) \mid S(t_0) = x]$ for all t_0.

By treating contract $C_i(\cdot, \cdot)$ as $C_{1,i}(\cdot, \cdot)$ and contract $C_{i+1}(\cdot, \cdot)$ as $C_{2,i}(\cdot, \cdot)$ in Theorem 2, we have the following result.

Corollary 2. *If $K_i = K$ and $\theta_i = \theta$ for all i, and the commodity price process $\{S(t) \colon t \geq 0\}$ is a homogeneous Markov Process, then*

$$H_i(u, v, s) \text{ is submodular in } (i, u), \text{ and in } (i, v) \text{ when } u = 0,$$
$$q_i(u, v, s) \uparrow_i .$$

4.2. Linearity

Based on previous results, we will show that the problem of computing the contract value and the optimal strategy can be reduced to the special case of when both u and v are multiples of M. In this case the optimal strategy will be a 'bang-bang' strategy, i.e. at each time grid, either we buy M, or we do not buy at all. To prove this, we need the following lemma.

Lemma 1. *Assume that $kM + jM + x + y \in R_i$, where k and j are integers, and that $0 \leq x, y \leq M$, then*

$$H_i(kM + x, jM + y, s) \text{ is linear in } x \in [0, M - y] \text{ and } x \in [M - y, M],$$
$$H_i(kM + (M - y), jM + y, s) \text{ is linear in } y \in [0, M],$$
$$H_i(kM, jM + y, s) \text{ is linear in } y \in [0, M].$$

Let $q_0 = q_i(kM + x, jM + y, s)$, then

$$q_0 \in \{0, x, x + y, M\} \text{ if } x + y \leq M ,$$
$$q_0 \in \{0, x + y - M, x, M\} \text{ if } x + y \geq M.$$

Proof. See Appendix A.

Remark. Lemma 1 shows that the optimal purchase amount would be among one of the four special values. If the commodity is counted unit by unit (e.g. packages) instead of having continuous form (e.g. electricity) then, by assuming that the commodities can be purchased with fractional amount, the optimal purchase amount will still be an integer as long as u, v, and M are integers. This is a strategy that is achievable under the constraint that the purchasing amount cannot be fractional. So both forms of commodity will have the same optimal strategy and contract value function. Because of this, we continue to assume that the commodity has continuous form.

From Lemma 1 we immediately find, when the obligatory and bonus amounts are both multiples of the single purchasing limit M, that the optimal purchasing amount is equal to either 0 or the maximum allowed amount M at that time.

Theorem 3. *For all feasible integers k and j, $q_i(kM, jM, s) = 0$ or M.*

Lemma 1 allows us to reduce the general swing contract to one in which both u and v are multiples of M.

Theorem 4. *Under the same assumptions as in Lemma 1, if $x + y \le M$ then*

$$\begin{aligned} H_i(kM + x, jM + y, s) = {} & \frac{M - x - y}{M} H_i(kM, jM, s) \\ & + \frac{y}{M} H_i(kM, (j+1)M, s) \\ & + \frac{x}{M} H_i((k+1)M, jM, s). \end{aligned}$$

If $x + y \ge M$ then

$$\begin{aligned} H_i(kM + x, jM + y, s) = {} & \frac{M - y}{M} H_i((k+1)M, jM, s) \\ & + \frac{M - x}{M} H_i(kM, (j+1)M, s) \\ & + \frac{x + y - M}{M} H_i((k+1)M, (j+1)M, s). \end{aligned}$$

So, once we know the contract value function $H_i(kM, jM, s)$ for all feasible integers k and j, we can calculate the value of other swing contracts as well.

Based on Theorems 3 and 4, we will assume that $M = 1$, and that $u = k$ and $v = j$ (both are integers) in the following. So the contract becomes similar to that of the American call option, though more complicated. The purchasing decision is now binary: either we buy or we wait till the next purchasing opportunity.

5. Properties on the spot market price

In Section 4 we showed how the contract constraints affect the optimal strategy and contract value function. All the results we have shown hold for all single-factor Markov price processes, except in Corollary 2 where we needed the price processes to be homogeneous. Now we show how the commodity price process affects the optimal strategy and the contract value function.

5.1. Preliminary

Before we proceed, it is convenient for us to introduce some basic concepts in probability and stochastic ordering.

For any random vector X, let F_X be its distribution function, i.e. $F_X(x) = \mathrm{P}(X \le x)$ for all x. Define $X \overset{\mathrm{D}}{=} Y$ if the random vectors X and Y have the same distribution function. Define

$$\{X(t) \colon t \in A\} \overset{\mathrm{D}}{=} \{Y(t) \colon t \in A\}$$

if

$$(X(t_1), \dots, X(t_n)) \overset{\mathrm{D}}{=} (Y(t_1), \dots, Y(t_n)) \quad \text{for all } t_1, \dots, t_n \in A.$$

https://doi.org/10.1239/jap/1208358947 Published online by Cambridge University Press

Definition. For two random variables X and Y, define $X \le_{\mathrm{st}} Y$ if

$$F_X(x) \ge F_Y(x) \quad \text{for all } x.$$

For a random variable set $\{X_z : z \ge 0\}$, define $X_z \uparrow_z^{\mathrm{st}}$ if

$$X_{z_1} \le_{\mathrm{st}} X_{z_2} \quad \text{for all } z_1 \le z_2.$$

Shaked and Shanthikumar [14, p. 4] provided the following result.

Lemma 2. *$X \le_{\mathrm{st}} Y$ is equivalent to $\mathrm{E}\, f(X) \le \mathrm{E}\, f(Y)$ for any nondecreasing function f.*

Definition. For a random sequence $\{X_n\}_{n\ge 1}$, X, define $X_n \xrightarrow{\mathrm{D}} X$ if $F_X(x) = \lim_{n\to\infty} F_{X_n}(x)$ at all continuous points of $F_X(x)$.

For all time grids t_i, we assume the following in the discussions below:

(a) $S_{i+1}^{i,0} = 0,\ S_{i+1}^{i,s} \uparrow_s^{\mathrm{st}}$;

(b) $\mathrm{E}\, S_{i+1}^{i,s} < \infty$ and continuous in s;

(c) $S_{i+1}^{i,s} \xrightarrow{\mathrm{D}} \infty$ as $s \to \infty$.

5.2. Monotonicity of the spot market price

Now we give the structural relationship between the optimal strategy and the spot market price.

Theorem 5. *On $k \in R_i$, the following assertions hold.*

(a) *If $\mathrm{E} \exp(-r\theta_{i+1}) S_{i+1}^{i,s} - s \downarrow_s$ for all i then $H_i(k, j, s) - (k + j)(s - K_i)$ is submodular in (k, s), and in (j, s) when $k = 0$; $q_i(k, j, s) \uparrow_s$.*

(b) *If $\mathrm{E} \exp(-r\theta_{i+1}) S_{i+1}^{i,s} - s \uparrow_s$ for all i then $H_i(k, 0, s) - k(s - K_i)$ is supermodular in (k, s); $q_i(k, 0, s) \downarrow_s$.*

(c) *If, for all i, $\mathrm{E} \exp(-r\theta_{i+1}) S_{i+1}^{i,s} - s = 0$ for all s then $H_i(k, 0, s) - k(s - K_i)$ and $q_i(k, 0, s)$ is the same for different s.*

Proof. See Appendix A.

5.3. Examples

Assume that $\{B_t : t \ge 0\}$ is a standard Brownian motion started at 0, having drift parameter 0, and volatility parameter 1. Let $\{S_t^s\}_{t\ge 0}$ be the market price process starting at s; we will omit the superscript 's' when it does not generate confusion. We will assume that $S_t^s \uparrow_{\mathrm{st}} s$ for all t. This assumption can be shown to be satisfied for all continuous Markov price processes, including all the examples provided below.

Now we provide some concrete market price process examples within which the conditions of Theorem 5 are satisfied. These examples and their extensions are widely used in various commodity market price process models (see Section 1 for references).

Example 1. (*Decaying price process.*) Suppose that the price process S_t is described by

$$\mathrm{d}S_t = (rS_t - \delta(t, S_t))\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}B_t,$$

where δ and σ are bivariate measurable functions. If $\delta(t, s) \uparrow_s$ then $\mathrm{E}\, \mathrm{e}^{-rt} S_t^s - s \downarrow_s$ for all t. If $\delta(t, s) \downarrow_s$ then $\mathrm{E}\, \mathrm{e}^{-rt} S_t^s - s \uparrow_s$ for all t.

These results can be easily proved by combining Lemma 2 and

$$\mathrm{E}\,\mathrm{e}^{-rt} S_t^s - s = -\int_0^t \mathrm{e}^{-rx}\,\mathrm{E}\,\delta(x, S_x^s)\,\mathrm{d}x.$$

When $\delta(t, s) = (\kappa(t) + r)s - \kappa(t)\mu(t)$, we have the following result.

Example 2. (*Mean reverting process.*) If the price process follows

$$\mathrm{d}S_t = \kappa(t)(\mu(t) - S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}B_t, \qquad \kappa(t) \geq 0,$$

where $\kappa(\cdot) \geq 0$ and $\mu(\cdot) \geq 0$, then $\mathrm{E}\,\mathrm{e}^{-rt} S_t^s - s \downarrow_s$ for all t.

When $\delta(t, s) = (r - \mu)s$ and $\sigma(t, s) = \sigma s$, we have the following result.

Example 3. (*Geometric Brownian motion.*) If the price process follows

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t,$$

where μ and σ are both constants, then $\mathrm{E}\,\mathrm{e}^{-rt} S_t^s - s \uparrow_s$ if $\mu \geq r$ and $\mathrm{E}\,\mathrm{e}^{-rt} S_t^s - s \downarrow_s$ if $\mu \leq r$.

5.4. Boundary results

When current price is high enough, the optimal value function and strategy becomes simple, as shown in Theorem 6. The results can be trivially proved by backward induction.

Theorem 6. *For $i \leq n - 1$ and $k + j \in R_i$, the following assertions hold.*

(a) *If* $\mathrm{E}\exp(-r\theta_{i+1}) S_{i+1}^{i,s} - s \to \infty$ *for all i then*

$$\lim_{s\to\infty}\left(H_i(k, j, s) - \sum_{h=0}^{k+j-1} \mathrm{E}\exp(-r(t_{n-h} - t_i))(S_{n-h}^{i,s} - K_{n-h})\right) = 0.$$

Therefore, $\lim_{s\to\infty} q_i(k, j, s) = 0$ *if* $k + j \in D_i$.

(b) *If* $\mathrm{E}\exp(-r\theta_{i+1}) S_{i+1}^{i,s} - s \to -\infty$ *for all i then*

$$\lim_{s\to\infty}\left(H_i(k, j, s) - \sum_{h=0}^{k+j-1} \mathrm{E}\exp(-r(t_{i+h} - t_i))(S_{i+h}^{s} - K_{i+h})\right) = 0.$$

Therefore, $\lim_{s\to\infty} q_i(k, j, s) = 1$.

6. Optimal strategy structure

Combining the results from the previous sections, we will provide a detailed structure of the optimal strategy in this section. Since the American call option (or more accurately, the Bermudan option) is a special case of the swing contract with $k = 0$ and $j = 1$, then the results we obtain here also apply to the American call option (with time-dependent strike prices).

6.1. General swing contract

We have assumed that once the commodity price becomes 0, all its future prices will stay at 0. In this case the optimal strategy can be determined easily.

Let $q_i^0(k) = q_i(k, 0, 0)$ for feasible k, clearly $q_i^0(k) = 0$ if and only if $\exp(-rt_i)K_i$ is among the k largest values of $\{\exp(-rt_{i+1})K_{i+1}, \dots, \exp(-rt_n)K_n\}$.

By Theorem 1 and Theorem 5, we immediately have the following result.

Lemma 3. *If* $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \uparrow_s$ *for all* i *then* $q_i^0(k+j) = 0$ *implies that* $q_i(k, j, s) = 0$ *for all* s.

Combining Theorems 1, 5, and 6, and Lemma 3, we can prove the following result.

Theorem 7. *When* $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \downarrow_s -\infty$ *for all* i *and* $k + j \in R_i$,

1. *if* $q_i^0(k) = 1$ *then* $q_i(k, j, s) = 1$ *for all* $s \geq 0$;
2. *if* $q_i^0(k) = 0$ *then* $q_i(k, j, s) = \mathbf{1}_{(s>s_i)}$ *for some* s_i.

When $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \uparrow_s \infty$ *for all* i *and* $k \in R_i$,

1. *if* $q_i^0(k) = 1$ *then* $q_i(k, 0, s) = \mathbf{1}_{(s\leq s_i)}$ *for some* s_i;
2. *if* $q_i^0(k) = 0$ *then* $q_i(k, 0, s) = 0$ *for all* $s \geq 0$.

When $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s = 0$ *for all* i, s, *and* $k + j \in R_i$,

1. *if* $q_i^0(k) = 1$ *then* $q_i(k, j, s) = 1$ *for all* $s \geq 0$;
2. *if* $q_i^0(k + j) = 0$ *then* $q_i(k, j, s) = 0$ *for all* $s \geq 0$;
3. *otherwise* $q_i(k, j, s) = \mathbf{1}_{(s>s_i)}$ *for some* s_i.

Note that the $s_i \equiv s_i(k, j)$ in Theorem 7 are finite and functions of (k, j). We omit this functional relationship where it is evident. Based on Theorem 1, $s_i(k, j) \downarrow_{(k,j)}$.

Theorem 7 implies the following result.

Corollary 3. *Assume that* $K_i = K$ *for all* i *and that* $k \in D_i$.

1. *If* $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} \geq s$ *for all* i *and* s *then* $q_i(k, 0, s) = q_i(0, k, s) = 0$ *for all* s.
2. *If* $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \downarrow_s -\infty$ *for all* i *then* $q_i(k, j, s) = \mathbf{1}_{(s>s_i)}$ *for some* s_i.
3. *If* $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} = s$ *for all* i *and* s *then* $q_i(k, j, s) = \mathbf{1}_{(s>s_i, k+j=n-i+1)}$ *for some* $s_i = s_i(k) \in (0, K]$ *and* $s_i(0) = K$.

6.2. American call option

In this subsection we assume that the market price process follows a geometric Brownian motion, and we are interested in an American call option with time-dependent strike prices.

When $\mu \geq r$ and the strike prices are equal at all times, the optimal strategy is well known: wait till maturity and then buy if and only if the market price is above the strike price; when the strike prices are different, we have the following 'interval' optimal strategy.

https://doi.org/10.1239/jap/1208358947 Published online by Cambridge University Press

Theorem 8. *If $\mu > r$, let $V_i(s) = V_i(0, 1, s)$ and $q_i(s) = q_i(0, 1, s)$. Then the following assertions hold.*

1. *$V_i(s) - s$ is convex in s and reaches a minimum at some $s_i \in (0, \infty)$.*
2. *If $V_i(s_i) \geq s_i - K_i$ then $q_i(s) = 0$ for all $s \geq 0$, and if $V_i(s_i) \leq s_i - K_i$ then $q_i(s) = \mathbf{1}_{(A_i < s < B_i)}$ for some (A_i, B_i) satisfying $0 < A_i < s_i < B_i < \infty$.*

Theorem 8 implies that, under some situations, it is optimal to exercise the option immediately if and only if the current price is within some interval.

Proof of Theorem 8. See Appendix A.

7. Conclusion

In this paper we proved that we can transform the general swing contract into an American style option that specifies the minimum and maximum numbers of exercising opportunities. We have shown the monotonicity relationship between the optimal strategy and the contract constraints as well as the market price processes. By examining the optimal strategy as the price goes to ∞, we found that the optimal strategies for the swing contract have simple threshold policies under some conditions on the market price process, and we gave widely used market price process examples in which these conditions were satisfied. For an American call option, when the discounted price process drifts upwards and the strike prices are time varying, we showed that the optimal strategy is of interval type.

In a separate paper by the authors (see [11]), we utilized the structure of the optimal strategy given in this paper to develop computational techniques for calculating the swing contract value function. In that paper we also obtained bounds on how far our estimator is from the true contract value. In the case when the commodity was purchased continuously (e.g. electricity) over time, we showed how the swing contract value could be approximated by its discrete counterpart. This paper also discussed the issue of a nonhomogeneous purchasing capacity and interest rate.

The techniques we developed for proving properties of the swing contract can be extended to other American style options with relatively few state variables as well. The swing contract we considered in this paper is of call type; another group of swing contracts are put type, where the contract owner has the right to sell commodity at specified strike prices. Parallel results can be derived for the put type contracts.

Appendix A.

Following the same proof as Ross [10, p. 6], we have the following result.

Lemma 4. *For a bivariate function $f(x, y)$, the following assertions hold.*

1. *If $f(x, y)$ is supermodular, and $a_x \uparrow_x$ and $b_x \uparrow_x$, then $a_x \leq b_x$ for all x. Let $f(x, y_x) = \max\{f(x, y) : y \in [a_x, b_x]\}$, then $y_x \uparrow_x$.*
2. *If $f(x, y)$ is submodular, $f(x, y_x) = \max\{f(x, y) : y \in [a, b]\}$, where a and b are constants. Then $y_x \downarrow_x$.*

For bivariate functions $f(x, y)$, $a(x, y)$, and $b(x, y)$, assume that

$$a(x, y) \leq b(x, y) \quad \text{and} \quad U(x, y) = [a(x, y), b(x, y)].$$

Let $g(x, y) = f(x-q(x, y), y) = \max\{f(x-q, y) : q \in U(x, y)]$, then we have the following result.

Lemma 5. (a) *If $f(x, y)$ is submodular and*

$$a(x_1, y_2) = \max(a(x_1, y_1), a(x_2, y_2) + x_1 - x_2) \quad \textit{for all } x_1 \leq x_2, y_1 \leq y_2, \tag{1}$$

$$b(x_2, y_1) = \min(b(x_1, y_1) + x_2 - x_1, b(x_2, y_2)) \quad \textit{for all } x_1 \leq x_2, y_1 \leq y_2, \tag{2}$$

then $g(x, y)$ is submodular in (x, y) and $q(x, y) \uparrow_y$.

(b) *If $f(x, y)$ is supermodular, and $a(x, y)$ and $b(x, y)$ do not depend on y, i.e. $a(x, y) = \hat{a}(x)$ and $b(x, y) = \hat{b}(x)$, then $\hat{a}(x) - x \downarrow_x$ and $\hat{b}(x) - x \downarrow_x$ imply that $g(x, y)$ is supermodular in (x, y), and $q(x, y) \downarrow_y$.*

Proof. Let $U(x, y) = [a(x, y), b(x, y)]$.

(a) Note that conditions (1) and (2) imply that

$$a(x, y) - x \downarrow_x, \qquad b(x, y) - x \downarrow_x, \qquad a(x, y) \uparrow_y, \quad \text{and} \quad b(x, y) \uparrow_y . \tag{3}$$

The submodularity of $f(x, y)$ implies that $f(x-q, y)$ is supermodular in (q, y), so $q(x, y) \uparrow_y$, by Lemma 4.

For any $x_1 < x_2$ and $y_1 < y_2$, if $x_1 - q(x_1, y_1) \geq x_2 - q(x_2, y_2)$ then

$$\begin{aligned} a(x_2, y_1) &\leq a(x_1, y_1) + x_2 - x_1 \quad \text{(by (6))} \\ &\leq q(x_1, y_1) + x_2 - x_1 \\ &\leq \min(b(x_1, y_1) + x_2 - x_1, q(x_2, y_2)) \\ &\leq \min(b(x_1, y_1) + x_2 - x_1, b(x_2, y_2)) \\ &= b(x_2, y_1) \quad \text{(by (2))}, \end{aligned}$$

so $q(x_1, y_1) + x_2 - x_1 \in U(x_2, y_1)$. Similarly, $q(x_2, y_2) + x_1 - x_2 \in U(x_1, y_2)$.

Let $t_1 = q(x_1, y_1) + x_2 - x_1$ and $t_2 = q(x_2, y_2) + x_1 - x_2$; therefore,

$$\begin{aligned} g(x_1, y_1) + g(x_2, y_2) &= f(x_1 - q(x_1, y_1), y_1) + f(x_2 - q(x_2, y_2)) \\ &= f(x_2 - t_1, y_1) + f(x_1 - t_2, y_2) \\ &\leq g(x_2, y_1) + g(x_1, y_2) \quad \text{(by definition)}. \end{aligned}$$

Conversely, if $x_1 - q(x_1, y_1) < x_2 - q(x_2, y_2)$, we can show, similar to the above proof, that $q(x_1, y_1) \in U(x_1, y_2)$ and $q(x_2, y_2) \in U(x_2, y_1)$. Thus,

$$\begin{aligned} &g(x_1, y_1) + g(x_2, y_2) \\ &\quad = f(x_1 - q(x_1, y_1), y_1) + f(x_2 - q(x_2, y_2)) \\ &\quad \leq f(x_2 - q(x_2, y_2), y_1) + f(x_1 - q(x_1, y_1), y_2) \quad \text{(by submodularity)} \\ &\quad \leq g(x_2, y_1) + g(x_1, y_2) \quad \text{(by definition)}. \end{aligned}$$

So, $g(x, y)$ is submodular.

(b) Similar to the proof in part (a) with some inequalities reversed, we can show part (b). This completes the proof.

It is easy to check that $f(x, y) \in \mathbb{F}_{12}$ if and only if, for all $h_1, h_2 > 0$,

$$f(x, y+h_2) + f(x+h_1+h_2, y) \le f(x+h_1, y+h_2) + f(x+h_1, y). \tag{4}$$

Symmetrically, $f(x, y) \in \mathbb{F}_{21}$ if and only if, for all $h_1, h_2 > 0$,

$$f(x+h_1, y) + f(x, y+h_1+h_2) \le f(x+h_2, y+h_1) + f(x, y+h_2). \tag{5}$$

The interrelationships between submodularity, concavity, and the $\mathbb{F}_{12}$ and $\mathbb{F}_{21}$ properties are given by the following result.

Lemma 6. *For a bivariate function $f(x, y)$, the following assertions hold.*

(a) *If $f(x, y) \in \mathbb{F}_{12} \bigcap \mathbb{F}_{21}$ then $f(x, r-x)$ is concave in $x \in [0, r]$.*

(b) *If $f(x, y) \in \mathbb{F}_{12}$ and if $f(x, y)$ is submodular then $f(x, y)$ is concave in x.*

(c) *If $f(x, y) \in \mathbb{F}_{21}$ and if $f(x, y)$ is submodular then $f(x, y)$ is concave in y.*

Proof. (a) By adding (4) and (5), we obtain

$$f(x, y+h_1+h_2) + f(x+h_1+h_2, y) \le f(x+h_1, y+h_2) + f(x+h_2, y+h_1).$$

This is equivalent to the concavity of $f(x, r-x)$.

Parts (b) and (c) can be proved similarly.

Lemma 7. (a) *If $f(x, y) \in \mathbb{F}_{12}$ and, for all $h_1, h_2 > 0$,*

$$\max(a(x, y+h_2), a(x+h_1+h_2, y) - h_1) = a(x+h_2, y), \tag{6}$$
$$\min(b(x+h_1+h_2, y), b(x, y+h_2) + h_1) = b(x+h_1, y+h_2), \tag{7}$$

then $g \in \mathbb{F}_{12}$ and $q(k, j) \in \mathbb{F}_1$.

(b) *If $f(x, y) \in \mathbb{F}_{12} \cap \mathbb{F}_{21}$, $f(x, y)$ is submodular, and $a(x, y) = \hat{a}(x)$ and $b(x, y) = \hat{b}(x+y)$, then $\hat{a}(x)$ and $\hat{b}(x)$ $\uparrow_x$ imply that $g(x, y) \in \mathbb{F}_{21}$.*

Proof. (a) If $f(x, y) \in \mathbb{F}_{12}$, let $f_z(x, q) = f(x-q, z-x)$ on $x \le z$, then $f_z(x, q)$ is supermodular in (x, q). From the definition of $q(x, y)$ we have

$$f_z(x, q(x, z-x)) = \max\{f_z(x, q) \colon a(x, z-x) \le q \le b(x, z-x)\}.$$

Also, conditions (6) and (7) imply that

$$\begin{aligned} a(x, y) &\in \mathbb{F}_1 \quad \text{and} \quad a(x+h, y) - h \downarrow_h, \\ b(x, y) &\in \mathbb{F}_1 \quad \text{and} \quad b(x+h, y) - h \downarrow_h . \end{aligned}$$

So, Lemma 4 implies that $q(x, z-x) \uparrow_x$; thus, $q(x, y) \in \mathbb{F}_1$.

For $h_1, h_2 > 0$, similar to the proof of Lemma 5, we can show that if $q(x, y+h_2) + h_1 \ge q(x+h_1+h_2, y)$ then $q(x, y+h_2) \in U(x+h_2, y)$ and $q(x+h_1+h_2, y) \in U(x+h_1, y+h_2)$. If otherwise then $q(x, y+h_2)+h_1 \in U(x+h_1, y+h_2)$ and $q(x+h_1+h_2, y)-h_1 \in U(x+h_2, y)$. Similar to the proof of Lemma 5, we can show that, in both cases,

$$g(x, y+h_2) + g(x+h_1+h_2, y) \le g(x+h_2, y) + g(x+h_1, y+h_2).$$

So, $g(x, y) \in \mathbb{F}_{12}$.

https://doi.org/10.1239/jap/1208358947 Published online by Cambridge University Press

(b) For any $h_1, h_2 > 0$, let $q_1 = q(x+h_2, y)$ and $q_2 = q(x, y+h_1+h_2)$. If $q_1 \geq q_2$ then (omit checking of feasibility)

$$\begin{aligned} & f(x-q_2, y+h_1+h_2) - f(x-q_2, y+h_2) \\ & \quad \leq f(x-q_1, y+h_1+h_2) - f(x-q_1, y+h_2) \quad \text{(by submodularity)} \\ & \quad \leq f(x-q_1+h_2, y+h_1) - f(x-q_1+h_2, y) \quad (\text{by } f \in \mathbb{F}_{21}). \end{aligned}$$

If $q_1 < q_2$ then (omit checking of feasibility)

$$\begin{aligned} & f(x-q_2, y+h_1+h_2) - f(x+h_2-q_2, y+h_1) \\ & \quad \leq f(x-q_1, y+h_1+h_2) - f(x+h_2-q_1, y+h_1) \quad (\text{by } f \in \mathbb{F}_{12}) \\ & \quad \leq f(x-q_1, y+h_1) - f(x-q_1+h_2, y) \quad (\text{by } f \in \mathbb{F}_{21}). \end{aligned}$$

Thus, in both case, we have

$$g(x, y+h_1+h_2) + g(x+h_2, y) \leq g(x, y+h_1) + g(x+h_2, y+h_1),$$

by definition of $g(x, y)$. Therefore, $g(x, y) \in \mathbb{F}_{21}$.

Proof of Theorem 1. The result is trivial for $i = n$. Suppose that $H_{i+1}(u, v, s) \in \mathbb{F}_{12} \cap \mathbb{F}_{21}$ and that $H_{i+1}(u, v, s)$ is submodular in (u, v). Then $V_i(u, v, s)$ satisfies the same properties. So, we can show that

$$V_i(x^+, y - x^-, s) \in \mathbb{F}_{12} \cap \mathbb{F}_{21} \text{ and is submodular in feasible } (x, y).$$

Lemmas 5 and 7 then give

$$H_i(u, v, s) \in \mathbb{F}_{12} \cap \mathbb{F}_{21} \text{ and is submodular in } (u, v).$$

Also, Lemmas 5 and 7 imply that $q(u, v, s) \uparrow_v$, $q(u, v, s) \in \mathbb{F}_1$, which gives $q(u, v, s) \uparrow_u$. The concavity results follow directly from Lemma 6.

Proof of Theorem 2. First we show that, for $u + h \in R_I$,

$$H_I^1(u, v, s) + H_I^2(u+h, v, s) \leq H_I^1(u+h, v, s) + H_I^2(u, v, s). \tag{8}$$

If $I = n$, we compare s and K_n at time n.

If $s \leq K_n$, suppose that the optimal purchasing amount for $C_{2,n}(u+h, v)$ and $C_{1,n}(u, v)$ are x and y, respectively (clearly $x = u + h$). Then $x - h$ and $y + h$ are feasible purchasing amounts for contracts $C_{2,n}(u, v)$ and $C_{1,n}(u+h, v)$, since $u + h \in R_n$ and $y \leq u$. Also, the remaining obligatory and bonus amounts for $C_{1,n}(u+h, v)$ and $C_{1,n}(u, v)$ will be equal. So (8) holds under the optimal strategy of $C_{1,n}(u+h, v)$ and $C_{2,n}(u, v)$.

If $s > K_n$, let x be the optimal purchasing amount for $C_{1,n}(u, v)$. If $x \leq M - h$ then we can buy an $x + h$ amount for $C_{1,n}(u+h, v)$; if $x > M - h$ then we can buy an M amount for $C_{1,n}(u+h, v)$ and its remaining obligatory amount becomes 0 (since $C_{2,n}(u+h, v)$ will purchase at least $u+h$ at time n for fulfilling obligation, implying that $u+h \leq M$), its remaining bonus amount is more than that of $C_{1,n}(u, v)$. In both cases it is easy to show that the total payoff for corresponding contracts on the left-hand side of (8) is less than or equal to the total payoff for corresponding contracts on the right-hand side at both time n and time $n+1$. So (8) holds again under the optimal strategy of $C_{1,n}(u+h, v)$.

For $I < n$. By purchasing the same amount for $C_{1,I}(u+h, v)$ (maybe suboptimal) as for $C_{1,I}(u, v)$ under the optimal strategy at time $\hat{T}$ and then using the concavity of $H_{I+1}(x^+, y - x^-, s)$ in $x \in [-y, (n-i)M]$ (Theorem 1), we obtain

$$V_I^2(u+h, v, s) + V_I^1(u, v, s) \leq V_I^2(u, v, s) + V_I^1(u+h, v, s).$$

Then (8) follows from Lemma 5. Similarly,

$$H_I^1(0, v, s) + H_I^2(0, v+h, s) \leq H_I^1(0, v+h, s) + H_I^2(0, v, s).$$

All the results then follow by backward induction with respect to i and Lemma 5.

Proof of Lemma 1. The results are trivial for $i = n$. Suppose that $H_{i+1}(u, v, s)$ satisfy the properties of this lemma. Then $V_i(u, v, s)$ also satisfies these piecewise linear properties.

So, for $k \geq 1$ and $x \in [0, M-y]$, we obtain $q_0 \in \{0, x, x+y, M\}$. Therefore,

$$\begin{aligned} H_i(kM+x, jM+y, s) = \max(&V_i(kM+x, jM+y, s), \\ &V_i(kM, jM+y, s) + x(s-K_i), \\ &V_i(kM-y, jM+y, s) + (x+y)(s-K_i), \\ &V_i((k-1)M+x, jM+y, s) + M(s-K_i)). \end{aligned}$$

By induction, all the above four parts are linear in $x \in [0, M-y]$, and so they are convex on that set. Since the maximal of convex functions are convex, $H_i(kM+x, jM+y, s)$ is therefore convex in $x \in [0, M-y]$. But we have shown that $H_i(u, v, s)$ is concave in u (by Theorem 1), so

$$H_i(kM+x, jM+y, s) \text{ is linear in } x \in [0, M-y].$$

Similarly, we can show that $H_i(0, jM+y, s)$ is linear in $y \in [0, M]$. From this we can prove that $H_i(x, jM+y, s)$ (that is, $k = 0$) is linear in $x \in [0, M-y]$.

The linearity and $q_0 \in \{0, (x+y)-M, x, M\}$ in $x \in [M-y, M]$ can be proved similarly.

For state $(kM+(M-y), jM+y, s)$, we have $q_0 \in \{0, M-y, M\}$; for state $(kM, jM+y, s)$, we have $q_0 \in \{0, y, M\}$. Then, similarly to the proof of the previous part, we can show the remaining results.

Proof of Theorem 5. The case in which

$$\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \downarrow_s \quad \text{or} \quad \mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s \uparrow_s$$

can be shown by backward induction and Lemma 5.

Now if $\mathrm{E}\exp(-r\theta_{i+1})S_{i+1}^{i,s} - s$ does not depend on s for all i then, from parts (a) and (b), we find that $q_i(k, 0, s)$ is both nondecreasing and nonincreasing in s, and that $\bar{H}_i(k, 0, s)$ is both submodular and supermodular in (k, s). Thus, both $q_i(k, 0, s)$ and $\bar{H}_i(k, 0, s) - \bar{H}_i(k-1, 0, s)$ do not depend on s for all feasible k. Since $\bar{H}_i(0, 0, s) = 0$, by induction on k upwardly, we can prove that $\bar{H}_i(k, 0, s)$ is constant for all k.

Proof of Theorem 8. The convexity of $V_i(s)$ (thus $V_i(s) - s$) is immediate by induction. We can show that $V_i(s)$ has derivative 0 at $s = 0$, therefore, $V_i(s) - s$ is decreasing around 0. Also, Theorem 6 implies that $\lim_{s\to\infty}(V_i(s) - s) = \infty$. So, there exists a unique minimum for $V_i(s) - s$ in $s \in (0, \infty)$. All the results then follow trivially.

References

[1] Ali, A. L., Mohamadreza, S. and Antony, W. (2001). A discrete valuation of swing options. *Canad. Appl. Math. Quat.* **9,** 35–73.
[2] Barbieri, A. and Garman, M. B. (1996). Putting a price on swings. *Energy Power Risk Manag.* **1**.
[3] Barbieri, A. and Garman, M. B. (1996). Understanding the valuation of swing contract. *Energy Power Risk Manag.* **1**.
[4] Carmona, R. and Touzi, N. (2003). Optimal multiple stopping and valuation of swing options. *Tech. Rep.* Available at http://www.cmap.polytechnique.fr/∼touzi/.
[5] Carmona, R. and Dayanic, S. (2003). Optimal multiple stopping of linear diffusions and swing options. *Tech. Rep.* Available at http://www.princeton.edu/∼sdayanik/.
[6] Deng, S. (1999). Stochastic models of energy commodity prices: mean reversion with jumps and spikes. Working paper, Georgia Institute of Technology. Available at http://www.ucei.berkeley.edu/.
[7] Jaillet, P., Ronn, E. and Tompaidis, S. (2003). Valuation of commodity-based swing options. *Manag. Sci.* **50,** 909–921.
[8] Keppo, J. (2004). Pricing of electricity swing options. *J. Derivatives* **11,** 26–43.
[9] Pilipovic, D. and Wengler, J. (1998). Getting into the swing. *Energy Power Risk Manag.* **2**.
[10] Ross, S. M. (1983). *Introduction to Stochastic Dynamic Programming*. Academic Press, Boston, MA.
[11] Ross, S. M. and Zhu, Z. (2005). Computing the value of swing contract. *Tech. Rep.*
[12] Schwartz, E. (1997). The stochastic behavior of commodity prices: implications for valuation and hedging. *J. Finance* **52,** 923–973.
[13] Schwartz, E. and James, S. (2000). Short-term variation and long-term dynamics of commodity prices. *Manag. Sci.* **46,** 893–911.
[14] Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and Their Applications*. Academic Press, Boston, MA.